# Contents

# Nets of real subspaces on homogeneous spaces and Algebraic Quantum Field Theory

Karl-Hermann Neeb

## Introduction

In these notes, we describe an interesting connection between unitary representations of Lie groups and nets of local algebras, as they appear in Algebraic Quantum Field Theory (AQFT). It is based on first translating the axioms for nets of operator algebras parameterized by regions in a spacetime manifold into those for nets of real subspaces, and then study this structure from a perspective based on geometry and representation theory of Lie groups.

This topic owes much of its fascination to the close relations between operator algebraic concepts, such as Kubo–Martin–Schwinger (KMS) conditions and spectral conditions, and the complex geometry related to unitary Lie group representations. To make this connection more concrete, suppose that $U_t = e^{itH}$ is a unitary one-parameter group on the complex Hilbert space $\mathcal{H}$, $H = H^*$ is its selfadjoint generator, and $\xi \in \mathcal{H}$. We are interested in analytic continuations of the orbit map

$$U^\xi \colon \mathbb{R} \to \mathcal{H}, \quad t \mapsto U_t\xi.$$

If a bounded analytic extension exists on the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$, then its range lies in an invariant subspace on which the operator $H$ is non-negative (spectral condition). This is rather restrictive, and it is much more common that $U^\xi$ only extends to the closure of a strip $\mathcal{S}_\beta = \{z \in \mathbb{C} \colon 0 < \operatorname{Im} z < \beta\}$. Here the most interesting context arises if the upper boundary values are coupled to the lower ones by a conjugation $J$ (an antilinear isometric involution) via

$$JU^\xi(i\beta + t) = U^\xi(t) \quad \text{for} \quad t \in \mathbb{R}.$$

Karl-Hermann Neeb
Department Mathematik, Friedrich–Alexander-Universität Erlangen-Nürnberg, Cauerstrasse 11, 91056 Erlangen, e-mail: neeb@math.fau.de

This is precisely the situation one finds in the modular theory of operator algebras if $\xi$ represents a KMS state (thermal state), and the case of positive spectrum corresponds to so-called ground states ([BGN20], [BN24], [NÓ19], [ANS25], [NR24]). Below we shall see that such conditions also specify so-called standard subspaces $\mathtt{V} \subseteq \mathcal{H}$ (for $\beta = \pi$) if $(U_t)_{t\in\mathbb{R}}$ is the corresponding modular group.

On the geometric side, an action $\sigma$ of a Lie group $G$ on a manifold $M$ often has a "complexification" in the sense that $M$ sits in the boundary of a complex manifold $\Xi$ that locally looks like a tube domain $\mathbb{R}^n + i\Omega \subseteq \mathbb{C}^n$, i.e., $\Omega \subseteq \mathbb{R}^n$ is a pointed open convex cone. In this context, one may also ask for extensions of orbit maps $\sigma^m \colon \mathbb{R} \to M, t \mapsto \exp(tx).m$ ($m \in M, x \in \mathfrak{g} = \mathbf{L}(G)$), to the upper half-plane $\sigma^m \colon \mathbb{C}_+ \to \Xi$, or to a strip $\sigma^m \colon \mathcal{S}_\beta \to \Xi$. In the latter case, we typically have an antiholomorphic involution $\tau_\Xi$ satisfying $\tau_\Xi(\sigma^m(i\beta + t)) = \sigma^m(t)$ for $t \in \mathbb{R}$. In the context of semisimple Lie groups, such situations are well-known for non-compactly causal symmetric spaces $M = G/H$, sitting in the boundary of the so-called complex crown of the Riemannian symmetric space $G/K$ ([GK02]). Then the existence of such analytic extensions specifies so-called wedge regions $W \subseteq M$, that can be characterized in many different ways ([NÓ23a, NÓ23b, NÓ23c]). Here the "imaginary tangent cone", specifying how $M$ sits in the boundary of $\Xi$, determines the causal structure on $M$. So $M$ carries similar geometric structures as the spacetimes in Mathematical Physics. Our goal is to connect the analytic extension phenomena in unitary group representations and the underlying geometry with structures in AQFT.

These notes consist of five main sections, followed by a discussion of perspectives and several appendices on background in various areas. In **Section 1** we discuss axioms for nets of local observables, as they appear in Algebraic Quantum Field Theory (AQFT). This involves a symmetry group $G$ (a connected Lie group) acting on a manifold $M$ (spacetime in the physics context) and, for each open subset $\mathcal{O} \subseteq M$, a von Neumann algebra $\mathcal{M}(\mathcal{O})$ on some complex Hilbert space $\mathcal{H}$, on which we also have a unitary representation $(U, \mathcal{H})$ of $G$, i.e., a continuous homomorphism $U \colon G \to \mathrm{U}(\mathcal{H})$.

Open subsets $\mathcal{O} \subseteq M$ may be considered as laboratories, in which experiments are performed that correspond to the evaluation of quantum observables, represented by hermitian elements of $\mathcal{M}(\mathcal{O})$. This leads to families, also called *nets*, of von Neumann algebras $(\mathcal{M}(\mathcal{O}))_{\mathcal{O}\subseteq M}$.

Before we turn to specific properties of such nets, we explain a key tool which is an important integredient in the modular theory of operator algebras: the Tomita–Takesaki Theorem (Theorem 1.22). Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and $\Omega \in \mathcal{H}$ be a non-zero vector which is *cyclic*, i.e., $\overline{\mathcal{M}\Omega} = \mathcal{H}$, and *separting*, i.e., the map $\mathcal{M} \to \mathcal{H}, M \mapsto M\Omega$, is injective. This means that the closed real subspace

$$\mathtt{V} := \mathtt{V}_{\mathcal{M},\Omega} := \overline{\mathcal{M}_h\Omega}, \qquad \mathcal{M}_h := \{M \in \mathcal{M} \colon M^* = M\},$$

is *cyclic*, i.e., $\mathtt{V} + i\mathtt{V}$ is dense in $\mathcal{H}$, and *separting*, i.e., $\mathtt{V} \cap i\mathtt{V} = \{0\}$ (cf. Lemma 1.20). We call such closed real subspaces *standard*. For every standard subspace $\mathtt{V} \subseteq \mathcal{H}$, the Tomita operator

$$T_{\mathtt{V}} \colon \mathcal{D}(T_{\mathtt{V}}) := \mathtt{V} + i\mathtt{V} \to \mathcal{H}, \qquad T_{\mathtt{V}}(v + iw) := v - iw$$

is closed, hence has a polar decomposition, i.e., $\Delta_{\mathtt{V}} := T_{\mathtt{V}}^* T_{\mathtt{V}}$ is a positive selfadjoint operator, and there exists an antilinear isometry $J_{\mathtt{V}}$ such that

$$T_{\mathtt{V}} = J_{\mathtt{V}} \Delta_{\mathtt{V}}^{1/2}$$

(see Definition 1.3 for details). Both factors are uniquely determined by $T_{\mathtt{V}}$. Then $(\Delta_{\mathtt{V}}^{it})_{t\in\mathbb{R}}$ is a unitary one-parameter group, and the main assertions of the Tomita–Takesaki Theorem are:

$$J\mathcal{M}J = \mathcal{M}' := \{A \in B(\mathcal{H}) \colon (\forall M \in \mathcal{M})\, AM = MA\},$$

and

$$\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M} \quad \text{for all} \quad t \in \mathbb{R}.$$

As a consequence, $\alpha_t(M) := \Delta^{it} M \Delta^{-it}$ defines a one-parameter group of automorphisms of $M$, called the *modular group*.

We are now ready to state the axioms of nets of local algebras. The axioms that we discuss here are:

- (Iso) **Isotony:** $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)$.
- (RS) **Reeh–Schlieder property:** There exists a unit vector $\Omega \in \mathcal{H}$ that is cyclic for $\mathcal{M}(\mathcal{O})$ if $\mathcal{O} \neq \emptyset$, i.e., $\mathcal{M}(\mathcal{O})\Omega$ is dense in $\mathcal{H}$ (cf. [RS61], [Ja00], [Ja00b]),
- (Cov) **Covariance:** $U_g \mathcal{M}(\mathcal{O}) U_g^{-1} = \mathcal{M}(g\mathcal{O})$ for $g \in G$.
- (Vac) **Invariance of the vacuum:** $U(g)\Omega = \Omega$ for $g \in G$.
- (BW) **Bisognano–Wichmann property:** There exists a Lie algebra element $h \in \mathfrak{g}$ and an open subset $W \subseteq M$ (called a wedge region), such that $\Omega$ is cyclic and separating for $\mathcal{M}(W)$, and the corresponding modular operator $\Delta$ of the standard subspace $\mathtt{V}_{\mathcal{M},\Omega} = \overline{\mathcal{M}_h \Omega}$ satisfies $\Delta^{-it/2\pi} = U(\exp th)$ for $t \in \mathbb{R}$. In this sense, the modular group is geometrically implemented by a one-parameter subgroup of $G$ (cf. [BDFS00], [BMS01], [HL82]).
- (Loc) **Locality:** There exists an open non-empty $G$-invariant subset $\mathcal{D}_{\mathrm{loc}} \subseteq M \times M$ such that $\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{D}_{\mathrm{loc}}$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)'$.
- (Add) **Additivity:** The von Neumann algebra $\mathcal{M}(\bigcup_j \mathcal{O}_j)$ is generated by the algebras $\mathcal{M}(\mathcal{O}_j), j \in J$.

A first step in our analysis is to simplify these structures by replacing the algebras $\mathcal{M}(\mathcal{O})$ by the real subspaces

$$\mathsf{H}(\mathcal{O}) := \mathtt{V}_{\mathcal{M}(\mathcal{O}),\Omega}.$$

Note that, whenever $\Omega$ is cyclic and separating for $\mathcal{M}(\mathcal{O})$, the corresponding modular operator and the modular involution are completely encoded by $\mathsf{H}(\mathcal{O})$.

We now formulate the basic axioms for the family $(\mathsf{H}(\mathcal{O}))_{\mathcal{O}\subseteq M}$:

(Iso) **Isotony**: $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)$.
(RS) **Reeh–Schlieder property:** $\mathsf{H}(\mathcal{O})$ is cyclic if $\mathcal{O} \neq \emptyset$.
(Cov) **Covariance:** $U_g\mathsf{H}(\mathcal{O}) = \mathsf{H}(g\mathcal{O})$ for $g \in G$.
(BW) **Bisognano–Wichmann property:** There exists a Lie algebra element $h \in \mathfrak{g}$ and an open connected subset $W \subseteq M$, such that $\mathsf{H}(W)$ is standard and the corresponding modular operator $\Delta$ satisfies $\Delta^{-it/2\pi} = U(\exp th)$ for $t \in \mathbb{R}$.
(Loc) **Locality:** There exists an open non-empty $G$-invariant subset $\mathcal{D}_{\rm loc} \subseteq M \times M$ such that $\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{D}_{\rm loc}$ implies $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)'$.
(Add) **Additivity:** $\mathsf{H}(\bigcup_j \mathcal{O}_j) = \overline{\sum_{j\in J} \mathsf{H}(\mathcal{O}_j)}$.

Our goal is to understand such nets and the requirements on the $G$-space $M$, its geometry, the structure of $G$ and the representation $(U, \mathcal{H})$ for which such nets exist. Eventually, one would like to “classify” all these nets in a suitable sense, but first one has to specify which structures we are dealing with. Key questions are:

(Q1) Which $G$-invariant structures do we need on $M$ as a fertile ground for nets of real subspaces?
(Q2) Which elements $h \in \mathfrak{g}$ can arise in the Bisognano–Wichmann (BW) condition?
(Q3) How to find the domains $W \subseteq M$ arising in the (BW) condition?
(Q4) For which unitary representations of $G$ are there isotone, covariant nets of real subspaces satisfying (RS) and (BW)?

In these notes we shall not go deeper into locality requirements, but see Section 5.6.1. We refer to [MNÓ26b] and [NÓ26] for recent progress in this direction. As we shall see below, these questions are highly intertwined, in particular when we discuss (Q4) in Section 3.

A key result, described in **Section 2**, answers (Q2), namely that $h$ has to be an *Euler element*, i.e., $\operatorname{ad} h$ is non-zero and diagonalizable with $\operatorname{Spec}(\operatorname{ad} h) \subseteq \{-1, 0, 1\}$. In the physical context of the Lorentz and Poincaré group, these are suitably normalized generators of Lorentz boosts.

In **Section 3** we further argue that it is natural to require $M$ to carry a *causal structure*, i.e., a field of pointed generating closed convex cones $C_m \subseteq T_m(M)$, invariant under the $G$-action. Given an Euler element $h$ and a causal structure on $M$, the natural candidates for $W$ are the connected components of the *positivity region*

$$W_M^+(h) = \Big\{ m \in M \colon \frac{d}{dt}\Big|_{t=0} \exp(th).m \in C_m^\circ \Big\} \tag{1}$$

of the vector field on $M$ corresponding to $h$. We discuss these structures for various examples. Since it will play an important role later in the construction of nets of real subspaces, we describe the compression semigroups

$$S_W := \{g \in G \colon g.W \subseteq W\}$$

for several types of wedge regions $W$. The most important examples of causal homogeneous spaces $M$ are causal symmetric spaces and causal flag manifolds (see Section 3.4 for more on flag manifolds).

In **Section 4** we turn to constructions of nets for a given antiunitary representation $(U, \mathcal{H})$ and an Euler element $h \in \mathfrak{g}$. This is motivated by the consequence of the Euler Element Theorem 2.3, according to which we may assume that the Lie algebra involution $\tau_h^{\mathfrak{g}} = e^{\pi i \operatorname{ad} h}$ integrates to a group involution $\tau_h$ (e.g., if $G$ is simply connected), so that we can form the group

$$G_{\tau_h} = G \rtimes \{\mathrm{id}_G, \tau_h\}$$

and assume that $U$ extends to an antiunitary representation of $G_{\tau_h}$. This specifies in particular a standard subspace $\mathtt{V} = \mathtt{V}(h, U)$ by

$$\Delta_{\mathtt{V}} = e^{2\pi i \cdot \partial U(h)} \quad \text{and} \quad J_{\mathtt{V}} = U(\tau_h) \tag{2}$$

(Definitions 1.3 and 2.20).

To find a net $\mathsf{H}$ satisfying (BW) with $\mathsf{H}(W) = \mathtt{V}$, it is instructive to observe that the elements of $\mathtt{V}$ are characterized by the (abstract) Kubo–Martin–Schwinger (KMS) condition: The orbit map $U^v(t) := U(\exp th)v$ extends analytically to the closure of the strip $\mathcal{S}_\pi = \{z \in \mathbb{C} \colon 0 < \operatorname{Im} z < \pi\}$, such that

$$U^v(\pi i) = J_{\mathtt{V}} v$$

(cf. Proposition 1.42).

This suggests to look for domains $W \subseteq M$ and a complex manifold $\Xi$ with $M \subseteq \partial \Xi$, on which $G$ acts by holomorphic maps, such that $W$ consists of elements $m \in M$ whose orbit map $\alpha^m(t) = \exp(th).m$ yields by analytic extension a map $\mathcal{S}_\pi \to \Xi$ satisfying $\alpha^m(\pi) = \overline{\tau}_h(m)$, where $\overline{\tau}_h$ is an antiholomorphic involution on $\Xi$, satisfying $\overline{\tau}_h(g.z) = \tau_h(g).\overline{\tau}_h(z)$ for $g \in G, z \in \Xi$. We call these points the *KMS points of* $M$.

For the case where $G$ is contained in its universal complexification $G_{\mathbb{C}}$, we describe in Section 4 conditions on a domain $\Xi \subseteq G_{\mathbb{C}}$ (crown domains for $G$), so that the following construction leads to nets. We start with a real subspace $\mathtt{F}$ of $J_{\mathtt{V}}$-fixed vectors $v \in \mathcal{H}$, whose orbit map $U^v \colon G \to \mathcal{H}$ extends analytically to a map $U^v \colon \Xi \to \mathcal{H}$, such that the limit

$$\beta^+(v) = \lim_{t \to \frac{\pi}{2}} U^v(\exp(-ith)) \tag{3}$$

exists in the space $\mathcal{H}^{-\infty}(U_h)$ of distribution vectors for the one-parameter group $U_h(t) = U(\exp th)$. We then have natural inclusions

$$\mathcal{H}^\infty \subseteq \mathcal{H}^\infty(U_h) \subseteq \mathcal{H} \subseteq \mathcal{H}^{-\infty}(U_h) \subseteq \mathcal{H}^{-\infty}$$

(see Appendix 7.5.2 for details) and

$$\mathtt{E} := \beta^+(\mathtt{F}) \subseteq \mathcal{H}^{-\infty}(U_h) \subseteq \mathcal{H}^{-\infty}$$

is a real subspace. For $\varphi \in C_c^\infty(G, \mathbb{R})$, the operator $U^{-\infty}(\varphi) = \int_G \varphi(g) U^{-\infty}(g)\, dg$ maps $\mathcal{H}^{-\infty}$ to $\mathcal{H}$. We thus obtain by

$$\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) := \overline{\mathrm{span}_{\mathbb{R}}\{U^{-\infty}(\varphi)\mathtt{E} \colon \varphi \in C_c^\infty(\mathcal{O}, \mathbb{R})\}} \subseteq \mathcal{H}, \tag{4}$$

a net of real subspaces on $G$ satisfying (Iso) and (Cov) for trivial reasons, but also (RS) and (BW). Here the main point is to show that $\mathsf{H}^G_{\mathtt{E}}(W^G) = \mathtt{V}$ holds for a suitable open subset $W^G \subseteq G$.

*Example.* Elementary particles in the sense of E. Wigner [Wgn39] (see also [Ni20]) are classified by irreducible unitary representations of the Poincaré group $G = \mathbb{R}^{1,d-1} \rtimes \mathrm{SO}_{1,d-1}(\mathbb{R})_e$. We write $\mathrm{V} := \mathbb{R}^{1,d-1}$ for the corresponding translation group. For scalar particles, the Hilbert space is of the form $\mathcal{H} = L^2(\mathbb{R}^{1,d-1}, \mu)$, where $\mu$ is a Lorentz invariant measure on the dual space $\mathrm{V}^*$ (often identified with V via the Lorentzian form). Here the space $\mathtt{E} = \mathbb{R}1$ of real-valued constant functions represents distribution vectors, and for test functions $\varphi \in C_c^\infty(\mathrm{V}, \mathbb{R})$, we have $U^{-\infty}(\varphi)\mathbf{1} = \widehat{\varphi}$ (Fourier transform). So the real subspace $\mathsf{H}^{\mathrm{V}}_{\mathtt{E}}(\mathcal{O})$ from (4) is generated by Fourier transforms of test functions supported in $\mathcal{O}$.

This leaves us with the question of how to find the crown domains $\Xi$ and real subspaces $\mathtt{F} \subseteq \mathcal{H}^J$. For semisimple groups, this can be done with the theory of crown domains for Riemannian symmetric spaces $G/K$. They provide natural domains $\Xi \subseteq G_{\mathbb{C}}$ to which orbit maps of $K$-finite vectors[1] of irreducible representations extend. A recent result by T. Simon ([Si24]) ensures that they have a sufficiently well-behaved boundary behavior at $\partial\Xi$ to ensure (3) and $\mathtt{E} = \beta^+(\mathtt{F}) \subseteq \mathcal{H}^{-\infty}(U_h)$. Here an important point is that **no further requirement beyond irreducibility** is needed for $U$ to obtain these nets, and they all descend in a natural way to the *non-compactly causal symmetric spaces* $M = G/H$, associated to the Euler element $h$ (cf. Sections 3.5, 5.3.2, [MNÓ23, Thm. 4.21]). An important example is de Sitter space $M = \mathrm{dS}^d \cong \mathrm{SO}_{1,d}(\mathbb{R})/\mathrm{SO}_{1,d-1}(\mathbb{R})$, for which the Euler element $h \in \mathfrak{so}_{1,d}(\mathbb{R})$ is the generator of a Lorentz boost.

**Section 5** develops a global perspective on these results. Here we are dealing with representations that are not necessarily irreducible. Starting

[1] Here $K \subseteq G$ is a maximal compact subgroup and $K$-finiteness means that $U(K)\xi$ is contained in a finite-dimensional subspace of $\mathcal{H}$.

with a homogeneous space $M = G/H$, a domain $W \subseteq M$ and an antiunitary representation $(U, \mathcal{H})$, we directly obtain two nets $\mathsf{H}_M^{\max}$ and $\mathsf{H}_M^{\min}$ on $M$, such that any net $\mathsf{H}$ on $M$ satisfying (Iso), (Cov) and $\mathsf{H}(W) = \mathtt{V} := \mathtt{V}(h, U)$ (cf. (2)) also satisfies

$$\mathsf{H}_M^{\min}(\mathcal{O}) \subseteq \mathsf{H}(\mathcal{O}) \subseteq \mathsf{H}_M^{\max}(\mathcal{O})$$

for every open subset $\mathcal{O} \subseteq M$ (Lemma 5.8). From this perspective, the question is, whether a net $\mathsf{H}$ satisfying (Iso), (Cov) and (BW) exists at all. This is equivalent to $\mathsf{H}^{\max}(W) = \mathtt{V}$, which in turn is equivalent to the inclusion of semigroups

$$S_W = \{g \in G\colon g.W \subseteq W\} \subseteq S_{\mathtt{V}} = \{g \in G\colon U(g)\mathtt{V} \subseteq \mathtt{V}\}. \tag{5}$$

The semigroup $S_W$ has already been described in Section 3 for several types of wedge regions. If $\ker U$ is discrete, then

$$S_{\mathtt{V}} = \exp(C_+) G_{\mathtt{V}} \exp(C_-) \quad \text{holds for} \quad C_\pm = \pm C_U \cap \mathfrak{g}_{\pm 1}(h), \tag{6}$$

where

$$C_U := \{x \in \mathfrak{g}\colon -i \cdot \partial U(x) \geq 0\}, \quad \text{with} \quad \partial U(x) = \frac{d}{dt}\Big|_{t=0} U(\exp tx),$$

is the *positive cone of the unitary representation* $U$, and $\mathfrak{g}_\lambda(h) = \ker(\lambda \mathbf{1} - \operatorname{ad} h)$ are the eigenspaces of $\operatorname{ad} h$ (Section 5.2).

If $G$ is semisimple and $M$ the non-compactly causal symmetric space associated to the Euler element $h$, then $S_W$ is a group, hence equal to $G_W$, so that (5) reduces to the inclusion $G_W \subseteq G_{\mathtt{V}}$, which boils down to the implication

$$g.W = W \Rightarrow U(g)J = JU(g),$$

which is equivalent to $\tau_h(g)^{-1} g \in \ker U$ for $g \in G_W$.

If $S_W$ is not a group, it is of the form

$$S_W = \exp(C_+) G_W \exp(C_-),$$

where the convex cones $C_\pm$ are specified in terms of an $\operatorname{Ad}(G)$-invariant cone $C_{\mathfrak{g}} \subseteq \mathfrak{g}$ by

$$C_\pm = \pm C_{\mathfrak{g}} \cap \mathfrak{g}_{\pm 1}(h).$$

Therefore (6) implies that $S_W \subseteq S_{\mathtt{V}}$ is equivalent to $G_W \subseteq G_{\mathtt{V}}$ and the *spectral condition*

$$C_\pm \subseteq C_U$$

on the representation $U$, i.e., the operators $-i\partial U(x)$ are positive for $x \in \pm C_\pm$. For the Poincaré group, acting on Minkowski space (Remark 1.29), this corresponds to the positivity of the energy.

We conclude these notes with a discussion of perspectives and open problems in **Section 6** and a number of appendices.

**Some history.** The starting point for the development that led to fruitful applications of modular theory in QFT was the Bisognano–Wichmann Theorem ([BW75, BW76]), asserting that the modular automorphisms $\alpha_t(M) = \Delta^{-it/2\pi} M \Delta^{it/2\pi}$ associated to the algebra $\mathcal{M}(W_R)$ of observables corresponding to the *Rindler wedge*

$$W_R = \{(x_0, x_1, \ldots, x_{d-1}) \colon x_1 > |x_0|\}$$

in $d$-dimensional Minkowski space $\mathbb{R}^{1,d-1}$, are implemented by the action of a one-parameter group of Lorentz boosts preserving $W_R$. This geometric implementation of modular automorphisms in terms of Poincaré transformations was a key step in a rich development based on the work of Borchers and Wiesbrock in the 1990s [Bo68, Bo92, Bo95, Bo97, Wi92, Wi93a, Wi93b, Wi98, GLW98]; see also [LRT78], [Lo82], [Fr85] and [Ha96]. They managed to distill the abstract essence from the Bisognano–Wichmann Theorem which led to a better understanding of the basic configurations of von Neumann algebras in terms of half-sided modular inclusions and modular intersections (see also [NÓ17, §4.4]). In his survey [Bo00], Borchers described how these concepts revolutionized quantum field theory. Subsequent developments can be found in [Ar99, Schr99, BGL02, Su05, Lo08, LW11, LL15, JM18, Mo18, MMTS21].

The condition that the modular group and conjugation of some local algebras correspond to geometric transformations, is also known as the condition of *Geometric Modular Action (GMA)*. In AQFT it serves as a selection principle for physical vacuum states, connecting Tomita–Takesaki theory to spacetime symmetries ([Bo93, Bo00], [Su96], [BFS99], [BDFS00], [BMS01], [SW03], [BS05], [LMaR09]).

The interplay between AQFT and spacetime symmetries led to renewed interest in causal symmetric spaces in relation with representation theory. Here causal symmetric spaces are considered as analogs of spacetime manifolds, where the causal cones $C_m \subseteq T_m(M)$ need not be Lorentzian. The irreducible Lorentzian symmetric spaces are de Sitter space $\mathrm{dS}^d$ (spacelike positively curved), anti-de Sitter space $\mathrm{AdS}^d$ (spacelike positively curved), and flat Minkowski space $\mathbb{R}^{1,d-1}$. Products of these manifolds with Riemannian symmetric spaces, such as spheres $\mathbb{S}^k$ and hyperbolic spaces, lead to natural Lorentzian causal symmetric spaces. This perspective allows to study various aspects of AQFT in a highly symmetric context without the need of an invariant Lorentzian form. For the classification of irreducible (non-flat) causal symmetric spaces we refer to the monograph [HÓ97] which builds on Ólafsson's classification [Ól91]. Note that there are also interesting homogeneous Lorentzian manifolds that are not symmetric, such as the Gödel Universe (cf. [Be24], [HN93]).

Another interesting class of causal manifolds has been studied by Mack and de Riese in [MdR07], the *simple spacetime manifolds*. These are the simply connected coverings $\widetilde{M}$ of the conformal compactifications $M$ of simple euclidean Jordan algebras (see [Ne26], [Be96] and the appendix to Section 3.4). The classical example is the *Einstein universe* $\widetilde{M} = \mathbb{R} \times \mathbb{S}^{d-1}$, as the simply connected covering of the conformal completion $(\mathbb{S}^1 \times \mathbb{S}^{d-1})/\{\pm\mathbf{1}\}$ of Minkowski space $\mathbb{R}^{1,d-1}$ (cf. [Ba25], [MNÓ26b]). The same class of manifolds has been studied by Günaydin in [Gu93] (see also the much earlier reference [Gu75]), where they are called *Jordan spacetimes*. The particular case of 4-dimensional Minkowski space, where $M \cong \mathrm{U}_2(\mathbb{C})$ carries a group structure, appears already in the work of Irving Segal [Se71, Se76]. The compact causal manifolds $M$ can also be characterized as those flag manifolds $G/P$ of a simple Lie group $G$ that carry a causal structure ([Ne26] and Theorem 3.25). In this context, the theory of open orbits of symmetric subgroups of $G$ in these spaces, so called *causal Makarevič spaces* ([Be96, Be98]), lead to natural "compactifications" of causal symmetric spaces ([Ne26]), and these are a key ingredient in holographic models (see [dB01], [St01], [BCW25], and also [Wo24] for a recent popular article relating to quantum cosmology).

An important link between QFT and geometry is the the Euler Element Theorem 2.3 ([MN24]) which implies that generators of modular groups, arising from local nets on causal homogeneous spaces by the Bisognano–Wichmann property, are Euler elements, and this answers our question (Q2). In the Lorentzian context, this means that they generate Lorentz boosts in suitable coordinates. This fact motivated the revision of the classification of causal symmetric spaces from the perspective of Euler elements in [MNÓ23]. It also suggested an abstract approach to modular pairs $(\Delta, J)$ from a purely Lie theoretic perspective (cf. Definition 2.19), that was started in [MN21] and exploited in the context of pairs of Euler nets and Spin–Statistics results in [MNÓ26a, MNÓ26b], building on [GL95].

At this point it is clear that there is a natural rich supply of causal homogeneous space on which one would like to understand local nets satisfying the Bisognano–Wichmann condition (BW). To understand the underlying geometry, one needs a good descriptions of the wedge regions, i.e., the connected components of the positivity regions $W_M^+(h)$ of Euler elements (cf. (1)). For compactly causal symmetric spaces, this was done in [NÓ23a], and in [NÓ23b] for non-compactly causal ones, for which the results were later refined to a classification of *modular geodesics* in [MNÓ24]. For causal Makarevič spaces we refer to [Ne26] and for causal flag manifolds to [MN26]. These results answer Question (Q3).

Progress concerning question (Q4), which representations of the group $G$ permit nets of real subspaces satisfying (Iso), (Cov), (RS) and (BW), was achieved rather stepwise. For the case where $M = G$ is a hermitian Lie groups and unitary highest weight representations, nets were constructed as in (4) in [NÓ21] and it also turned out that this construction can be used for causal flag manifolds ([MN26]). A major step was the unexpected insight, that for any connected simple Lie group $G$ and any antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$, many nets with the above four properties can be constructed on the associated non-compactly causal symmetric spaces (see [FNÓ25a], and [FNÓ25b] and [BEM98] for more more specific information concerning de Sitter space). For non-semisimple Lie groups, the results are still equite incomplete, but see [BN25] and [Oeh22a, Oeh23] for natural strategies.

**How to read these notes?** Each of the five sections has a main part and appendices. The appendices contain more details and discussion of related issues. So they can be skipped on first reading.

The following list of notations will be used throughout, if not specified otherwise.

### Notation

- $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.
- $\mathcal{H}$ denotes a complex Hilbert space. The scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ is linear in the second and antilinear in the first argument.
- Strips in the complex plane:

$$\mathcal{S}_\beta = \{z \in \mathbb{C} \colon 0 < \operatorname{Im} z < \beta\} \quad \text{and} \quad \mathcal{S}_{\pm\beta} = \{z \in \mathbb{C} \colon |\operatorname{Im} z| < \beta\}$$

- The neutral element of a Lie group $G$ is denoted $e$, and $G_e$ is the identity component.
- Left and right multiplications on $G$ are denoted by $\lambda_g(x) = gx$ and $\rho_g(x) = xg$.
- The Lie algebra of a Lie group $G$ is denoted $\mathbf{L}(G)$ or $\mathfrak{g}$.
- For an involutive automorphism $\sigma$ of $G$, we write $G^\sigma = \{g \in G \colon \sigma(g) = g\}$ for the subgroup of fixed points and $G_\sigma := G \rtimes \{\mathrm{id}_G, \sigma\}$ for the corresponding group extension.
- $\mathrm{AU}(\mathcal{H})$ is the group of unitary or antiunitary operators on a complex Hilbert space $\mathcal{H}$.
- An antiunitary representation of $G_\sigma$ is a homomorphism $U \colon G_\sigma \to \mathrm{AU}(\mathcal{H})$ with $U(G) \subseteq \mathrm{U}(\mathcal{H})$ for which the involution $J := U(\sigma)$ is antiunitary, i.e., a *conjugation*. We denote representations as pairs $(U, \mathcal{H})$.
- If $G$ is a group acting on a set $M$ and $W \subseteq M$ a subset, then the stabilizer subgroup of $W$ in $G$ is denoted $G_W := \{g \in G \colon g.W = W\}$, and the compression semigroup by $S_W := \{g \in G \colon g.W \subseteq W\}$.
- If $\mathfrak{g}$ is a Lie algebra and $h \in \mathfrak{g}$, then $\mathfrak{g}_\lambda(h) = \ker(\operatorname{ad} h - \lambda \mathbf{1})$ is the $\lambda$-eigenspace of $\operatorname{ad} h$ and $\mathfrak{g}^\lambda(h) = \bigcup_k \ker(\operatorname{ad} h - \lambda \mathbf{1})^k$ is the generalized $\lambda$-eigenspace.
- An element $x$ of a Lie algebra $\mathfrak{g}$ is called
  - ◇ *hyperbolic*, if $\operatorname{ad} x$ is diagonalizable over $\mathbb{R}$.
  - ◇ *elliptic* or *compact*, if $\operatorname{ad} x$ is semisimple with purely imaginary spectrum, i.e., $\overline{e^{\mathbb{R} \operatorname{ad} x}}$ is a compact subgroup of $\mathrm{Aut}(\mathfrak{g})$.

- For a Lie subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$, we write $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{s}) = \langle e^{\mathrm{ad}\,\mathfrak{s}} \rangle \subseteq \mathrm{Aut}(\mathfrak{g})$ for the subgroup generated by $e^{\mathrm{ad}\,\mathfrak{s}}$. We call $\mathfrak{s}$ *compactly embedded* if the group $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{s})$ has compact closure.
- We write $\mathcal{E}(\mathfrak{g})$ for the set of *Euler elements* $h \in \mathfrak{g}$, i.e., $\mathrm{ad}\,h$ is non-zero and diagonalizable with $\mathrm{Spec}(\mathrm{ad}\,h) \subseteq \{-1, 0, 1\}$.
- We call $h$ *symmetric* if $-h \in \mathcal{O}_h := \mathrm{Inn}(\mathfrak{g})h$. We write $\tau_h := e^{\pi i\,\mathrm{ad}\,h} \in \mathrm{Aut}(\mathfrak{g})$ for the involution of $\mathfrak{g}$ specified by $h$.
- A *causal $G$-space* is a smooth $G$-space $M$, endowed with a $G$-invariant *causal structure*, i.e., a field $(C_m)_{m\in M}$ of pointed generating closed convex cones $C_m \subseteq T_m(M)$.
- For a unitary representation $(U, \mathcal{H})$ of $G$ we write:
  - ◇ $\partial U(x) = \frac{d}{dt}\big|_{t=0} U(\exp tx)$ for the infinitesimal generator of the unitary one-parameter group $(U(\exp tx))_{t\in\mathbb{R}}$ in the sense of Stone's Theorem.
  - ◇ $\mathtt{d}U \colon \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$ for the representation of the Lie algebra $\mathfrak{g}$ on the space $\mathcal{H}^\infty$ of smooth vectors. Then $\partial U(x) = \overline{\mathtt{d}U(x)}$ (operator closure) for $x \in \mathfrak{g}$.
- For a $*$-algebra $\mathcal{M} \subseteq B(\mathcal{H})$, we write $\mathcal{M}_h := \{A \in \mathcal{M} \colon A^* = A\}$ for the real subspace of hermitian elements, and for $\Omega \in \mathcal{H}$, we put $\mathtt{V}_{\mathcal{M},\Omega} := \overline{\mathcal{M}_h \Omega}$.
- For a, possibly unbounded, operator $T \colon \mathcal{D}(T) \to \mathcal{H}$, we write $\mathcal{R}(T) := T(\mathcal{H})$ for its range and $\mathcal{N}(T) := \ker(T)$ for its kernel.

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# 1 Nets of operator algebras and AQFT

In this section we discuss standard subspaces and their connections with von Neumann algebras. This provides the background for the translation between nets of local algebras and the corresponding nets of standard subspaces.

These axioms can be discussed on two levels: the level of operator algebras and the level of standard subspaces. We start in Section 1.1 with the concept of a standard subspace $\mathtt{V}$ of a complex Hilbert space $\mathcal{H}$. In particular we show that they can be parametrized by pairs $(\Delta, J)$, where $\Delta > 0$ is a selfadjoint operator and $J$ is a *conjugation* (an antilinear isometric involution) satisfying the modular relation $J\Delta J = \Delta^{-1}$. This is the key data that appears in the modular theory of operator algebras (Tomita–Takesaki Theorem 1.22). Section 1.2 discusses some finer tools, including the "dual" standard subspace $\mathtt{V}'$ and criteria for equality of standard subspaces. Many examples of standard subspaces have natural descriptions as graphs of closed operators. This perspective is explored further in Section 1.3. In Section 1.4 we then turn to the level of operator algebras. We take a closer look at the Tomita–Takesaki Theorem and some key examples. Section 1.5 contains a sufficient criterion for the equality of two von Neumann subalgebras $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{M}$ in terms of the corresponding real subspaces. Such tools are important for translating between the order structure on real subspaces and von Neumann algebras. After these preparations, we turn in Section 1.6 to nets of local algebras and of real subspaces, providing more background and context. At this point we know how to pass from nets of local algebras to nets of real subspaces, but there are also various natural constructions in the opposite direction. Since we shall not need this below, we only briefly explain the simplest one of these, related to the symmetric/bosonic Fock space in Section 1.7. We conclude this section with some appendices in Section 1.8. These appendices provide background information that is not used in the main text, but may provide some illuminating background on the constructions in the following sections.

## 1.1 Standard subspaces of Hilbert spaces

In this subsection, we introduce the key concept of a standard subspace $\mathtt{V}$ of a complex Hilbert space $\mathcal{H}$. Standard subspaces are "slanted" real forms in the sense that $\mathtt{V} + i\mathtt{V}$ is dense in $\mathcal{H}$ and $\mathtt{V} \cap i\mathtt{V} = \{0\}$. As we shall see below, they are parametrized by pairs $(\Delta, J)$, where $\Delta > 0$ is a selfadjoint operator and $J$ is a conjugation (an antilinear isometric involution) satisfying the modular relation $J\Delta J = \Delta^{-1}$. Standard subspaces appear naturally in the modular theory of operator algebras (Tomita–Takesaki Theorem 1.22) and also in antiunitary representations of Lie groups, where they correspond to antiunitary representations of the one-dimensional multiplicative group

$$\mathbb{R}^\times \cong \mathbb{R} \times \{\pm 1\}.$$

This establishes an important link between operator algebras and antiunitary representations.

**Definition 1.1** (a) A closed real subspace $\mathtt{V} \subseteq \mathcal{H}$ is called

- *separating* if $\mathtt{V} \cap i\mathtt{V} = \{0\}$,
- *cyclic* if $\mathtt{V} + i\mathtt{V}$ is dense in $\mathcal{H}$,
- *standard* if it is cyclic and separating.

We write $\mathrm{Stand}(\mathcal{H})$ for the set of standard subspaces of $\mathcal{H}$.

(b) For a separating subspace $\mathtt{V}$, we define the antilinear *Tomita involution*

$$T_{\mathtt{V}} \colon \mathtt{V} + i\mathtt{V} \to \mathcal{H}, \quad T_{\mathtt{V}}(v+iw) = v - iw \quad \text{for} \quad v, w \in \mathtt{V}.$$

(c) We write $\gamma(v,w) := \operatorname{Im}\langle v, w\rangle$ for the canonical symplectic form on $\mathcal{H}$. For a real subspace $\mathtt{V} \subseteq \mathcal{H}$, we define its *symplectic orthogonal space* by

$$\mathtt{V}' := \mathtt{V}^{\perp_\gamma} = \{w \in \mathcal{H} \colon \operatorname{Im}\langle v, w\rangle = 0\} = i\mathtt{V}^{\perp_{\mathbb{R}}},$$

where $\mathtt{V}^{\perp_{\mathbb{R}}}$ is the real orthogonal space of $\mathtt{V}$ with respect to the real-valued scalar product $\operatorname{Re}\langle v, w\rangle$. Note that $\langle \mathtt{V}, \mathtt{V}'\rangle \subseteq \mathbb{R}$.

**Lemma 1.2** *If $\mathtt{V}$ is standard, then $T_{\mathtt{V}}$ is closed and densely defined.*

***Proof*** As $\mathtt{V}$ is cyclic, the operator $T_{\mathtt{V}}$ is densely defined. To see that the graph of $T_{\mathtt{V}}$ is closed, suppose that $\xi_n = a_n + ib_n$ is a sequence in $\mathcal{D}(T_{\mathtt{V}}) = \mathtt{V} + i\mathtt{V}$ with $a_n, b_n \in \mathtt{V}$, such that $(\xi_n, T_{\mathtt{V}}\xi_n) = (a_n + ib_n, a_n - ib_n) \to (\xi, \eta)$ in $\mathcal{H} \times \mathcal{H}$. As $\mathtt{V}$ is closed,

$$a_n = \frac{1}{2}(a_n + ib_n + (a_n - ib_n)) = \frac{1}{2}(\xi_n + T_{\mathtt{V}}\xi_n) \to \frac{1}{2}(\xi + \eta) =: a \in \mathtt{V},$$

and

$$b_n = \frac{1}{2i}(a_n + ib_n - (a_n - ib_n)) = \frac{1}{2i}(\xi_n - T_{\mathtt{V}}\xi_n) \to \frac{1}{2i}(\xi - \eta) =: b \in \mathtt{V}.$$

Therefore $\xi = a + ib \in \mathcal{D}(T_{\mathtt{V}})$ satisfies $T_{\mathtt{V}}\xi = a - ib = \eta$. This means that $T_{\mathtt{V}}$ is closed. □

**Definition 1.3** We have seen in Lemma 1.2 that, for every standard subspace $\mathtt{V} \subseteq \mathcal{H}$, the Tomita operator

$$T_{\mathtt{V}} \colon \mathcal{D}(T_{\mathtt{V}}) := \mathtt{V} + i\mathtt{V} \to \mathcal{H}, \qquad T_{\mathtt{V}}(v + iw) := v - iw$$

is closed, hence has a polar decomposition ([Sch12, Thm. 7.2], [SZ79, Thm. 9.29][2]), i.e.,

$$\Delta_{\mathtt{V}} := T_{\mathtt{V}}^* T_{\mathtt{V}}$$

is a positive selfadjoint operator, and there exists an antilinear isometry $J_{\mathtt{V}}$ such that

$$T_{\mathtt{V}} = J_{\mathtt{V}} \Delta_{\mathtt{V}}^{1/2}.$$

[2] To obtain the polar decomposition of a closed operator $T$, the main step is to show that the operator $T^*T$ is selfadjoint. Then the unique positive square root $|T| := \sqrt{T^*T}$ satisfies $\| \, |T|\xi\| = \|T\xi\|$ for all $\xi \in \mathcal{D}(T)$, which easily leads to a partial isometry $U$ from the range $\overline{\mathcal{R}(|T|)} = \mathcal{N}(|T|)^\perp = \mathcal{N}(T)^\perp$ of $|T|$ to the range $\overline{\mathcal{R}(T)}$ of $T$ with $T = U|T|$.

The isometry $J_{\mathtt{V}}$ is defined on all of $\mathcal{H}$ because $\Delta_{\mathtt{V}}$ has dense range, which in turn follows from $\mathcal{R}(\Delta_{\mathtt{V}})^\perp = \ker(\Delta_{\mathtt{V}}) = \ker(T_{\mathtt{V}}) = \{0\}$. The relation

$$J_{\mathtt{V}}\Delta_{\mathtt{V}}^{1/2} = T_{\mathtt{V}} = T_{\mathtt{V}}^{-1} = \Delta_{\mathtt{V}}^{-1/2}J_{\mathtt{V}}^{-1} = J_{\mathtt{V}}^{-1}(J_{\mathtt{V}}\Delta_{\mathtt{V}}^{-1/2}J_{\mathtt{V}}^{-1})$$

and the uniqueness of the polar decomposition now imply $J_{\mathtt{V}}^2 = \mathbf{1}$ and the *modular relation*

$$J_{\mathtt{V}}\Delta_{\mathtt{V}}J_{\mathtt{V}} = \Delta_{\mathtt{V}}^{-1}. \tag{7}$$

The unitary one-parameter group $(\Delta_{\mathtt{V}}^{it})_{t\in\mathbb{R}}$ is called the *modular group of* $\mathtt{V}$. It has the important property that it preserves $\mathtt{V}$ (Remark 1.4(b)) and its true importance is revealed in the Tomita–Takesaki Theorem 1.22.

*Remark 1.4* (a) The modular group $\Delta_{\mathtt{V}}^{it}$ commutes with the antiunitary conjugation $J_{\mathtt{V}}$. In fact, the antilinearity of $J_{\mathtt{V}}$ implies that

$$J_{\mathtt{V}}\Delta_{\mathtt{V}}^{z}J_{\mathtt{V}} = \Delta_{\mathtt{V}}^{-\overline{z}} \quad \text{for} \quad z \in \mathbb{C}.$$

In view of [NÓ15, Prop. 3.1], a unitary one-parameter group $(U_t = e^{itH})_{t\in\mathbb{R}}$ commutes with some conjugation $J$ if and only if $H$ is *symmetric* in the sense that there exists a unitary involution $S$ satisfying $SHS^{-1} = -H$.
(b) The fact that the operators $\Delta_{\mathtt{V}}^{it}$ commute with $J_{\mathtt{V}}$ implies that they also commute with $T_{\mathtt{V}}$, hence leave $\mathtt{V}$ invariant.

**Definition 1.5** We write $\mathrm{Stand}(\mathcal{H})$ for the set of standard subspaces of the complex Hilbert space $\mathcal{H}$ and $\mathrm{Mod}(\mathcal{H})$ for the set of all pairs $(\Delta, J)$, where $J$ is a conjugation and $\Delta > 0$ a positive selfadjoint operator satisfying the modular relation $J\Delta J = \Delta^{-1}$.

**Proposition 1.6** *The map*

$$\mathrm{Mod}(\mathcal{H}) \to \mathrm{Stand}(\mathcal{H}), \quad (\Delta, J) \mapsto \mathrm{Fix}(J\Delta^{1/2})$$

*is a bijection. Its inverse is given by* $\mathtt{V} \mapsto (\Delta_{\mathtt{V}}, J_{\mathtt{V}})$.

***Proof*** ([Lo08, Prop. 3.2]) To see that we obtain a bijection, suppose that $(\Delta, J)$ is a pair of modular objects, i.e., a positive operator and a conjugation, satisfying the modular relation (7). Then $T := J\Delta^{1/2}$ is a closed, densely defined antilinear involution and

$$\mathtt{V} := \mathrm{Fix}(T) := \{\xi \in \mathcal{D}(T) \colon T\xi = \xi\} = \{\xi \in \mathcal{D}(\Delta^{1/2}) = \mathcal{D}(T) \colon \Delta^{1/2}\xi = J\xi\}$$

is a standard subspace with $J_{\mathtt{V}} = J$ and $\Delta_{\mathtt{V}} = \Delta$. Here closedness of $T$ follows from the closedness of the selfadjoint operator $\Delta^{1/2}$, and this implies the closedness of the subspace $\mathrm{Fix}(T) = \{v \in \mathcal{D}(T) \colon Tv = v\}$ of fixed points of $T$. □

## 1.2 More background on standard subspaces

In this subsection we collect some observations concerning standard subspaces that we shall use later on.

**Lemma 1.7** *The map $\mathtt{V} \mapsto \mathtt{V}'$ has the following properties:*

(a) $\mathtt{V}'' = \mathtt{V}$.
(b) $\mathtt{V}$ *is cyclic if and only if* $\mathtt{V}'$ *is separating.*
(c) $\mathtt{V}$ *is standard if and only if* $\mathtt{V}'$ *is standard.*
(d) $T_{\mathtt{V}}^* = T_{\mathtt{V}'}$*, i.e.,* $\mathcal{D}(T_{\mathtt{V}}^*) = \mathtt{V}' + i\mathtt{V}'$ *and* $\langle T_{\mathtt{V}}\xi, \eta\rangle = \overline{\langle \xi, T_{\mathtt{V}'}\eta\rangle}$ *for* $\xi \in \mathtt{V} + i\mathtt{V}$ *and* $\eta \in \mathtt{V}' + i\mathtt{V}'$.
(e) $\Delta_{\mathtt{V}'} = \Delta_{\mathtt{V}}^{-1}$ *and* $J_{\mathtt{V}'} = J_{\mathtt{V}}$.
(f) $J_{\mathtt{V}}\mathtt{V} = \mathtt{V}'$.
(g) $\mathtt{V} \cap \mathtt{V}' = \mathrm{Fix}(\Delta_{\mathtt{V}}) \cap \mathtt{V} = \mathrm{Fix}(J_{\mathtt{V}}) \cap \mathtt{V}$.

***Proof***(a) follows immediately from the Hahn–Banach Theorem. Alternatively, we can use that $\mathtt{V}' = i\mathtt{V}^{\perp_{\mathbb{R}}}$ and that multiplication with $i$ is isometric, to obtain $\mathtt{V}'' = i^2(\mathtt{V}^{\perp_{\mathbb{R}}})^{\perp_{\mathbb{R}}} = \mathtt{V}$.

(b) The subspace $(\mathtt{V} + i\mathtt{V})' = \mathtt{V}' \cap i\mathtt{V}'$ vanishes if and only if $\mathtt{V}'$ is separating if and only if $\mathtt{V}$ is cyclic.

(c) If $\mathtt{V}$ is standard, then (b) implies that $\mathtt{V}'$ is separating. That $\mathtt{V}'$ is also cyclic follows from (b) and $(\mathtt{V}')' = \mathtt{V}$ being separating. Hence $\mathtt{V}'$ is standard if $\mathtt{V}$ has this property. If $\mathtt{V}'$ is standard, then we now see with (a) that $\mathtt{V} = \mathtt{V}''$ is also standard.

(d) First we show that $T_{\mathtt{V}'} \subseteq T_{\mathtt{V}}^*$. In fact, for $a, b \in \mathtt{V}'$ and $v, w \in \mathtt{V}$, we derive from $\langle \mathtt{V}, \mathtt{V}'\rangle \subseteq \mathbb{R}$ that

$$\begin{aligned}\langle T_{\mathtt{V}'}(a + ib), v + iw\rangle &= \langle a - ib, v + iw\rangle = \langle a, v\rangle - \langle b, w\rangle + i(\langle b, v\rangle + \langle a, w\rangle)\\ &= \overline{\langle a + ib, v - iw\rangle} = \overline{\langle a + ib, T_{\mathtt{V}}(v + iw)\rangle} = \langle T_{\mathtt{V}}^*(a + ib), v + iw\rangle.\end{aligned}$$

Next we observe that, for $\xi \in \mathtt{V}$ and $\eta \in \mathcal{D}(T_{\mathtt{V}}^*)$, we have

$$\langle \xi, T_{\mathtt{V}}^*\eta\rangle = \overline{\langle T_{\mathtt{V}}\xi, \eta\rangle} = \overline{\langle \xi, \eta\rangle}.$$

From the equality of real and imaginary part, we derive that

$$T_{\mathtt{V}}^*\eta - \eta \in \mathtt{V}^{\perp_{\mathbb{R}}} = i\mathtt{V}' \quad \text{and} \quad T_{\mathtt{V}}^*\eta + \eta \in \mathtt{V}'.$$

Therefore $\eta \in \mathtt{V}' + i\mathtt{V}' = \mathcal{D}(T_{\mathtt{V}'})$, and hence $T_{\mathtt{V}'} = T_{\mathtt{V}}^*$.

(e) From (d) we derive with Exercise 1.56 that

$$T_{\mathtt{V}'} = (T_{\mathtt{V}})^* = (J_{\mathtt{V}}\Delta_{\mathtt{V}}^{1/2})^* \overset{1.56}{=} \Delta_{\mathtt{V}}^{1/2}J_{\mathtt{V}}^* = \Delta_{\mathtt{V}}^{1/2}J_{\mathtt{V}} = J_{\mathtt{V}}\Delta_{\mathtt{V}}^{-1/2}.$$

Thus (e) follows from the uniqueness of the polar decomposition.

(f) If $v \in \mathtt{V}$, then

$$T_{\mathtt{V}'}J_{\mathtt{V}}v = J_{\mathtt{V}}\Delta_{\mathtt{V}}^{-1/2}J_{\mathtt{V}}v = \Delta_{\mathtt{V}}^{1/2}v = J_{\mathtt{V}}v$$

shows that $J_{\mathtt{V}}\mathtt{V} \subseteq \mathtt{V}'$. Likewise $J_{\mathtt{V}}\mathtt{V}' = J_{\mathtt{V}'}\mathtt{V}' \subseteq \mathtt{V}'' = \mathtt{V}$, so that $\mathtt{V}' \subseteq J_{\mathtt{V}}\mathtt{V}$, and thus $\mathtt{V}' = J_{\mathtt{V}}\mathtt{V}$.

(g) For $v \in \mathtt{V}$ we have $\Delta_{\mathtt{V}}^{1/2}v = J_{\mathtt{V}}v$. If $\Delta_{\mathtt{V}}v = v$, then $\Delta_{\mathtt{V}}^{1/2}v = v$ by functional calculus, so that $J_{\mathtt{V}}v = v$ as well, and this implies with (f) that $v \in \mathtt{V} \cap \mathtt{V}'$. If, conversely, $v \in \mathtt{V} \cap \mathtt{V}'$, then (d) shows that $\Delta_{\mathtt{V}}v = T_{\mathtt{V}}^*T_{\mathtt{V}}v = T_{\mathtt{V}}v = v$, and $J_{\mathtt{V}}v = v$ follows as above. □

**Lemma 1.8** ([Lo08, Prop. 3.11]) *Let $U_t = e^{itA}$ be a unitary one-parameter group on $\mathcal{H}$, and $f\colon \mathbb{R} \to \mathbb{C}$ a locally bounded Borel measurable function. If $\mathcal{D} \subseteq \mathcal{D}(f(A))$ is a $U$-invariant linear subspace dense in $\mathcal{H}$, then it is a core for $f(A)$, i.e., the graph of $f(A)$ is the closure of its restriction to $\mathcal{D}$.*

***Proof*** We factorize $f = f_0 f_1$ with $f_0(\mathbb{R}) \subseteq \mathbb{T}$ and $f_1 \geq 0$, so that $f(A) = f_0(A)f_1(A)$. Then $f_0(A)$ is bounded and $\mathcal{D} \subseteq \mathcal{D}(f(A)) = \mathcal{D}(f_1(A))$. It therefore suffices to show that $\mathcal{D}$ is a core for $B := f_1(A)$, resp., that the graph $\Gamma(B_0)$ of $B_0 := B|_{\mathcal{D}}$ is dense in the graph of $B$. This is equivalent to $B_0$ being essentially selfadjoint.

Replacing $B_0$ by its closure, whose domain is also $U$-invariant, we may assume that $B_0$ is closed and we have to show that $B_0 = B$. As $B$ is selfadjoint, it suffices to verify that $\mathcal{R}(B_0 + i\mathbf{1})$ is dense in $\mathcal{H}$. So let $v \in \mathcal{R}(B_0 + i\mathbf{1})^{\perp}$. We have to show that $v = 0$.

The closed subspace $\Gamma(B_0) \subseteq \mathcal{H}^2$ is invariant under the diagonal action of the operators $(U_t)_{t\in\mathbb{R}}$, hence also under the operators $U(\varphi) = \int_{\mathbb{R}} \varphi(t)U_t\,dt$ for $\varphi \in L^1(\mathbb{R})$. In view of the relation $U(\varphi) = \widehat{\varphi}(A)$, these include the operators $\psi(A), \psi \in \mathcal{S}(\mathbb{R},\mathbb{R})$ (Schwartz functions). For all $w \in \mathcal{D}$ and $\psi \in \mathcal{S}(\mathbb{R},\mathbb{R})$, we thus have

$$v \perp (B_0 + i\mathbf{1})\psi(A)\mathcal{D} = (f_1(A) + i\mathbf{1})\psi(A)\mathcal{D}.$$

If $\psi$ has compact support, then the operator $f_1(A)\psi(A)$ is bounded because $f_1$ is locally bounded, hence bounded on the compact subset $\mathrm{supp}(\psi)$. So the density of $\mathcal{D}$ in $\mathcal{H}$ implies that $v \perp (B + i\mathbf{1})\psi(A)\mathcal{H}$. This in turn implies that

$$\psi(A)v \in \mathcal{R}(B + i\mathbf{1})^{\perp} = \{0\}.$$

Choosing $\psi_n$ in such a way that $0 \leq \psi_n \leq 1$ and $\psi_n|_{[-n,n]} = 1$, then we see that the relation $0 = \psi_n(A)v \to v$ entails that $v = 0$. □

**Lemma 1.9** (Equality Lemma) *Let $\mathsf{H}_1 \subseteq \mathtt{V} \subseteq \mathsf{H}_2$ be closed real subspaces such that $\mathtt{V}$ is standard, $\mathsf{H}_1$ is cyclic and $\mathsf{H}_2$ separating. If $\Delta_{\mathtt{V}}^{it}\mathsf{H}_j = \mathsf{H}_j$ holds for all $t \in \mathbb{R}$, then $\mathsf{H}_1 = \mathtt{V} = \mathsf{H}_2$.*

***Proof*** ([Lo08, Prop. 3.10]) Our assumptions imply that $\mathsf{H}_1$ and $\mathsf{H}_2$ are both standard because $\mathsf{H}_1$ inherits from $\mathtt{V}$ that it is separating, and $\mathsf{H}_2$ that it is cyclic. So $\mathsf{H}_1 + i\mathsf{H}_1 = \mathcal{D}(T_{\mathsf{H}_1}) = \mathcal{D}(\Delta_{\mathsf{H}_1}^{1/2})$ is a dense subspace of $\mathcal{H}$, invariant under the modular group $U_t = \Delta_{\mathtt{V}}^{it}$, $t \in \mathbb{R}$. This subspace is contained in $\mathtt{V} + i\mathtt{V} = \mathcal{D}(T_{\mathtt{V}}) = \mathcal{D}(\Delta_{\mathtt{V}}^{1/2})$, hence a core of $\Delta_{\mathtt{V}}^{1/2}$ by Lemma 1.8, and therefore also a core of $T_{\mathtt{V}}$. Since $T_{\mathtt{V}}$ is an extension of $T_{\mathsf{H}_1}$, the closedness of $T_{\mathsf{H}_1}$ implies that $T_{\mathsf{H}_1} = T_{\mathtt{V}}$, so that $\mathsf{H}_1 = \mathtt{V}$.

To deal with $\mathsf{H}_2$, we note that $\mathsf{H}_2' \subseteq \mathtt{V}'$ is cyclic by Lemma 1.7(b). Our assumption now implies that the cyclic subspace $\mathsf{H}_2'$ is invariant under the modular group of $\mathtt{V}'$, and the first part of the proof thus entails $\mathsf{H}_2' = \mathtt{V}'$. Finally, $\mathsf{H}_2 = \mathsf{H}_2'' = \mathtt{V}'' = \mathtt{V}$ (Lemma 1.7). □

**Proposition 1.10** *Let* $\mathtt{V}_1$ *and* $\mathtt{V}_2$ *be two standard subspaces with* $\Delta_{\mathtt{V}_1} = \Delta_{\mathtt{V}_2}$*. If* $\mathtt{V}_1 \cap \mathtt{V}_2$ *is cyclic, then* $\mathtt{V}_1 = \mathtt{V}_2$*.*

***Proof*** The subspace $\mathtt{V} := \mathtt{V}_1 \cap \mathtt{V}_2$ is invariant under the modular group $\Delta_{\mathtt{V}_1}^{i\mathbb{R}} = \Delta_{\mathtt{V}_2}^{i\mathbb{R}}$. Hence the assertion follows from the Equality Lemma. □

**Lemma 1.11** *For a standard subspace* $\mathtt{V} \subseteq \mathcal{H}$*, the following are equivalent:*

(a) $\mathtt{V} = \mathtt{V}'$.
(b) $\Delta_{\mathtt{V}} = \mathbf{1}$.
(c) $\mathtt{V} \subseteq \mathtt{V}'$.

***Proof*** (a) ⇒ (b): From Lemma 1.7(e) we know that $\Delta_{\mathtt{V}'} = \Delta_{\mathtt{V}}^{-1}$. Therefore $\mathtt{V} = \mathtt{V}'$ implies $\Delta_{\mathtt{V}} = \Delta_{\mathtt{V}}^{-1}$, so that $\Delta_{\mathtt{V}} = \mathbf{1}$ follows from the Spectral Theorem.
(b) ⇒ (a): If $\Delta_{\mathtt{V}} = \mathbf{1}$, then $T_{\mathtt{V}} = J_{\mathtt{V}}$ implies $\mathtt{V}' = J\mathtt{V} = T_{\mathtt{V}}\mathtt{V} = \mathtt{V}$ (Lemma 1.7).
(c) ⇔ (a): We have to show that $\mathtt{V} \subseteq \mathtt{V}'$ implies equality. If $J_{\mathtt{V}}\mathtt{V}' = \mathtt{V} \subseteq \mathtt{V}'$, then applying $J_{\mathtt{V}} = J_{\mathtt{V}'}$ implies $\mathtt{V}' = J_{\mathtt{V}}\mathtt{V} \subseteq J_{\mathtt{V}}\mathtt{V}' = \mathtt{V} \subseteq \mathtt{V}'$, hence equality. □

## 1.3 Standard subspaces and graphs

In many situations standard subspaces are naturally described in terms of graphs. We briefly introduce this perspective in this subsection. This provides in particular natural realizations of the dense complex subspace $\mathtt{V} + i\mathtt{V}$ of $\mathcal{H}$. We also relate this perspective to the geometry of the strip, the complex upper half plane and the unit disc in $\mathbb{C}$.

Let $\mathtt{V} \subseteq \mathcal{H}$ be a standard subspace and recall that $\mathtt{V} + i\mathtt{V} = \mathcal{D}(\Delta^{1/2})$. The natural Hilbert space structure on this dense subspace of $\mathcal{H}$ is obtained from the isomorphism with the graph

$$\Gamma(\Delta^{1/2}) = \{(v, \Delta^{1/2}v) \colon v \in \mathcal{D}(\Delta^{1/2})\} \subseteq \mathcal{H} \oplus \mathcal{H},$$

which is a closed subspace.

**Proposition 1.12** *Let* $\mathcal{H}$ *be a complex Hilbert space. Consider the complex structure on* $\mathcal{H}^{\oplus 2}$*, defined by* $I(v,w) := (iv, -iw)$ *and a densely defined operator*

$$A \colon \mathcal{D}(A) \to \mathcal{H} \quad \text{with closed graph} \quad \mathtt{V} := \Gamma(A) \subseteq \mathcal{H}^{\oplus 2}.$$

*Then the following assertions hold:*

(a) *The graph* $\mathtt{V}$ *is separating if and only if* $A$ *is injective.*
(b) *The graph* $\mathtt{V}$ *is cyclic if and only if* $A$ *has dense range.*
(c) *The graph* $\mathtt{V}$ *is standard if and only if* $A$ *is injective with dense range.*
(d) *If* $\mathtt{V}$ *is standard, then its Tomita operator is given by* $T_{\mathtt{V}}(v,w) = (A^{-1}w, Av)$*, and if* $A = U|A|$ *is the polar decomposition of* $A$*, then*

$$\Delta_{\mathtt{V}} = A^*A \oplus (A^{-1})^*A^{-1} \quad \text{and} \quad J_{\mathtt{V}}(v,w) = (U^{-1}w, Uv).$$

(e) *If* $A > 0$ *is strictly positive, then its graph* $\mathtt{V} \subseteq \mathcal{H}^{\oplus 2}$ *is a standard subspace with*

$$\Delta_{\mathtt{V}} = A^2 \oplus A^{-2} \quad \text{and} \quad J_{\mathtt{V}}(v,w) = (w,v).$$

***Proof*** (a) We first observe that

$$I\mathtt{V} = \{(iv, -iAv) \colon v \in \mathcal{D}(A)\} = \{(v, -Av) \colon v \in \mathcal{D}(A)\} = \Gamma(-A),$$

and thus

$$\mathtt{V} \cap I\mathtt{V} = \Gamma(A) \cap \Gamma(-A) = \ker(A) \oplus \{0\}. \tag{8}$$

Therefore $\mathtt{V}$ is separating if and only if $A$ is injective.
(b) Next we observe that

$$\Gamma(A)^{\perp_{\mathbb{R}}} = \{(-A^*v, v) \colon v \in \mathcal{D}(A^*)\} = \tau_{\rm flip}\Gamma(-A^*) =: \Gamma^{\rm flip}(-A^*),$$

arises by applying the flip involution $\tau(v,w) = (w,v)$ to the graph of $-A^*$. So

$$(\mathtt{V}+I\mathtt{V})^{\perp_{\mathbb{R}}} = \mathtt{V}^{\perp_{\mathbb{R}}} \cap I\mathtt{V}^{\perp_{\mathbb{R}}} = \Gamma^{\rm flip}(-A^*) \cap I\Gamma^{\rm flip}(-A^*) = \Gamma^{\rm flip}(-A^*) \cap \Gamma^{\rm flip}(A^*)$$

is trivial if and only if $A^*$ is injective, which is equivalent to $\mathcal{R}(A) = A\mathcal{D}(A)$ being dense.
(c) follows from (a) and (b).
(d) To identify the corresponding modular objects, we claim that

$$\mathtt{V} + I\mathtt{V} = \mathcal{D}(A) \oplus \mathcal{D}(A^{-1}).$$

Clearly, $\Gamma(\pm A) \subseteq \mathcal{D}(A) \oplus \mathcal{R}(A) = \mathcal{D}(A) \oplus \mathcal{D}(A^{-1})$, so that "$\subseteq$" holds. For the converse, let $v \in \mathcal{D}(A)$, $w \in \mathcal{D}(A^{-1})$ and put $u := A^{-1}w$. Then

$$(v,w) = (v, Au) = \Big(\frac{v+u}{2}, A\frac{v+u}{2}\Big) + \Big(\frac{v-u}{2}, -A\frac{v-u}{2}\Big) \in \mathtt{V} + I\mathtt{V}.$$

The domain of the modular operator $T_{\mathtt{V}}$ is $\mathtt{V} + I\mathtt{V} = \mathcal{D}(A) \oplus \mathcal{D}(A^{-1})$. On this domain the prescription

$$T(v,w) := (A^{-1}w, Av)$$

defines an $I$-antilinear involution with $\mathrm{Fix}(T) = \Gamma(A) = \mathtt{V}$. This implies that $T = T_{\mathtt{V}}$ is the Tomita operator of the standard subspace $\mathtt{V}$.

It is easy to see that the adjoint operator is given by

$$T^*(v,w) = (A^*w, (A^{-1})^*v) \quad \text{with domain} \quad \mathcal{R}(A^*) \oplus \mathcal{D}(A^*).$$

We thus obtain

$$\Delta_{\mathtt{V}}(v,w) = (T^*T)(v,w) = T^*(A^{-1}w, Av) = (A^*Av, (A^{-1})^*A^{-1}w),$$

and therefore $\Delta_{\mathtt{V}} = |A|^2 \oplus |A^{-1}|^2$. Finally, we obtain

$$\begin{aligned} J_{\mathtt{V}}(v,w) &= T_{\mathtt{V}}\Delta_{\mathtt{V}}^{-1/2}(v,w) = T_{\mathtt{V}}(|A|^{-1}v, |A^{-1}|^{-1}w) \\ &= (A^{-1}|A^{-1}|^{-1}w, A|A|^{-1}v) = (U^{-1}w, Uv). \end{aligned}$$

(e) is a special case of (d). □

**Example 1.13** Let $\mathtt{V} \subseteq \mathcal{H}$ be a standard subspace. Then $\mathtt{V} + i\mathtt{V} = \mathcal{D}(\Delta^{1/2})$ and the embedding

$$\mathtt{V}_{\mathbb{C}} \to \Gamma(\Delta^{1/2}), \quad (v,w) \mapsto (v + iw, \Delta^{1/2}(v + iw))$$

identifies $\mathtt{V}_{\mathbb{C}}$ with a standard subspace of $\mathcal{H}^{\oplus 2}$, endowed with the complex structure $I(v,w) = (iv, -iw)$. Its modular operator takes the form

$$\Delta_{\mathtt{V}_{\mathbb{C}}} = \Delta_{\mathtt{V}} \oplus \Delta_{\mathtt{V}}^{-1}.$$

**Example 1.14** (Standard subspaces for the translation representation) We consider $\mathcal{H} = L^2(\mathbb{R})$, $\beta > 0$, and the standard subspace $\mathtt{V} \subseteq L^2(\mathbb{R})$, specified by

$$Jf = \overline{f} \quad \text{and} \quad (\Delta^{-it/2\beta}f)(x) = f(x+t), \quad x,t \in \mathbb{R}.$$

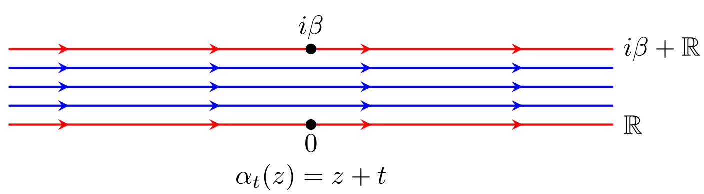



Figure 1: The translation flow on the strip $\mathcal{S}_\beta$.

Then $\mathcal{D}(\Delta^{1/2})$ consists of the space of boundary values of elements of the Hardy space

$$H^2(\mathcal{S}_\beta) := \Big\{F \in \mathcal{O}(\mathcal{S}_\beta) \colon \sup_{0<y<\beta} \|F(\cdot + iy)\|_2 < \infty\Big\}$$

(cf. [Go69, Prop 5.1]). For $f \in \mathcal{D}(\Delta^{1/2})$ we then have (almost everywhere in the sense of $L^2$-functions)

$$(\Delta^{1/2} f)(x) = f(x + i\beta)$$

(the upper boundary values on $\mathbb{R} + i\beta$), so that $f$ is fixed by $J\Delta^{1/2}$ if and only if $f^\sharp = f$, where

$$f^\sharp(x) := \overline{f(x+i\beta)} \quad \text{for} \quad x \in \mathbb{R}.$$

This shows that

$$\mathtt{V} = \{f \in \mathcal{D}(\Delta^{1/2}) \colon f^\sharp = f\}. \tag{9}$$

Endowed with the graph topology, we have $\mathcal{D}(\Delta^{1/2}) \cong \Gamma(\Delta^{1/2})$, and this further leads to

$$\Gamma(\Delta^{1/2}) \cong H^2(\mathcal{S}_\beta) \subseteq L^2(\mathbb{R})^{\oplus 2},$$

where we identify $H^2(\mathcal{S}_\beta)$ via the boundary value map $F \mapsto (F|_{\mathbb{R}}, F|_{\mathbb{R}+i\beta})$ with a closed subspace of $L^2(\mathbb{R})^{\oplus 2}$.

In this picture, the Tomita involution $T_{\mathtt{V}}$ corresponds to the involution on $H^2(\mathcal{S}_\beta)$, given by

$$f^\sharp(z) = \overline{f(\beta i + \overline{z})} \quad \text{for} \quad z \in \mathcal{S}_\beta, \tag{10}$$

and the lower boundary value map thus induces an isometry

$$H^2(\mathcal{S}_\beta)^\sharp := \{f \in H^2(\mathcal{S}_\beta) \colon f^\sharp = f\} \to \mathtt{V}, \quad f \mapsto f|_{\mathbb{R}} \tag{11}$$

(cf. [NÓ17, Ex. 3.16]). On the pairs $(f_1, f_2) = (f, \Delta^{1/2} f) \in \Gamma(\Delta^{1/2}) \subseteq L^2(\mathbb{R})^{\oplus 2}$ of boundary values of elements of $H^2(\mathcal{S}_\beta)$, the involution $\sharp$ then takes the form

$$(f_1, f_2)^\sharp = (\overline{f_2}, \overline{f_1}).$$

**Example 1.15** (Standard subspaces for the dilation representation)
On $\mathcal{H} = L^2(\mathbb{R}_+)$ we consider the standard subspace $\mathtt{V} \subseteq L^2(\mathbb{R}_+)$, specified by

$$Jf := \overline{f} \quad \text{and} \quad (\Delta^{-it/2\pi} f)(x) = e^{t/2} f(e^t x), \quad x, t \in \mathbb{R}.$$

Then $\mathcal{D}(\Delta^{1/2})$ consists of the space of boundary values of elements of the Hardy space

$$H^2(\mathbb{C}_+) := \Big\{F \in \mathcal{O}(\mathbb{C}_+) \colon \sup_{0<y<\beta} \|F(\cdot + iy)\|_2 < \infty\Big\}.$$

This can be derived from the corresponding statement on the strip $\mathcal{S}_\pi$ (Example 1.14) by using the fact that the exponential map $\mathcal{S}_\pi \to \mathbb{C}_+, z \mapsto e^x$ is biholomorphic, which implies that

$$\Gamma \colon H^2(\mathbb{C}_+) \to H^2(\mathcal{S}_\pi), \quad \Gamma(F)(z) := e^{z/2} F(e^z)$$

defines a unitary map (cf. [ANS25, §2]). It satisfies

$$e^{t/2}\Gamma(F)(e^t z) = e^{(z+t)/2}F(e^{z+t}) = \Gamma(F)(z+t),$$

hence intertwines the operators $\Delta^{-it/2\pi}$ with the translations on $H^2(\mathcal{S}_\pi)$. We further have

$$\Gamma(F)^\sharp(z) = \overline{\Gamma(F)(\pi i + \overline{z})} = e^{z/2 - \frac{\pi i}{2}}\overline{F(-e^{\overline{z}})} = \Gamma(F^\sharp)(z)$$

for

$$F^\sharp(z) := -i\overline{F(-\overline{z})}.$$

This shows that $\mathtt{V} \subseteq L^2(\mathbb{R}_+)$ is the real subspace of boundary values of

$$H^2(\mathbb{C}_+)^\sharp = \{F \in H^2(\mathbb{C}_+) \colon \overline{F(-\overline{z})} = iF(z)\}.$$

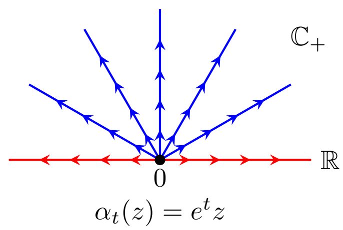



Figure 2: The dilation flow on the upper half plane $\mathbb{C}_+$.

**Example 1.16** (Standard subspaces for the hyperbolic flow on the disc) We now consider $L^2(\mathbb{S}^1)$ and the hyperbolic flow on $\mathbb{S}^1$, the boundary of the complex unit disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$, given by

$$\alpha_t(z) = \frac{\cosh(t/2)z + \sinh(t/2)}{\sinh(t/2)z + \cosh(t/2)}.$$

It fixes $\pm 1$ and acts on $\mathbb{D}$ as shown the Figure 3 below.

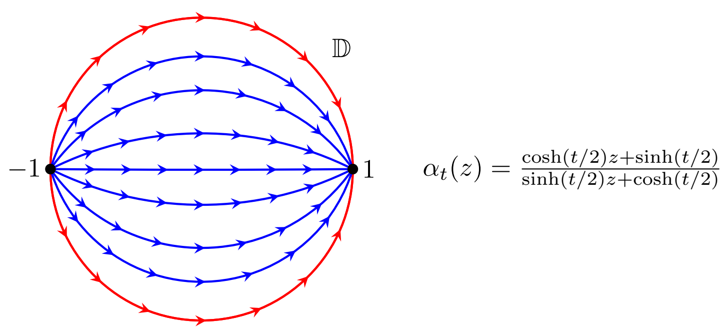



Figure 3: The hyperbolic flow on the unit disc.

In particular, we have $\alpha_t(0) = \tanh(t/2)$, and the orbit map $\alpha^0(t) = \tanh(t/2)$, considered as a map on the horizontal line $\mathbb{R} + \frac{\pi i}{2} \subseteq \mathcal{S}_\pi$, extends to a biholomorphic map

$$\psi \colon \mathcal{S}_\pi \to \mathbb{D}, \quad z \mapsto \tanh\Big(\frac{1}{2}\Big(z - \frac{\pi i}{2}\Big)\Big) = \tanh\Big(\frac{z}{2} - \frac{\pi i}{4}\Big),$$

and this map yields a unitary operator

$$\Gamma \colon H^2(\mathbb{D}) \to H^2(\mathcal{S}_\pi), \quad \Gamma(F)(z) := \sqrt{\psi'(z)}F(\psi(z)))$$

(cf. [ANS25, §2]).

The map $\psi$ maps $\mathbb{R} \subseteq \partial\mathcal{S}_\pi$ to the lower half circle $\mathbb{S}^1_- \subseteq \partial\mathbb{D}$, hence also induces a unitary operator $\Gamma_- \colon L^2(\mathbb{S}^1_-) \to L^2(\mathbb{R})$. It also satisfies $\psi(z + t) = \alpha_t\psi(z)$, hence intertwines the translation action of $\mathbb{R}$ on $H^2(\mathcal{S}_\pi)$ with a unitary one-parameter group $(\Delta^{-it/2\pi})_{t\in\mathbb{R}}$ on $L^2(\mathbb{S}^1_-)$. Accordingly, $\mathcal{D}(\Delta^{1/2})$ can naturally be identified with the space of boundary values of elements of the Hardy space $H^2(\mathbb{D})$ on the lower half-circle $\mathbb{S}^1_-$.

By analytic continuation it follows easily that $\psi(\overline{z}+\pi i) = \overline{\psi(z)}$ for $z \in \mathcal{S}_\pi$, so that

$$F^\sharp(z) := \overline{F(\overline{z})}$$

is a conjugation on $H^2(\mathbb{D})$ which is intertwined by $\Gamma$ with the corresponding involution on $H^2(\mathcal{S}_\pi)$. This shows that $\mathtt{V} \subseteq L^2(\mathbb{S}^1_-)$ is the real subspace of boundary values of

$$H^2(\mathbb{D})^\sharp = \{F \in H^2(\mathbb{D}) \colon \overline{F(z)} = F(\overline{z})\}.$$

## 1.4 Modular theory and the Tomita–Takesaki Theorem

The correspondence between modular objects and standard subspaces is the core of the modular theory of operator algebras. It is the key structure in the Tomita–Takesaki Theorem discussed below.

In the modular theory of operator algebras one studies pairs $(\mathcal{M}, \Omega)$, consisting of a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ and a cyclic separating unit vector $\Omega \in \mathcal{H}$. The Tomita–Takesaki Theorem (Theorem 1.22) then leads to a positive selfadjoint operator $\Delta$ and an antiunitary involution $J$ satisfying the $J\mathcal{M}J = \mathcal{M}'$, and $\alpha_t(M) := \Delta^{it}M\Delta^{-it}$ defines automorphisms of $\mathcal{M}$ (see [BR87], [Su87], [Su05], [Bl06]). Modular operators and their spectra are the key tool in the classification of factors (=simple von Neumann algebras) and in the characterization of von Neumann algebras by their natural cones by A. Connes [Co73, Co74, Co75, CS78].

In the context of nets of local algebra $\mathcal{M}(\mathcal{O})$, whenever $\Omega$ is cyclic and separating for $\mathcal{M}(\mathcal{O})$, we obtain corresponding modular objects $(\Delta_{\mathcal{O}}, J_{\mathcal{O}})$. This connection between the Araki–Haag–Kastler theory of local observables

([Ha96], [Ar99]) and modular theory leads naturally to antiunitary group representations and the perspective explored in Section 4.

**Definition 1.17** For a subset $S \subseteq B(\mathcal{H})$, we write

$$S' := \{A \in B(\mathcal{H}) \colon (\forall M \in S)\, AM = MA\}$$

for its *commutant*. It is a closed subalgebra and $*$-invariant if $S$ has this property.

A *von Neumann algebra* is a $*$-invariant complex subalgebra $\mathcal{M} \subseteq B(\mathcal{H})$, satisfying $\mathcal{M} = \mathcal{M}''$. For a von Neumann algebra $\mathcal{M}$, a unit vector $\Omega \in \mathcal{H}$ is called

- *cyclic,* if $\mathcal{M}\Omega$ is dense in $\mathcal{H}$.
- *separating,* if the orbit map $\mathcal{M} \to \mathcal{H}, M \mapsto M\Omega$ is injective,
- *standard*, if it is cyclic and separating.

**Lemma 1.18** *$\Omega \in \mathcal{H}$ is cyclic for $\mathcal{M}$ if and only if it is separating for $\mathcal{M}'$.*

***Proof*** Suppose first that $\Omega$ is cyclic for $\mathcal{M}$. For $A \in \mathcal{M}'$ with $A\Omega = 0$, we then obtain $A\mathcal{M}\Omega = \mathcal{M}A\Omega = \{0\}$, and since $\mathcal{M}\Omega$ is dense in $\mathcal{H}$, it follows that $A = 0$. So $\Omega$ is separating for $\mathcal{M}'$.

Suppose, conversely, that $\Omega$ is separating for $\mathcal{M}'$. Let $P \colon \mathcal{H} \to \mathcal{H}$ be the orthogonal projection onto $\overline{\mathcal{M}\Omega}$. Then $P \in \mathcal{M}'$ and $(\mathbf{1} - P)\Omega = 0$ imply $\mathbf{1} = P$, so that $\Omega$ is cyclic for $\mathcal{M}$. $\square$

**Definition 1.19** For $\Omega \in \mathcal{H}$ and $\mathcal{M} \subseteq B(\mathcal{H})$, we consider the closed real subspace

$$\mathtt{V} := \mathtt{V}_{\mathcal{M},\Omega} := \overline{\mathcal{M}_h\Omega} \subseteq \mathcal{H}, \tag{12}$$

where $\mathcal{M}_h := \{M \in \mathcal{M} \colon M^* = M\}$ is the real subspace of hermitian elements in $\mathcal{M}$.

**Lemma 1.20** *The following assertions hold for a von Neumann algebra $\mathcal{M} \subseteq B(\mathcal{H})$ and a unit vector $\Omega \in \mathcal{H}$.*

(a) *$\mathtt{V}_{\mathcal{M},\Omega}$ is cyclic if and only if $\Omega$ is cyclic for $\mathcal{M}$.*
(b) *$\mathtt{V}_{\mathcal{M},\Omega}$ is separating if and only if $\Omega$ is separating for the restriction of $\mathcal{M}$ to the cyclic subspace $\overline{\mathcal{M}\Omega}$, i.e., for $A \in \mathcal{M}$ the relation $A\Omega = 0$ implies $A\mathcal{M}\Omega = \{0\}$.*
(c) *$\mathtt{V}_{\mathcal{M},\Omega}$ is standard if and only if $\Omega$ is a standard vector for $\mathcal{M}$.*
(d) *$\mathtt{V}_{\mathcal{M}',\Omega} \subseteq (\mathtt{V}_{\mathcal{M},\Omega})'$.*
(e) *$\mathcal{Z}(\mathcal{M})_h\Omega \subseteq \mathtt{V}_{\mathcal{M},\Omega} \cap (\mathtt{V}_{\mathcal{M},\Omega})'$.*

Note that $\mathtt{V}_{\mathcal{M},\Omega}$ being separating only contains information on the representation of $\mathcal{M}$ on the cyclic subspace $\mathcal{K} := \overline{\mathcal{M}\Omega} \subseteq \mathcal{H}$, but not on the representation of $\mathcal{M}$ on $\mathcal{K}^\perp$. If $\mathcal{H} = \mathbb{C}^2$, $\mathcal{M} \cong \mathbb{C}^2$ is the subalgebra of diagonal operators, and $\Omega = \mathbf{e}_1$, then $\mathtt{V}_{\mathcal{M},\Omega} = \mathbb{R}\mathbf{e}_1$ is separating, but $\Omega$ is **not** separating for $\mathcal{M}$. This subtlety does not play a role for (c) because we also assume cyclicity.

***Proof*** (a) follows immediately from the definitions.
(b) Let $\mathcal{K} := \overline{\mathcal{M}\Omega}$. Suppose first that $\Omega$ is separating for the von Neumann algebra $\mathcal{M}_\mathcal{K} := (\mathcal{M}|_\mathcal{K})''$. Then $\Omega$ is cyclic for the commutant $\mathcal{M}_\mathcal{K}' \subseteq B(\mathcal{K})$ of $\mathcal{M}|_\mathcal{K}$ (Lemma 1.18). We have for $A \in \mathcal{M}_{\mathcal{K},h}$ and $B \in \mathcal{M}'_{\mathcal{K},h}$ the relation

$$\langle A\Omega, B\Omega\rangle = \langle \Omega, AB\Omega\rangle = \langle \Omega, BA\Omega\rangle = \langle B\Omega, A\Omega\rangle,$$

so that

$$\langle \mathtt{V}_{\mathcal{M},\Omega}, \mathtt{V}_{\mathcal{M}'_\mathcal{K},\Omega}\rangle \subseteq \mathbb{R}. \tag{13}$$

We conclude that

$$\mathtt{V}_{\mathcal{M},\Omega} \cap i\mathtt{V}_{\mathcal{M},\Omega} \subseteq \mathtt{V}^\perp_{\mathcal{M}'_\mathcal{K},\Omega} \cap \mathcal{K} = (\mathcal{M}'_\mathcal{K}\Omega)^\perp \cap \mathcal{K} = \{0\},$$

i.e., $\mathtt{V}_{\mathcal{M},\Omega}$ is separating.

Now we assume, conversely, that $\mathtt{V}_{\mathcal{M},\Omega}$ is separating. We show that $\Omega$ is separating for $\mathcal{M}_\mathcal{K}$. So let $A \in \mathcal{M}_\mathcal{K}$ with $A\Omega = 0$. For $B \in \mathcal{M}_\mathcal{K}$, we then have

$$A^*B\Omega = (A^*B + B^*A)\Omega \in \mathtt{V}_{\mathcal{M},\Omega},$$

so that $A^*\mathcal{M}_\mathcal{K}\Omega \subseteq \mathtt{V}_{\mathcal{M},\Omega}$ is a complex linear subspace, hence trivial because $\mathtt{V}_{\mathcal{M},\Omega}$ is separating. Thus $A^*\mathcal{K} = \{0\}$, and this implies that $A = 0$. Therefore $\Omega$ is separating for $\mathcal{M}_\mathcal{K}$.
(c) follows from (a) and (b).
(d) follows directly from (13).
(e) follows from $\mathcal{Z}(\mathcal{M}) \subseteq \mathcal{M} \cap \mathcal{M}'$ and (d). □

*Remark 1.21* (a) Cyclic vectors play an important role in representation theory because every ∗-representation on a Hilbert space is a direct sum of cyclic representations. Moreover, representations with cyclic unit vector $\Omega$ can be reconstructed completely from the corresponding state

$$\omega\colon \mathcal{M} \to \mathbb{C}, \quad \omega(M) := \langle \Omega, M\Omega\rangle.$$

The map $\iota\colon \mathcal{H} \to \mathcal{M}^*, \iota(v)(M) := \langle \Omega, Mv\rangle$ is injective and intertwines the representation on $\mathcal{H}$ with the right translation representation on $\mathcal{M}^*$, defined by

$$(A\lambda)(M) := \lambda(MA).$$

The Hilbert space structure on $\iota(\mathcal{H})$, for which $\iota$ is isometric, is given by

$$\langle \iota(M\Omega), \iota(N\Omega)\rangle = \omega(M^*N),$$

exhibiting $\iota(\mathcal{H})$ as a reproducing kernel Hilbert space of linear functionals $f \in \mathcal{M}^*$, satisfying

$$f(M) = \langle M^*\Omega, f\rangle \quad \text{for} \quad M \in \mathcal{M}, f \in \iota(\mathcal{H})$$

(cf. [Ne00, Ch. I]).

(b) If $\Omega$ is standard, then the orbit map $\pi^\Omega \colon \mathcal{M} \to \mathcal{H}, M \mapsto M\Omega$ is a dense linear embedding of $\mathcal{M}$ into $\mathcal{H}$, so that we may consider $\mathcal{H}$ as the completion of $\mathcal{M}$ with respect to the scalar product $\langle M, N\rangle := \omega(M^*N)$.

That $\Omega$ is separating corresponds to the property of the state $\omega$ that $\omega(M^*M) = 0$ implies $M = 0$ ($\omega$ is *faithful*). One can show that all normal faithful states on a von Neumann algebra (cf. Appendix 7.1) lead to equivalent GNS representations, called the *standard form representation* ([Bl06, Thm. III.2.6.7]). For more details on these issues, see also the discussion of *symmetric form representations* in [NÓ17], [Bl06], [BGN20, §3.1] and Remark 7.4.

**Theorem 1.22** (Tomita–Takesaki Theorem; Tomita, 1967; Takesaki, 1970) *Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and $\Omega \in \mathcal{H}$ be a standard vector for $\mathcal{M}$. Then $\mathtt{V} := \mathtt{V}_{\mathcal{M},\Omega} := \overline{\mathcal{M}_h\Omega}$ is a standard subspace. The corresponding modular objects $(\Delta, J)$ satisfy*

(a) *$J\mathcal{M}J = \mathcal{M}'$ and $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$ for $t \in \mathbb{R}$.*
(b) *$J\Omega = \Omega$, $\Delta\Omega = \Omega$ and $\Delta^{it}\Omega = \Omega$ for $t \in \mathbb{R}$.*
(c) *For $M \in \mathcal{Z}(\mathcal{M}) := \mathcal{M} \cap \mathcal{M}'$, the center of $\mathcal{M}$, we have $JMJ = M^*$ and $\Delta^{it}M\Delta^{-it} = M$ for $t \in \mathbb{R}$.*

It follows in particular, that

$$\alpha_t(A) := \Delta^{it}A\Delta^{-it} \tag{14}$$

defines a one-parameter group of automorphisms of $\mathcal{M}$, called the *modular automorphism group associated to $\Omega$.*

***Proof*** By Lemma 1.20(c), $\mathtt{V}$ is a standard subspace. The standard subspace $\mathtt{V}$ already provides $\Delta$ and $J$. The main work consists in the verification of (a), for which we refer to [BR87, Thm. 2.5.14]. We only sketch how to obtain (b) and (c).
(b) From $\langle \Omega, \mathcal{M}_h\Omega\rangle \subseteq \mathbb{R}$, it follows that $\Omega \in \mathtt{V} \cap \mathtt{V}'$, so that Lemma 1.7(g) implies that $J$ and $\Delta$ fix $\Omega$. Now (b) follows from functional calculus.
(c) Let $\mathcal{Z} := \mathcal{M} \cap \mathcal{M}'$ denote the center of $\mathcal{M}$ and $\mathcal{M}'$. It suffices to show that any hermitian $Z \in \mathcal{Z}$ commutes with $J$ and $\Delta^{it}$ for $t \in \mathbb{R}$. In view of (a), $JZJ \in \mathcal{Z}$, and

$$JZJ\Omega = JZ\Omega = Z\Omega$$

follows from $Z\Omega \in \mathtt{V} \cap \mathtt{V}' \subseteq \mathrm{Fix}(J)$ (Lemma 1.7(g) and Lemma 1.20(e)). Using that $\Omega$ is separating for $\mathcal{M}$, we obtain $JZJ = Z$. We likewise have $\Delta^{it}Z\Delta^{-it} \in \mathcal{M}$ by (a) and $\Delta^{it}\Omega = \Omega$ and $\Delta^{it}Z\Omega = Z\Omega$ by Lemma 1.7(g). Therefore

$$\Delta^{it}Z\Delta^{-it}\Omega = \Delta^{it}Z\Omega = Z\Omega,$$

and $\Omega$ being separating yields $\Delta^{it}Z\Delta^{-it} = Z$. □

An approach to the Tomita–Takesaki Theorem through bounded operators can be found in [RvD77]. For a rather general approach to modular operators for pairs of subspaces of real Hilbert spaces, we refer to [NZ24].

The passage to the commutant of an algebra translates easily into the symplectic orthogonal space $\mathtt{V}'$ (cf. Definition 1.1).

**Lemma 1.23** *For a standard vector $\Omega$ of $\mathcal{M}$, we have $(\mathtt{V}_{\mathcal{M},\Omega})' = \mathtt{V}_{\mathcal{M}',\Omega}$.*

***Proof*** Let $J = J_{\mathcal{M},\Omega}$ and $\mathtt{V} := \mathtt{V}_{\mathcal{M},\Omega}$. In view of $J\Omega = \Omega$ and $J\mathcal{M}J = \mathcal{M}'$ (Theorem 1.22), Lemma 1.7(f) yields

$$\mathtt{V}' \overset{1.7(f)}{=} J\mathtt{V} = J\overline{\mathcal{M}_h\Omega} = \overline{\mathcal{M}'_hJ\Omega} = \overline{\mathcal{M}'_h\Omega} = \mathtt{V}_{\mathcal{M}',\Omega},$$

and this implies the assertion. □

**Examples 1.24** (a) Let $\mathcal{H} = L^2(X, \mathfrak{S}, \mu)$ for a $\sigma$-finite measure space $(X, \mathfrak{S}, \mu)$ and $\mathcal{M} = L^\infty(X, \mathfrak{S}, \mu)$, acting on $\mathcal{H}$ by multiplication operators. Then the normal states of $\mathcal{M}$ (cf. Appendix 7.1) are of the form

$$\omega_h(f) = \int_X fh\,d\mu,$$

where $0 \leq h$ satisfies $\int_X h\,d\mu = 1$. Such a state is faithful if and only if $h \neq 0$ holds $\mu$-almost everywhere. Then $\Omega := \sqrt{h} \in \mathcal{H}$ is a corresponding standard unit vector. Let $\mathtt{V} = \mathtt{V}_{\mathcal{M},\Omega}$ be the corresponding standard subspace. As it consists of real-valued functions, we obtain $T_{\mathtt{V}}(f) = \overline{f}$, which is isometric and therefore $T_{\mathtt{V}} = J$ and $\Delta_{\mathtt{V}} = \mathbf{1}$.
(b) Let $\mathcal{H} = B_2(\mathcal{K})$ be the space of Hilbert–Schmidt operators on the complex separable Hilbert space $\mathcal{K}$ and consider the von Neumann algebra $\mathcal{M} = B(\mathcal{K})$, acting on $\mathcal{H}$ by left multiplications. Then $\mathcal{M}' \cong B(\mathcal{K})^{\mathrm{op}}$, the opposite algebra, acting by right multiplications. Normal states of $\mathcal{M}$ are of the form

$$\omega_S(A) = \mathrm{tr}(AS), \quad \text{where} \quad 0 \leq S \quad \text{with} \quad \mathrm{tr}\,S = 1.$$

Such a state is faithful if and only if $\ker S = \{0\}$ (which requires $\mathcal{K}$ to be separable), and then $\Omega := \sqrt{S} \in \mathcal{H}$ is a cyclic separating unit vector. Then $T_{\mathtt{V}}(M\Omega) = M^*\Omega = (\Omega M)^*$ implies that

$$JA = A^* \quad \text{and} \quad \Delta(A) = \Omega^2 A \Omega^{-2} = SAS^{-1} \quad \text{for} \quad A \in B_2(\mathcal{K}).$$

(c) The prototypical pair $(\Delta, J)$ of a modular operator and a modular conjugation arises from the regular representation of a locally compact group $G$ on the Hilbert space $\mathcal{H} = L^2(G, \mu_G)$ with respect to a left Haar measure $\mu_G$. Here the modular operator is given by the multiplication

$$\Delta f = \Delta_G \cdot f,$$

where $\Delta_G\colon G \to \mathbb{R}_+^\times$ is the modular function of $G$ (cf. (188)), and the modular conjugation is given by

$$(Jf)(g) = \Delta_G(g)^{-\frac{1}{2}}\overline{f(g^{-1})}.$$

Accordingly, we have for $T = J\Delta^{1/2}$:

$$(Tf)(g) = \Delta_G(g)^{-1}\overline{f(g^{-1})} = f^*(g).$$

The corresponding von Neumann algebra is the algebra $\mathcal{M} \subseteq B(L^2(G,\mu_G))$, generated by the left regular representation. If $M_f h = f * h$ is the left convolution with $f \in C_c(G)$, then the value of the corresponding normal weight $\omega$ on $\mathcal{M}$ (Remark 7.4) is given by $\omega(M_f) = f(e)$, so that $\omega$ corresponds to evaluation in $e$, which is defined on the weakly dense subalgebra of $\mathcal{M}$, coming from the representation of the convolution algebra $(C_c(G), *)$.

*Remark 1.25* Theorem 1.22(c) asserts that the modular group and $J$ commute with all central projections, and this entails that the whole structure adapts to the canonical central disintegration

$$\mathcal{M} = \int_X^\oplus \mathcal{M}_x \, d\mu(x)$$

of $\mathcal{M}$, for which $\mathcal{Z}(\mathcal{M}) = L^\infty_{\rm loc}(X, \mathfrak{S}, \mu)$ are the scalar decomposable operators on a locally finite measure space $(X, \mathfrak{S}, \mu)$, and almost every von Neumann algebra $\mathcal{M}_x$ is a factor, i.e., $\mathcal{Z}(\mathcal{M}_x) = \mathbb{C}\mathbf{1}$ (cf. Examples 7.2(b) and [MN24, §5.4] for more details).

So the modular groups are "direct integrals" of modular groups of factors. For factors, the modular operators and their spectra are a key tool in Connes' classification of factors and in the characterization of von Neumann algebras in terms of their natural cones by A. Connes [Co73, Co74, Co75] (see also [NÓ17, §4.4] and [BR87]).

## 1.5 Pairs of von Neumann algebras

**Proposition 1.26** ([Lo08, Prop. 3.24]) *Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra with standard vector $\Omega$.*

(a) *If $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{M}$ are von Neumann algebras with $\mathtt{V}_{\mathcal{N}_1,\Omega} \subseteq \mathtt{V}_{\mathcal{N}_2,\Omega}$, then $\mathcal{N}_1 \subseteq \mathcal{N}_2$.*
(b) *If $\mathcal{N}$ is a von Neumann algebra commuting with $\mathcal{M}$ and $\mathtt{V}_{\mathcal{N},\Omega} = \mathtt{V}'_{\mathcal{M},\Omega}$, then $\mathcal{N} = \mathcal{M}'$.*

***Proof*** (a) Let $A \in \mathcal{N}_1$ be selfadjoint. As $\mathcal{N}_{1,h}\Omega \subseteq \overline{\mathcal{N}_{2,h}\Omega}$, there exists a sequence of hermitian elements $A_n \in \mathcal{N}_2$ with $A_n\Omega \to A\Omega$. Then $A_nA'\Omega \to$

$AA'\Omega$ for every $A' \in \mathcal{M}'$. Thus $A_n \to A$ strongly on the dense subspace $\mathcal{M}'\Omega$. Since the hermitian operators $A_n$ and $A$ are bounded, and $\Omega$ is separating, hence cyclic for $\mathcal{M}'$, the dense subspace $\mathcal{M}'\Omega$ is a common core for all of them. With [RS73, Thm. VIII.25] it now follows that $A_n \to A$ holds in the strong resolvent sense, i.e., that $(i\mathbf{1} + A_n)^{-1} \to (i\mathbf{1} + A)^{-1}$ in the strong operator topology. This implies that $(i\mathbf{1} + A)^{-1} \in \mathcal{N}_2$, which entails $A \in \mathcal{N}_2$.
(b) From $\mathcal{N} \subseteq \mathcal{M}'$ and $\mathtt{V}_{\mathcal{N},\Omega} = \mathtt{V}'_{\mathcal{M},\Omega} = \mathtt{V}_{\mathcal{M}',\Omega}$ (Lemma 1.23) we derive with (a) that $\mathcal{N} = \mathcal{M}'$. □

**Corollary 1.27** *Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra and $\Omega \in \mathcal{H}$ separating for $\mathcal{M}$. To every von Neumann subalgebra $\mathcal{N} \subseteq \mathcal{M}$ we associate the closed real subspace $\mathtt{V}_{\mathcal{N}} := \overline{\mathcal{N}_h\Omega}$. Then $\mathtt{V}_{\mathcal{N}_1} = \mathtt{V}_{\mathcal{N}_2}$ implies $\mathcal{N}_1 = \mathcal{N}_2$ for $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{M}$.*

Note that the subspace $\mathtt{V}_{\mathcal{N}}$ is standard in $\overline{\mathcal{N}\Omega}$. It is also standard in $\mathcal{H}$ if $\Omega$ is cyclic for $\mathcal{N}$.

## 1.6 The axioms for nets of local observables

States of quantum mechanical systems are represented by one-dimensional subspaces $\mathbb{C}\Omega$ of a complex Hilbert space $\mathcal{H}$ and selfadjoint elements $A \in B(\mathcal{H})$ represent observables. The evaluation of an observable in a state $[\Omega] := \mathbb{C}\Omega$ corresponds to the evaluation of the corresponding state

$$\omega(A) = \frac{\langle \Omega, A\Omega \rangle}{\langle \Omega, \Omega \rangle}.$$

For some systems, the observables are restricted to selfadjoint elements of a proper von Neumann subalgebra $\mathcal{M} \subsetneq B(\mathcal{H})$. We refer to Varadarajan [Va85] and Mackey [Ma78] for more details on this perspective on quantum mechanics.

In Algebraic Quantum Field Theory (AQFT) one starts with a "spacetime manifold" $M$, which, in the simplest case is Minkowski space $M = \mathbb{R}^{1,d-1}$. We write its elements as pairs

$$x = (x_0, \mathbf{x}) = (x_0, x_1, \ldots, x_{d-1})$$

and define the Lorentzian form by

$$\beta(x, y) = x_0 y_0 - \mathbf{x}\mathbf{y} = x_0 y_0 - x_1 y_1 - \cdots - x_{d-1} y_{d-1}. \tag{15}$$

We call $x \in \mathbb{R}^{1,d-1}$ *timelike* if $\beta(x,x) > 0$, *lightlike* if $\beta(x,x) = 0$, and *spacelike* if $\beta(x,x) < 0$. The open convex cone

$$\mathrm{V}_+ := \{x \in \mathbb{R}^{1,d-1} \colon x_0 > 0, \beta(x,x) > 0\}$$

is called the *positive light cone*. Positive timelike vectors are possible tangent vectors $\gamma'(t)$ to *worldlines* $\gamma\colon \mathbb{R} \to M$ of massive particles and positive lightlike vectors are tangent vectors to light-rays (moving with the speed of light). *Causal curves* are specified by $\gamma'(t) \in \overline{\mathrm{V}_+}$ for every $t$, i.e., they correspond to movements not faster than light.

**Examples 1.28** (see also Example 3.11 for these examples)
(a) There are also curved homogeneous spacetimes, such as *de Sitter space*

$$\mathrm{dS}^d = \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d} \colon x_0^2 - \mathbf{x}^2 = -1\}.$$

It describes a model of a spherical (positively curved) expanding universe. It is a hypersurface in the $(d+1)$-dimensional Minkowski space $\mathbb{R}^{1,d}$. For $x \in \mathrm{dS}^d$, the tangent space $T_x(\mathrm{dS}^d)$ can be identified with the hyperplane

$$x^{\perp_\beta} = \{y \in \mathbb{R}^{1,d} \colon \beta(x, y) = 0\}.$$

Since $x$ is spacelike, the restriction of $\beta$ as in (15) to this hyperplane is Lorentzian, and this specifies a Lorentzian metric on $\mathrm{dS}^d$. The causal structure on $\mathrm{dS}^d$ is specified by

$$C_x = \overline{\mathrm{V}_+} \cap T_x(\mathrm{dS}^d).$$

(b) *Anti-de Sitter space* is the hypersurface

$$\mathrm{AdS}^d = \{(x_1, x_2, \mathbf{x}) \in \mathbb{R}^{2,d-1} \colon x_1^2 + x_2^2 - \mathbf{x}^2 = 1\}$$

in $\mathbb{R}^{2,d-1}$, endowed with the symmetric bilinear form

$$\gamma(x, y) = x_1 y_1 + x_2 y_2 - \mathbf{x}\mathbf{y} \quad \text{for} \quad x = (x_1, x_2, \mathbf{x}) \in \mathbb{R}^{2,d-1}. \tag{16}$$

Again, for $x \in \mathrm{AdS}^d$, the tangent space $T_x(\mathrm{AdS}^d)$ can be identified with the hyperplane

$$x^{\perp_\gamma} = \{y \in \mathbb{R}^{2,d-1} \colon \gamma(x, y) = 0\}.$$

Since $\gamma(x, x) = 1$, the restriction of $\gamma$ to this hyperplane is Lorentzian, and it is easy to verify that $\mathrm{AdS}^d$ is time-orientable (there exists a continuous selection of "positive" light cones) (cf. [NÓ23a, §11]). In fact, we can use the matrix $D \in \mathfrak{so}_{2,d-1}(\mathbb{R})$ defined by

$$D(x_1, x_2, \ldots, x_{d+1}) = (-x_2, x_1, 0, \cdots, 0).$$

It generates the subgroup $\mathrm{SO}_2(\mathbb{R})$, acting on $\mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2\}$, and for every $x \in \mathrm{AdS}^d$, the corresponding vector $Dx \in T_x(\mathrm{AdS}^d)$ is non-zero timelike, hence defines a causal orientation. We conclude that there exists a causal structure on $\mathrm{AdS}^d$, for which

$$C_{\mathbf{e}_1} = \{(0, x_2, \mathbf{x}) \colon x_2 \geq \sqrt{\mathbf{x}^2}\}.$$

One can also argue by the connectedness of the stabilizer group $(\mathrm{SO}_{2,d-1}(\mathbb{R})_e)^{\mathbf{e}_1} \cong \mathrm{SO}_{1,d-1}(\mathbb{R})_e$ to see that it leaves both light cones in $T_{\mathbf{e}_1}(\mathrm{AdS}^d)$ invariant.

For a family of von Neumann algebras $\mathcal{M}(\mathcal{O}) \subseteq B(\mathcal{H})$ and a unitary representation $(U, \mathcal{H})$ of a Lie group $G$ on $\mathcal{H}$, acting also on $M$, we consider the following axioms:

(Iso) **Isotony:** $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)$.
(RS) **Reeh–Schlieder property:** There exists a unit vector $\Omega \in \mathcal{H}$ that is cyclic for $\mathcal{M}(\mathcal{O}), \mathcal{O} \neq \emptyset$.
(Cov) **Covariance:** $U_g \mathcal{M}(\mathcal{O}) U_g^{-1} = \mathcal{M}(g\mathcal{O})$ for $g \in G$.
(Vac) **Invariance of the vacuum:** $U(g)\Omega = \Omega$ for $g \in G$.
(BW) **Bisognano–Wichmann property:** There exists a Lie algebra element $h \in \mathfrak{g}$ and a connected open subset $W \subseteq M$ (called a wedge region), such that $\Omega$ is cyclic and separating for $\mathcal{M}(W)$, and the corresponding modular operator $\Delta = \Delta_{\mathcal{M}(W),\Omega}$ is given by

$$\Delta = e^{2\pi i \cdot \partial U(h)}, \quad \text{i.e.,} \quad \Delta^{-it/2\pi} = U(\exp th) \quad \text{for} \quad t \in \mathbb{R}.$$

(Loc) **Locality:** There exists an open non-empty $G$-invariant subset $\mathcal{D}_{\mathrm{loc}} \subseteq M \times M$ (the locality domain) such that $\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{D}_{\mathrm{loc}}$ implies $\mathcal{M}(\mathcal{O}_1) \subseteq \mathcal{M}(\mathcal{O}_2)'$.
(Add) **Additivity:** The von Neumann algebra $\mathcal{M}(\bigcup_j \mathcal{O}_j)$ is generated by the algebras $\mathcal{M}(\mathcal{O}_j), j \in J$ (cf. [SW87]).

*Remark 1.29* These axioms are an abstract form of the axioms imposed on nets of local algebras on Minkowski space $M = \mathbb{R}^{1,d-1}$ and the *Poincaré group*

$$G = \mathbb{R}^{1,d-1} \rtimes \mathrm{SO}_{1,d-1}(\mathbb{R})_e,$$

acting by affine isometries. We now explain the differences, resp., the specifics of the Minkowski case.
(a) Here $h$ is a generator of a Lorentz boost:

$$h.(x_0, x_1, \ldots, x_{d-1}) = (x_1, x_0, 0, \ldots, 0), \tag{17}$$

and the corresponding wedge region is the *Rindler wedge*

$$W_R := \{x \in \mathbb{R}^{1,d-1} \colon x_1 > |x_0|\}, \tag{18}$$

the set of all points $x$, where the linear vector field $x \mapsto h.x$ is positive timelike. The corresponding one-parameter group of $G$ consists of Lorentz boosts

$$e^{th} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \oplus \mathbf{1}_{\mathbb{R}^{d-2}}.$$

(b) The physical interpretation of the Reeh–Schlieder condition is that every state can be measured with arbitrary precision in any laboratory $\mathcal{O}$.

(c) In AQFT one sometimes assumes, in addition to (Vac), the "irreducibility condition" that the fixed point space $\mathcal{H}^G$ of $G$ is one-dimensional, i.e., $\mathcal{H}^G = \mathbb{C}\Omega$. We refer to [MN24, §5.4] for results concerning direct integral decompositions reducing to this case.
(d) For Minkowski space, the subset $\mathcal{D}_{\rm loc} \subseteq M \times M$ is the set of spacelike pairs

$$\{(x,y) \in \mathbb{R}^{1,d-1} \times \mathbb{R}^{1,d-1} \colon \beta(x-y, x-y) < 0\}$$

for the Lorentzian form $\beta(x,y) = x_0 y_0 - \mathbf{x}\mathbf{y}$. These are the pairs of spacetime events that cannot "exchange" information traveling not faster than light:

$$\mathcal{D}_{\rm loc} = G.\{(x,0) \colon x \in \mathbb{R}^{1,d-1}, \beta(x,x) < 0\} = G.(\mathbb{R}_{>0}\mathbf{e}_1 \times \{0\}).$$

As a consequence, observables in $\mathcal{O}_1$ and $\mathcal{O}_2$ are not subject to an uncertaintly relation if

$$\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{D}_{\rm loc}.$$

To make this more precise, we recall that, for two selfadjoint operators $A_1$ and $A_2$, commuting is equivalent to the non-existence of uncertainties in common measurements (Exercise 1.55). Then there exists a spectral measure $P$ on $\mathbb{R}^2$ with

$$A_1 = \int_{\mathbb{R}^2} x_1 \, dP(x) \quad \text{and} \quad A_2 = \int_{\mathbb{R}^2} x_2 \, dP(x).$$

As a consequence, states can be localized simultaneously with respect to $A_1$ and $A_2$ with arbitrary precision.

The monographs of Varadarajan [Va85] and Mackey [Ma78] are excellent references for the connection between observables in Quantum Physics and selfadjoint operators. We also recommend the recent paper [Ba24] by J. Baez on Jordan and Lie structures related to classical and quantum observables.

We would like to understand the configurations specified by the $G$-action on $M$, the geometry of $M$, the unitary representation $U \colon G \to \mathrm{U}(\mathcal{H})$ and the von Neumann algebras $\mathcal{M}(\mathcal{O})$, satisfying these axioms. As the algebra structure of the local algebras $\mathcal{M}(\mathcal{O})$ only enters through the modular groups, it makes sense to strip it off to simplify the situation, with the hope that we arrive at more tractable structures.

So we consider the family

$$\mathsf{H}(\mathcal{O}) = \mathsf{V}_{\mathcal{M}(\mathcal{O}),\Omega} = \overline{\mathcal{M}(\mathcal{O})_h \Omega} \subseteq \mathcal{H} \tag{19}$$

of closed real subspaces. If $\Omega$ is standard for $\mathcal{M}(\mathcal{O})$, then $\mathsf{H}(\mathcal{O})$ is standard (Lemma 1.20(c)), and the corresponding modular objects can be recovered from $\mathsf{H}(\mathcal{O})$ (Definition 1.3). So we do not lose any information on them.

The axioms for the algebras $\mathcal{M}(\mathcal{O})$ thus turn into the following axioms for the net $\mathsf{H}(\mathcal{O})$ of real subspaces:

(Iso) **Isotony**: $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)$

(RS) **Reeh–Schlieder property:** $\mathsf{H}(\mathcal{O})$ is cyclic if $\mathcal{O} \neq \emptyset$.
(Cov) **Covariance:** $U_g \mathsf{H}(\mathcal{O}) = \mathsf{H}(g\mathcal{O})$ for $g \in G$.
(BW) **Bisognano–Wichmann property:** There exists a Lie algebra element $h \in \mathfrak{g}$ and an open connected subset $W \subseteq M$, such that $\mathsf{H}(W)$ is standard and the corresponding modular operator is

$$\Delta_{\mathsf{H}(W)} = e^{2\pi i \cdot \partial U(h)}, \quad \text{i.e.,} \quad \Delta_{\mathsf{H}(W)}^{-it/2\pi} = U(\exp th), t \in \mathbb{R}.$$

(Loc) **Locality:** There exists an open non-empty $G$-invariant subset $\mathcal{D}_{\rm loc} \subseteq M \times M$ such that $\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{D}_{\rm loc}$ implies $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)'$.
(Add) **Additivity:** $\mathsf{H}(\bigcup_j \mathcal{O}_j) = \overline{\sum_{j\in J} \mathsf{H}(\mathcal{O}_j)}$.

*Remark 1.30* (a) The covariance condition (Cov) for real subspaces follows from the $G$-invariance of $\Omega$ and the covariance condition $U_g \mathcal{M}(\mathcal{O}) U_g^{-1} = \mathcal{M}(g\mathcal{O})$.
(b) The subspace $\mathsf{H}(M)$ is $G$-invariant by (Cov) and cyclic by (RS). If it is also separating, hence standard, then its modular operator $\Delta_{\mathsf{H}(M)}$ and the conjugation $J_M := J_{\mathsf{H}(M)}$ commute with $U(G)$. If (BW) holds, then the Equality Lemma 1.9 implies $\mathsf{H}(W) = \mathsf{H}(M)$, and thus $h$ is central in $\mathfrak{g}$, provided $\ker U$ is discrete. This shows that $\mathsf{H}(M)$ cannot be standard if the net is not very degenerate.

*Remark 1.31* (Localization in wedge regions) The Equality Lemma 1.9 also has the interesting consequence, that, whenever $\mathcal{O} \subseteq W$ is a non-empty open subset invariant under $\exp(\mathbb{R}h)$, and $\mathsf{H}(\mathcal{O})$ is cyclic (for example if (RS) is satisfied), then $\mathsf{H}(\mathcal{O}) = \mathsf{H}(W)$. This applies in particular to any flow-invariant neighborhood of a given flow line $\exp(\mathbb{R}h).p_0 \subseteq W$.

In the context of operator algebras, this fact is closely related to the interpretation of any such flow line as the worldline of an "observer". For Minkowski space (Example 3.9) this is literally true because these flow lines in the Rindler wedge correspond to uniformly accelerated observers.

## 1.7 Operator algebras on the symmetric Fock spaces

The passage from a net of algebras $\mathcal{M}(\mathcal{O})$ to a net of real subspace $\mathsf{H}(\mathcal{O})$ (which is similar to a forgetful functor) can be "inverted" (in the spirit of an adjoint functor) by procedures of second quantization assigning operator algebras $\Gamma(\mathsf{H})$ to real subspaces $\mathsf{H} \subseteq \mathcal{H}$ (see also [Ar63] and [NÓ17]). Therefore any result on nets of real subspaces can be transformed into a result on nets of local algebras obtained by second quantization (see also [NÓ17, Rem. 4.10]). We note, however, that most second quantization procedures (such as the bosonic and fermionic one) are "free" in the sense that they do not take interaction between particles into account. For further reading on Fock spaces (bosonic and fermionic) and second quantization, we refer to

[SiB74], [Ne10], [Ot95, §3], [BR87], [BSZ92], [Os73]. For a recent systematic construction of twisted second quantization functors, we refer to [CSL23]. In any case, **the requirements on $G$, the $G$-action on $M$, and the antiunitary representation of $G_{\tau_h}$ are the same for the existence of nets of real subspaces and nets of von Neumann algebras.**

As far as the symmetries and the modular groups are concerned, the algebra axioms are faithfully represented by the axioms for their associated real subspaces. Even inclusions are rather well-behaved (cf. Proposition 1.26).

### 1.7.1 Weyl operators on the symmetric Fock space

In this subsection, we consider the symmetric (bosonic) Fock space

$$S(\mathcal{H}) := \widehat{\bigoplus}_{k=0}^{\infty} S^k(\mathcal{H})$$

of the complex Hilbert space $\mathcal{H}$. Here consider $S^k(\mathcal{H}) \subseteq \mathcal{H}^{\otimes n}$ as the subspace of fixed points for the permutation representation $(\rho, \mathcal{H}^{\otimes n})$ of the symmetric group $S_n$ on the Hilbert space tensor product $\mathcal{H}^{\otimes n}$, in which the scalar product is determined by

$$\langle v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_n \rangle = \langle v_1, w_1 \rangle \cdots \langle v_n, w_n \rangle.$$

We write

$$P_+ \colon \mathcal{H}^{\otimes n} \to S^n(\mathcal{H}), \quad P_+ := \frac{1}{n!} \sum_{\sigma \in S_n} \rho(\sigma)$$

for the orthogonal projection and define the symmetric product by

$$v_1 \cdots v_n := P_+(v_1 \otimes \cdots \otimes v_n).$$

The inner products of such elements are given by

$$\begin{aligned} \langle v_1 \cdots v_n, w_1 \cdots w_n \rangle &= \langle P_+(v_1 \otimes \cdots \otimes v_n), P_+(w_1 \otimes \cdots \otimes w_n) \rangle \\ &= \langle P_+(v_1 \otimes \cdots \otimes v_n), w_1 \otimes \cdots \otimes w_n \rangle \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \langle v_{\sigma(1)}, w_1 \rangle \cdots \langle v_{\sigma(n)}, w_n \rangle \end{aligned} \tag{20}$$

**Lemma 1.32** (Bases of $S^n(\mathcal{H})$) *Let $(\mathbf{e}_j)_{j \in J}$ be an ONB in $\mathcal{H}$. For $\mathbf{m} \in \mathbb{N}_0^{(J)}$, the set of finitely supported functions $J \to \mathbb{N}_0, j \mapsto m_j$, considered as multiindices. We put $|\mathbf{m}| := \sum_{j \in J} m_j$. Then the elements*

$$\mathbf{e}^{\mathbf{m}} := \mathbf{e}_{j_1}^{m_{j_1}} \cdots \mathbf{e}_{j_k}^{m_{j_k}} \quad \textit{for} \quad \{j \in J \colon m_j \neq 0\} = \{j_1, \ldots, j_k\}$$

*with $|\mathbf{m}| = n$ form an orthogonal basis of $S^n(\mathcal{H})$ satisfying*

$$\|\mathbf{e}^{\mathbf{m}}\|^2 = \frac{\mathbf{m}!}{n!} \quad \textit{for} \quad \mathbf{m}! := \prod_{j \in J} m_j!.$$

***Proof*** Expanding products $v_1 \vee \cdots \vee v_n \in S^n(\mathcal{H})$ with respect to the orthonormal basis $(\mathbf{e}_j)_{j \in J}$, we see that the finite products

$$\mathbf{e}_{j_1} \vee \cdots \vee \mathbf{e}_{j_n}, \quad j_1, \ldots, j_n \in J,$$

form a total subset of $S^n(\mathcal{H})$. This proves that the $\mathbf{e}^{\mathbf{m}}$, $|\mathbf{m}| = n$, form a total subset of $S^n(\mathcal{H})$. Next we observe that, for

$$\mathbf{e}^{\mathbf{m}} = \mathbf{e}_{j_1}^{m_{j_1}} \vee \cdots \vee \mathbf{e}_{j_k}^{m_{j_k}} \quad \text{and} \quad \mathbf{e}^{\mathbf{n}} = \mathbf{e}_{j_1}^{n_{j_1}} \vee \cdots \vee \mathbf{e}_{j_\ell}^{n_{j_\ell}}$$

to have a non-zero scalar product we need that $m_j = n_j$ for every $j \in J$. Therefore the family $(\mathbf{e}^{\mathbf{m}})_{|\mathbf{m}|=n}$ is orthogonal in $S^n(\mathcal{H})$. From (20) we further obtain that

$$\langle \mathbf{e}^{\mathbf{m}}, \mathbf{e}^{\mathbf{m}} \rangle = \frac{m_{j_1}! \cdots m_{j_k}!}{n!} = \frac{\mathbf{m}!}{n!},$$

which complete the proof of the lemma. □

We want to define natural unitary operators on $S(\mathcal{H})$, called the *Weyl operators*. They will form a unitary representation of the Heisenberg group (cf. (25) below). We start by observing that, for every $v \in \mathcal{H}$, the series

$$\operatorname{Exp}(v) := \sum_{n=0}^{\infty} \frac{1}{n!} v^n, \tag{21}$$

defines an element in $S(\mathcal{H})$ and that by

$$\langle v^n, w^n \rangle = n! \langle v, w \rangle^n \quad \text{and} \quad \|v^n\| = \sqrt{n!} \|v\|^n$$

([NÓ17, §6.1]), the scalar product of two exponentials is given by

$$\langle \operatorname{Exp}(v), \operatorname{Exp}(w) \rangle = \sum_{n=0}^{\infty} \frac{n!}{(n!)^2} \langle v, w \rangle^n = e^{\langle v, w \rangle}.$$

**Lemma 1.33** $\operatorname{Exp}(\mathcal{H})$ *is total in* $S(\mathcal{H})$*, i.e., it spans a dense subspace.*

***Proof*** Let $\mathcal{K} \subseteq S(\mathcal{H})$ be the closed subspace generated by $\operatorname{Exp}(\mathcal{H})$. We consider the unitary representation of the circle group $\mathbb{T} \subseteq \mathbb{C}^\times$ on $S(\mathcal{H})$ by

$$U_z(v_1 \vee \cdots \vee v_n) := z^n (v_1 \vee \cdots \vee v_n) \quad \text{for} \quad n \in \mathbb{N}_0, v_j \in \mathcal{H}.$$

In particular we have

$$U_z \operatorname{Exp}(v) = \operatorname{Exp}(zv) \quad \text{for} \quad z \in \mathbb{T}, v \in \mathcal{H}. \tag{22}$$

The decomposition $S(\mathcal{H}) = \widehat{\bigoplus}_{n\in\mathbb{N}_0} S^n(\mathcal{H})$ is the eigenspace decomposition with respect to the operators $U_z$ and it is easy to see that the action of $\mathbb{T}$ on $S(\mathcal{H})$ has continuous orbit maps (Exercise 1.46). For $\xi \in S(\mathcal{H})$ with $\xi = \sum_{n=0}^\infty \xi_n$ and $\xi_n \in S^n(\mathcal{H})$, we have $U_z\xi = \sum_{n=0}^\infty z^n\xi_n$, so that

$$\xi_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-2\pi int} U_{e^{it}}\xi\, dt$$

(observe the analogy with Fourier coefficients). It follows that, for $\xi \in \mathcal{K}$, the existence of the above Riemann integral in the closed subspace $\mathcal{K}$, which is invariant under $(U_z)_{z\in\mathbb{T}}$ by (22), implies $\xi_n \in \mathcal{K}$. We conclude that $v^n \in \mathcal{K}$ for $v \in \mathcal{H}$ and $n \in \mathbb{N}_0$. Therefore it suffices to observe that the subset $\{v^n \colon v \in \mathcal{H}\}$ is total in $S^n(\mathcal{H})$ (Exercise 1.54). ☐

For $v, x \in \mathcal{H}$ we have

$$\langle \mathrm{Exp}(v + x), \mathrm{Exp}(w + x)\rangle = e^{\langle v+x,w+x\rangle} = e^{\langle v,w\rangle} e^{\langle x,w\rangle + \frac{\|x\|^2}{2}} e^{\langle v,x\rangle + \frac{\|x\|^2}{2}},$$

so that there exists a well-defined and uniquely determined unitary operator $U(x)$ on $S(\mathcal{H})$ satisfying

$$U(x)\,\mathrm{Exp}(v) = e^{-\langle x,v\rangle - \frac{\|x\|^2}{2}}\,\mathrm{Exp}(v + x) \quad \text{for} \quad x, v \in \mathcal{H} \tag{23}$$

(Exercise 1.52; the surjectivity of $U(x)$ follows from the totality of $\mathrm{Exp}(\mathcal{H})$). A direct calculation then shows that

$$U(x)U(y) = e^{-i\,\mathrm{Im}\langle x,y\rangle} U(x + y) \quad \text{for} \quad x, y \in \mathcal{H}. \tag{24}$$

In fact, for $v \in \mathcal{H}$, we have

$$\begin{aligned} U(x)U(y)\,\mathrm{Exp}(v) &= U(x)e^{-\langle y,v\rangle - \frac{\|y\|^2}{2}}\,\mathrm{Exp}(v + y) \\ &= e^{-\langle y,v\rangle - \frac{\|y\|^2}{2}} e^{-\langle x,v+y\rangle - \frac{\|x\|^2}{2}}\,\mathrm{Exp}(v + y + x) \\ &= e^{-\langle x+y,v\rangle} e^{-\frac{\|y\|^2}{2} - \frac{\|x\|^2}{2} - \langle x,y\rangle}\,\mathrm{Exp}(v + y + x) \end{aligned}$$

and

$$\begin{aligned} U(x + y)\,\mathrm{Exp}(v) &= e^{-\langle x+y,v\rangle - \frac{\|x+y\|^2}{2}}\,\mathrm{Exp}(v + y + x) \\ &= e^{-\langle x+y,v\rangle - \frac{\|x\|^2}{2} - \frac{\|y\|^2}{2} - \mathrm{Re}\langle x,y\rangle}\,\mathrm{Exp}(v + y + x) \end{aligned}$$

The relation (24) shows that the map $U \colon (\mathcal{H}, +) \to \mathrm{U}(S(\mathcal{H}))$ is not a group homomorphism. Instead, we have to replace the additive group $(\mathcal{H}, +)$ by the *Heisenberg group*

$$\mathrm{Heis}(\mathcal{H}) := \mathbb{T} \times \mathcal{H} \quad \text{with} \quad (z, v)(z', v') := (zz'e^{-i\,\mathrm{Im}\langle v,v'\rangle}, v + v') \tag{25}$$

to obtain a unitary representation

$$\widehat{U}\colon \operatorname{Heis}(\mathcal{H}) \to \mathrm{U}(S(\mathcal{H})) \quad \text{by} \quad \widehat{U}(z,v) := zU(v).$$

The operators

$$W(v) := U\big(iv/\sqrt{2}\big), \qquad v \in \mathcal{H},$$

are called *Weyl operators*. They satisfy the *Weyl relations*

$$W(v)W(w) = e^{-i\operatorname{Im}\langle v,w\rangle/2}W(v+w) \quad \text{for} \quad v,w \in \mathcal{H}. \tag{26}$$

They are an exponentiated form of the "canonical commutation relations" for the corresponding infinitesimal generators.

The *Weyl algebra*

$$W(\mathcal{H}) := C^*(\{W(v)\colon v \in \mathcal{H}\}) \subseteq B(S(\mathcal{H}))$$

is the $C^*$-subalgebra of $B(S(\mathcal{H}))$ generated by the Weyl operators. It plays an important role in Quantum (Statistical) Mechanics and Quantum Field Theory. This is partly due to the fact that it is a simple $C^*$-algebra (all ideals are trivial), which implies in particular that all its representations are faithful. Closely related is its universal property: If $\mathcal{A}$ is a unital $C^*$-algebra and $\varphi\colon \mathcal{H} \to \mathrm{U}(\mathcal{A})$ a map satisfying the Weyl relations in the form

$$\varphi(v)\varphi(w) = e^{-i\operatorname{Im}\langle v,w\rangle/2}\varphi(v+w) \quad \text{for} \quad v,w \in \mathcal{H}, \tag{27}$$

then there exists a unique homomorphism $\Phi\colon W(\mathcal{H}) \to \mathcal{A}$ of unital $C^*$-algebras with $\Phi \circ W = \varphi$. An excellent discussion of the Weyl algebra and its properties can be found in the monograph [BR96] which also describes the physical applications in great detail.

### 1.7.2 From real subspaces to von Neumann algebras

In this subsection, we describe a mechanism that associates to real subspaces of a Hilbert space $\mathcal{H}$ von Neumann algebras on the symmetric Fock space $S(\mathcal{H})$. This construction plays an important role in recent developments in Algebraic Quantum Field Theory (AQFT) because it provides natural links between the geometric structure of spacetime and operator algebras (see in particular [Ar99, Lo08, Le15]). It has also been of great interest for the classification of von Neumann factors because it provides very controlled constructions of factors whose type can be determined in some detail ([AW63, AW68]).

We write

$$\gamma(v,w) := \operatorname{Im}\langle v,w\rangle \quad \text{for} \quad v,w \in \mathcal{H}$$

and observe that $\gamma$ is skew-symmetric and non-degenerate, so that the underlying real Hilbert space $\mathcal{H}^{\mathbb{R}}$ carries the structure of a *symplectic vector space* $(\mathcal{H}^{\mathbb{R}}, \gamma)$.

Using the Weyl operators, we associate to every real linear subspace $V \subseteq \mathcal{H}$ a von Neumann subalgebra

$$\mathcal{R}(V) := W(V)'' = \{W(v) \colon v \in V\}'' \subseteq B(S(\mathcal{H})). \tag{28}$$

**Lemma 1.34** *We have*

(i) $\mathcal{R}(V) \subseteq \mathcal{R}(W)'$ *if and only if* $V \subseteq W'$.
(ii) $\mathcal{R}(V)$ *is commutative if and only if* $V \subseteq V'$.
(iii) $\mathcal{R}(\mathcal{H}) = B(S(\mathcal{H}))$, *i.e., the representation of* $\mathrm{Heis}(\mathcal{H})$ *on* $S(\mathcal{H})$ *is irreducible.*
(iv) $\mathcal{R}(V) = \mathcal{R}(\overline{V})$.
(v) $\Omega = \mathrm{Exp}(0) \in S(\mathcal{H})$ *is cyclic for* $\mathcal{R}(V)$ *if and only if* $V + iV$ *is dense in* $\mathcal{H}$.
(vi) $\Omega = \mathrm{Exp}(0) \in S(\mathcal{H})$ *is separating for* $\mathcal{R}(V)$ *if and only if* $\overline{V} \cap i\overline{V} = \{0\}$.

***Proof*** (i) follows directly from the Weyl relations (26).
(ii) follows from (i).
(iii) follows from [BR96, Prop. 5.2.4(3)].
(iv) follows from the fact that $\mathcal{H} \to B(S(\mathcal{H})), v \mapsto W_v$ is strongly continuous and $\mathcal{R}(V)$ is closed in the weak operator topology.
(v) Let $\mathcal{K} := \overline{V + iV}$. Then $\mathcal{R}(V)\Omega \subseteq S(\mathcal{K})$, so that $\Omega$ cannot be cyclic if $\mathcal{K} \neq \mathcal{H}$.

Suppose, conversely, that $\mathcal{K} = \mathcal{H}$ and that $f \in (\mathcal{R}(V)\Omega)^{\perp}$. Then the holomorphic function $\widehat{f}(v) := \langle f, \mathrm{Exp}(v)\rangle$ on $\mathcal{H}$ vanishes on $iV$, hence also on $V + iV$, and since this subspace is dense in $\mathcal{H}$, we obtain $f = 0$ because $\mathrm{Exp}(\mathcal{H})$ is total in $S(\mathcal{H})$.
(vi) In view of (iv), we may assume that $V$ is closed. Let $0 \neq w \in \mathcal{K} := V \cap iV$. To see that $\Omega$ is not separating for $\mathcal{R}(V)$, it suffices to show that, for the one-dimensional Hilbert space $\mathcal{H}_0 := \mathbb{C}w$, the vector $\Omega$ is not separating for $\mathcal{R}(\mathbb{C}w) = B(S(\mathbb{C}w))$ (see (iii)). This is obviously the case because $\dim S(\mathbb{C}w) > 1$.

Suppose, conversely, that $\mathcal{K} = \{0\}$. As $\mathcal{K} = V'' \cap (iV'') = (V' + iV')'$, it follows that $V' + iV'$ is dense in $\mathcal{H}$. By (v), $\Omega$ is cyclic for $\mathcal{R}(V')$ which commutes with $\mathcal{R}(V)$. Therefore $\Omega$ is separating for $\mathcal{R}(V)$ (Lemma 1.18). $\square$

**Theorem 1.35** ([Ar63]) (Araki's Duality Theorem) *For closed real subspaces* $V, W, V_j$ *of* $\mathcal{H}$, *the following assertions hold:*

(i) $\mathcal{R}(V) \subseteq \mathcal{R}(W)$ *if and only if* $V \subseteq W$.
(ii) $\mathcal{R}\big(\bigcap_{j \in J} V_j\big) = \bigcap_{j \in J} \mathcal{R}(V_j)$.
(iii) $\mathcal{R}(V)' = \mathcal{R}(V')$ *(Duality).*
(iv) $\mathcal{Z}(\mathcal{R}(V)) = \mathcal{R}(V \cap V')$ *and* $\mathcal{R}(V)$ *is a factor if and only if* $V \cap V' = \{0\}$.

***Proof*** We only comment on some of these statements:
(i) That $V \subseteq W$ implies $\mathcal{R}(V) \subseteq \mathcal{R}(W)$ is clear, but the converse is non-trivial. It can be derived from the duality property (iii), which is the main result of [Ar63].
(ii) here "$\subseteq$" is obvious.
(iii) is a deep theorem.
(iv) follows from (ii) and (iii) via $\mathcal{R}(V \cap V') = \mathcal{R}(V) \cap \mathcal{R}(V)' = \mathcal{Z}(\mathcal{R}(V))$.□

The preceding theorem asserts in particular that

- $\mathcal{R}(V)$ is a factor if and only if $V \cap V' = \{0\}$. This means that the form $\gamma|_{V \times V}$ is non-degenerate, i.e., that $(V, \gamma)$ is a symplectic vector space.

Subspaces with this property are easy to construct. In [Ar64b] many results on the types of the so-obtained factors can be found. In particular, it is shown that factors of type II do not occur, and [Ar64] provides an explicit criterion for $\mathcal{R}(V)$ to be of type I. "Generically", the so-obtained factors are of type III. We refer to [Sa71] for more on types of von Neumann algebras.

## 1.8 Appendices to Section 1

The appendices to this section provide complementary information that is not used in the arguments in the main text, but which concerns issues providing more background on our constructions.

### 1.8.1 Endomorphisms of standard subspaces and von Neumann algebras

In this appendix we briefly discuss the difference between unitary endomorphisms of von Neumann algebras and the corresponding endomorphisms of associated standard subspaces. Concretely, let $\Omega \in \mathcal{H}$ be a standard vector for the von Neumann algebra $\mathcal{M}$, let $\mathtt{V} = \mathtt{V}_{\mathcal{M},\Omega}$ be the corresponding standard subspace, and let $G \subseteq \mathrm{U}(\mathcal{H})$ be a subgroup. Then the following example shows that the inclusion

$$S_{\mathcal{M},\Omega} = \{g \in G \colon g\mathcal{M}g^{-1} \subseteq \mathcal{M}, g\Omega = \Omega\} \subseteq S_{\mathtt{V},\Omega} = \{g \in G \colon g\mathtt{V} \subseteq \mathtt{V}, g\Omega = \Omega\}$$

may be proper.

**Examples 1.36** [3] (a) We consider the Hilbert space $\mathcal{H} := M_n(\mathbb{C})$ of matrices, endowed with the Hilbert–Schmidt scalar product $\langle A, B\rangle := \mathrm{tr}(A^*B)$. By matrix multiplications from the left, we obtain a von Neumann subalgebra $\mathcal{M} \subseteq B(\mathcal{H})$, isomorphic to $M_n(\mathbb{C})$, and its commutant $\mathcal{M}'$ consists of right

[3] We thank Yoh Tanimoto for the discussion that led to this example.

multiplications (cf. Example 1.24(b)). The unit vector $\Omega := \frac{1}{\sqrt{n}}\mathbf{1}_n$ is cyclic and separating, and the corresponding standard subspaces for $\mathcal{M}$ and $\mathcal{M}'$ coincide with

$$\mathtt{V}_{\mathcal{M}} = \mathtt{V}_{\mathcal{M}'} = \mathrm{Herm}_n(\mathbb{C})$$

of hermitian matrices. Now $\theta(A) := A^\top$ defines a unitary operator on $\mathcal{H}$, preserving $\Omega$ and the standard subspace $\mathtt{V}_{\mathcal{M}} = \mathtt{V}_{\mathcal{M}'}$, and satisfying $\theta\mathcal{M}\theta^{-1} = \mathcal{M}'$. For $G = \mathrm{U}(\mathcal{H})$, we therefore have $S_{\mathtt{V},\Omega} \neq S_{\mathcal{M},\Omega}$.
(b) In the situation above, when $\mathcal{M}$ is given, the $G$-orbit of $\mathcal{M}$ in the space of von Neumann subalgebras of $B(\mathcal{H})$ can be identified with the homogeneous space $G/G_{\mathcal{M}}$, where $G_{\mathcal{M}} = \{g \in G \colon g\mathcal{M}g^{-1} = \mathcal{M}\}$ is the stabilizer of $\mathcal{M}$. Similarly, we obtain with the stabilizer group $G_{\mathtt{V}} = \{g \in G \colon g\mathtt{V} = \mathtt{V}\}$ and embedding

$$G/G_{\mathtt{V}} \hookrightarrow \mathrm{Stand}(\mathcal{H}), \quad gG_{\mathtt{V}} \mapsto g\mathtt{V}.$$

The discrepancy between both spaces comes from the fact that the von Neumann algebra $\mathcal{M}$ need not be invariant under the stabilizer group $G_{\mathtt{V}}$ of $\mathtt{V}$.

Related questions have been analyzed by Y. Tanimoto in [Ta10]. He refines the picture by considering the closed convex cone

$$\mathtt{V}_{\mathcal{M}}^+ = \overline{\{M\Omega \colon 0 \leq M = M^* \in \mathcal{M}\}} \subseteq \mathtt{V}_{\mathcal{M}},$$

which leads to the inclusions

$$S_{\mathcal{M},\Omega} \hookrightarrow S_{\mathtt{V}_{\mathcal{M}}^+,\Omega} = \{g \in G \colon g\mathtt{V}_{\mathcal{M}}^+ \subseteq \mathtt{V}_{\mathcal{M}}^+, g\Omega = \Omega\} \subseteq S_{\mathtt{V}_{\mathcal{M}},\Omega}.$$

The semigroup $S_{\mathtt{V}_{\mathcal{M}}^+,\Omega}$ appears to be much closer to $S_{\mathcal{M},\Omega}$ than $S_{\mathtt{V},\Omega}$. From [Ta10, Thm. 2.10] it follows in particular that, if $\mathcal{M}$ is purely infinite, then $S_{\mathtt{V}_{\mathcal{M}}^+,\Omega} = S_{\mathcal{M},\Omega}$. Note that the examples under Examples 1.36(a) are of finite type.

Let $\mathcal{M}_*$ denote the predual of the von Neumann algebra $\mathcal{M}$ (the space of normal linear functionals) and $\mathcal{M}_*^+$ the convex cone of positive normal functionals. In this context, it is also interesting to note that the map

$$\mathtt{V}_{\mathcal{M}}^+ \to \mathcal{M}_*^+, \quad \xi \mapsto \omega_\xi, \qquad \omega_\xi(M) = \langle \xi, M\xi \rangle$$

is bijective by [Ko80, Thm. 1.2]. Accordingly, every element $g \in S_{\mathtt{V}_{\mathcal{M}}^+}$ induces a continuous map on the convex cone $\mathcal{M}_*^+$. We refer to [Ta10] and [Co74] for more details and to [Da96] for some interesting results on endomorphisms of von Neumann algebras.

### 1.8.2 Positive definite functions on $\mathbb{R}$ satisfying a KMS condition

This subsection has only illustrative character. It explains how the KMS condition that classically appears in the context of KMS states for $C^*$-algebraic dynamical systems ([Ku57], [MS59], [HHW67], [BB94], [AA+20], [Ku02], [PW78], [Str08], [Ni23]),[4] can be formulated independently of $C^*$-algebras as a condition for functions on $\mathbb{R}$ with values in spaces of bilinear forms.

**Definition 1.37** Let $V$ be a real vector space and $\mathrm{Bil}(V)$ be the space of real bilinear maps $V \times V \to \mathbb{C}$. A function $\psi\colon \mathbb{R} \to \mathrm{Bil}(V)$ is said to be *positive definite* if the kernel $\psi(t-s)(v,w)$ on $\mathbb{R} \times V$ is positive definite. We say that a positive definite function $\psi\colon \mathbb{R} \to \mathrm{Bil}(V)$ satisfies the *KMS condition* for $\beta > 0$ if $\psi$ extends to a function $\overline{\mathcal{S}_\beta} \to \mathrm{Bil}(V)$ which is pointwise continuous and pointwise holomorphic on the interior $\mathcal{S}_\beta$, and satisfies

$$\psi(i\beta + t) = \overline{\psi(t)} \quad \text{for} \quad t \in \mathbb{R}. \tag{29}$$

The central idea in the classification of positive definite functions satisfying a KMS condition is to relate them to standard subspaces. A key result is the following characterization of the KMS condition in terms of standard subspaces ([NÓ19, Thm. 2.6]). Here we write $\mathrm{Bil}^+(V) \subseteq \mathrm{Bil}(V)$ for the convex cone of all those bilinear forms $f$ for which the sesquilinear extension to $V_\mathbb{C} \times V_\mathbb{C}$ is positive semidefinite.

**Theorem 1.38** (Characterization of the KMS condition) *Let $V$ be a real vector space and $\psi\colon \mathbb{R} \to \mathrm{Bil}(V)$ be a pointwise continuous positive definite function. Then the following are equivalent:*

(i) *$\psi$ satisfies the KMS condition for $\beta > 0$.*
(ii) *There exists a standard real subspace $\mathtt{V}$ in a Hilbert space $\mathcal{H}$ and a linear map $j\colon V \to \mathtt{V}$ such that*

$$\psi(t)(v,w) = \langle j(v), \Delta^{-it/\beta} j(w)\rangle \quad \textit{for} \quad t \in \mathbb{R}, v, w \in V. \tag{30}$$

(iii) *There exists a $\mathrm{Bil}^+(V)$-valued regular Borel measure $\mu$ on $\mathbb{R}$ satisfying*

$$\psi(t) = \int_\mathbb{R} e^{it\lambda}\, d\mu(\lambda), \qquad \textit{where} \qquad d\mu(-\lambda) = e^{-\beta\lambda} d\overline{\mu}(\lambda).$$

*If these conditions are satisfied, then the function $\psi\colon \overline{\mathcal{S}_\beta} \to \mathrm{Bil}(V)$ is pointwise bounded.*

The equivalence of (i) and (ii) in the preceding theorem describes the tight connection between the KMS condition and the modular objects associated to a standard real subspace. Part (iii) provides an integral representation that can be viewed as a classification result.

**Corollary 1.39** *For a standard subspace $\mathtt{V} \subseteq \mathcal{H}$ and the modular operator $\Delta_\mathtt{V}$, the function*

[4] The fundamental KMS equilibrium condition originated in the context of analytic properties of Green's functions in [Ku57] and [MS59] and was formulated in the present form by Haag, Hugenholtz and Winnink in [HHW67].

$$\psi\colon \mathbb{R} \to \mathrm{Bil}(\mathtt{V}), \quad \psi(t)(v,w) := \langle v, \Delta_{\mathtt{V}}^{-it/2\pi} w\rangle$$

*satisfies the KMS condition for* $\beta = 2\pi$*.*

*Remark 1.40* (KMS states of $C^*$-algebras) Important special cases arise from $C^*$-dynamical systems $(\mathcal{A}, \mathbb{R}, \alpha)$, where $\mathcal{A}$ is a $C^*$-algebra and $\alpha\colon \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ defines a strongly continuous $\mathbb{R}$-action on $\mathcal{A}$. Let

$$V := \mathcal{A}_h := \{A \in \mathcal{A}\colon A^* = A\}$$

and consider an $\alpha$-invariant state $\omega$ on $\mathcal{A}$. Such a state is a *$\beta$-KMS state* if and only if

$$\psi\colon \mathbb{R} \to \mathrm{Bil}(\mathcal{A}_h), \quad \psi(t)(A,B) := \omega(A\alpha_t(B))$$

satisfies the KMS condition for $\beta > 0$ (cf. [NÓ15, Prop. 5.2], [RvD77, Thm. 4.10], [BR96]). If $(\pi_\omega, U^\omega, \mathcal{H}_\omega, \Omega)$ is the corresponding covariant GNS representation of $(\mathcal{A}, \mathbb{R})$, then

$$\omega(A) = \langle \Omega, \pi_\omega(A)\Omega\rangle \quad \text{for} \quad A \in \mathcal{A} \quad \text{and} \quad U_t^\omega \Omega = \Omega \quad \text{for} \quad t \in \mathbb{R}.$$

For $A, B \in \mathcal{A}_h$, we thus obtain

$$\begin{aligned}\psi(t)(A,B) &= \omega(A\alpha_t(B)) = \langle \Omega, \pi_\omega(A\alpha_t(B))\Omega\rangle \\ &= \langle \Omega, \pi_\omega(A)U_t^\omega \pi_\omega(B) U_{-t}^\omega \Omega\rangle = \langle \pi_\omega(A)\Omega, U_t^\omega \pi_\omega(B)\Omega\rangle\end{aligned}$$

The corresponding standard real subspace of $\mathcal{H}_\omega$ is $\mathtt{V}_{\mathcal{A},\Omega} := \overline{\pi_\omega(\mathcal{A}_h)\Omega}$. Here we use that the KMS condition implies that $\Omega$ is a separating vector for the von Neumann algebra $\pi_\omega(\mathcal{A})''$ (cf. [Si23] and [BR87]).

### 1.8.3 KMS vectors for 1-parameter groups

In this subsection, we collect some general tools concerning holomorphic extensions of orbit maps of one-parameter groups on locally convex spaces to strips in the complex plane. They are instrumental in formulating Kubo–Martin–Schwinger (KMS) boundary conditions that are related to the construction of standard subspaces (see Definition 1.37).

**Definition 1.41** Let $(U_t)_{t\in\mathbb{R}}$ be a one-parameter subgroup of $\mathrm{GL}(\mathcal{Y})$ for a topological vector space $\mathcal{Y}$ and $J$ an antilinear operator on $\mathcal{Y}$, commuting with $(U_t)_{t\in\mathbb{R}}$. We write $\mathcal{Y}_{\mathrm{KMS}}$ for the subspace of those $y \in \mathcal{Y}$, whose orbit map

$$U^v\colon \mathbb{R} \to \mathcal{Y}, t \mapsto U_t v$$

extends to a continuous map on $\overline{\mathcal{S}_\pi} := \mathbb{R} + i[0,\pi]$, holomorphic on the interior $\mathcal{S}_\pi$, such that[5]

[5] By equivariance, it actually suffices that $U^v(\pi i) = Jv$.

$$U^v(\pi i + t) = JU^v(t) = JU_t v \quad \text{for} \quad t \in \mathbb{R}. \tag{31}$$

We call the elements of this space *KMS vectors* (with respect to $U$ and $J$).

To connect with standard subspaces, we first derive a characterization of the elements of a standard subspace $\mathtt{V}$ specified by the pair $(\Delta, J)$ as the space $\mathcal{H}_{\rm KMS}$ for the unitary one-parameter group $(\Delta^{\frac{-it}{2\pi}})_{t\in\mathbb{R}}$ and the conjugation $J$.

**Proposition 1.42** *Let $\mathtt{V} \subseteq \mathcal{H}$ be a standard subspace with modular objects $(\Delta, J)$. For $\xi \in \mathcal{H}$, we consider the orbit map $\alpha^\xi \colon \mathbb{R} \to \mathcal{H}, t \mapsto \Delta^{-it/2\pi}\xi$. Then the following are equivalent:*

(a) $\xi \in \mathtt{V}$.
(b) $\xi \in \mathcal{D}(\Delta^{1/2})$ *with* $\Delta^{1/2}\xi = J\xi$.
(c) *The orbit map $\alpha^\xi \colon \mathbb{R} \to \mathcal{H}$ extends to a continuous map $\overline{\mathcal{S}_\pi} \to \mathcal{H}$ which is holomorphic on $\mathcal{S}_\pi$ and satisfies $\alpha^\xi(\pi i) = J\xi$.*
(d) *There exists an element $\eta \in \mathcal{H}^J$ whose orbit map $\alpha^\eta$ extends to a continuous map $\overline{\mathcal{S}_{\pm\pi/2}} \to \mathcal{H}$ which is holomorphic on the interior and satisfies $\alpha^\eta(-\pi i/2) = \xi$.*

***Proof*** The equivalence of (a) and (b) follows from the definition of $\Delta$ and $J$.
(b) $\Rightarrow$ (c): If $\xi \in \mathcal{D}(\Delta^{1/2})$, then $\xi \in \mathcal{D}(\Delta^z)$ for $0 \le \operatorname{Re} z \le 1/2$, so that the map

$$f \colon \overline{\mathcal{S}_\pi} \to \mathcal{H}, \quad f(z) := \Delta^{-\frac{iz}{2\pi}}\xi$$

is defined. Let $P$ denote the spectral measure of the selfadjoint operator

$$H := -\frac{1}{2\pi}\log\Delta \quad \text{and let} \quad P^\xi = \langle \xi, P(\cdot)\xi\rangle$$

denote the corresponding positive measure on $\mathbb{R}$ defined by $\xi \in \mathcal{H}$. Then [NÓ18, Lemma A.2.5] shows that

$$\mathcal{L}(P^\xi)(2\pi) = \int_{\mathbb{R}} e^{-2\pi\lambda}\, dP^\xi(\lambda) = \|e^{-\pi H}\xi\|^2 = \|\Delta^{1/2}\xi\|^2 < \infty.$$

This implies that the kernel

$$\begin{aligned}\langle f(w), f(z)\rangle &= \langle \Delta^{-\frac{iw}{2\pi}}\xi, \Delta^{-\frac{iz}{2\pi}}\xi\rangle = \langle \xi, \Delta^{-\frac{i(z-\overline{w})}{2\pi}}\xi\rangle \\ &= \langle \xi, e^{(z-\overline{w})iH}\xi\rangle = \mathcal{L}(P^\xi)\Big(\frac{z-\overline{w}}{i}\Big)\end{aligned}$$

is continuous on $\overline{\mathcal{S}_\pi} \times \overline{\mathcal{S}_\pi}$ by the Dominated Convergence Theorem, holomorphic in $z$, and antiholomorphic in $w$ on the interior ([Ne00, Prop. V.4.6]). This implies (c) because it shows that $f$ is holomorphic on $\mathcal{S}_\pi$ ([Ne00, Lemma A.III.1]) and continuous on $\overline{\mathcal{S}_\pi}$ (Exercise 1.46).
(c) $\Rightarrow$ (d): For $\alpha^\xi \colon \overline{\mathcal{S}_\pi} \to \mathcal{H}$ as in (c), we have

$$J\alpha^{\xi}(z) = \alpha^{\xi}(\pi i + \overline{z}) \tag{32}$$

by analytic continuation, so that

$$\eta := \alpha^{\xi}(\pi i/2) \in \mathcal{H}^J \quad \text{with} \quad \alpha^{\eta}(z) = \alpha^{\xi}\Big(z + \frac{\pi i}{2}\Big).$$

(d) ⇒ (b): We abbreviate $\mathcal{S} := \mathcal{S}_{\pm\pi/2}$. The kernel $K(z,w) := \langle \alpha^{\eta}(w), \alpha^{\eta}(z)\rangle$ is continuous on $\overline{\mathcal{S}} \times \overline{\mathcal{S}}$ and holomorphic in $z$ and antiholomorphic in $w$ on $\mathcal{S}$. It also satisfies $K(z+t,w) = K(z, w-t)$ for $t \in \mathbb{R}$. Hence there exists a continuous function $\varphi$ on $\overline{\mathcal{S}}$, holomorphic on $\mathcal{S}$, such that

$$K(z,w) = \varphi\Big(\frac{z - \overline{w}}{2}\Big).$$

For $t \in \mathbb{R}$, we then have $\varphi(t) = \langle \eta, \alpha^{\eta}(2t)\rangle = \int_{\mathbb{R}} e^{2it\lambda}\, dP^{\eta}(\lambda)$, so that [NÓ18, Lemma A.2.5] yields $\mathcal{L}(P^{\eta})(\pm\pi) < \infty$ and $\eta \in \mathcal{D}(\Delta^{\pm 1/4})$. This implies that $\alpha^{\eta}(z) = \Delta^{-iz/2\pi}\eta$ for $z \in \overline{\mathcal{S}}$.

From $\xi = \alpha^{\eta}(-\pi i/2) = \Delta^{-1/4}\eta$ we derive that

$$\alpha^{\xi}(z) = \alpha^{\eta}\Big(z - \frac{\pi i}{2}\Big) = \Delta^{-iz/2\pi}\xi \quad \text{for} \quad z \in \overline{\mathcal{S}_{\pi}}.$$

Further, $J\eta = \eta$ implies

$$J\alpha^{\xi}(z) = J\alpha^{\eta}\Big(z - \frac{\pi i}{2}\Big) = \alpha^{\eta}\Big(\overline{z} + \frac{\pi i}{2}\Big) = \alpha^{\xi}(\pi i + \overline{z}).$$

For $z = 0$, we obtain in particular $J\xi = \alpha^{\xi}(\pi i) = \Delta^{1/2}\xi$. □

In [BN24] we study for an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ the space $\mathcal{H}^{-\infty}_{\mathrm{KMS}} := (\mathcal{H}^{-\infty})_{\mathrm{KMS}}$ of KMS distribution vectors (see Appendix 7.5 for details), on which we have the one-parameter group $U^{-\infty}(\exp th)$ generated by the Euler element $h$ and the action of the conjugation $J^{-\infty} = U^{-\infty}(\tau_h)$.

Our general results imply in particular the following theorem, which is a key tool to verify that nets of real subspaces satisfy the Bisognano–Wichmann property $\mathsf{H}(W) \subseteq \mathtt{V}$.

**Theorem 1.43** *Let $(U, \mathcal{H})$ be an antiunitary representation of $G_{\tau_h}$ and $\mathtt{V} \subseteq \mathcal{H}$ the standard subspace specified by $\Delta_{\mathtt{V}} = e^{2\pi i \cdot \partial U(h)}$ and $J_{\mathtt{V}} = U(\tau_h)$. Then the following assertions hold:*

(a) $\mathcal{H}^{-\infty}_{\mathrm{KMS}} := (\mathcal{H}^{-\infty})_{\mathrm{KMS}}$ *is a weak-∗ closed subspace of* $\mathcal{H}^{-\infty}$.
(b) $\mathcal{H}^{-\infty}_{\mathrm{KMS}} \cap \mathcal{H} = \mathtt{V}$.
(c) $\mathtt{V}$ *is weak-∗ dense in* $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$.

***Proof*** (a)-(c) follow from [BN24, Thm. 6.2], [BN24, Thm. 6.4], and [BN24, Thm. 6.5], respectively. □

### 1.8.4 Boundary values for one-parameter groups

In this section, we collect some useful facts on boundary values of analytically extended orbit maps of unitary one-parameter groups $(U_t)_{t\in\mathbb{R}}$ and a conjugation $J$, commuting with $U$. The main point is to identify the subspace $\mathcal{H}^{-\infty}_{\rm KMS}$ of distribution vectors, satisfying the KMS condition (cf. Definition 1.41) with elements of the real subspace $\mathcal{H}^J_{\rm temp}$, specified in terms of the spectral measure $P$ of $U$.

Let $P$ be the uniquely determined spectral measure on $\mathbb{R}$ for which

$$U_t = \int_{\mathbb{R}} e^{itx}\, dP(x), \quad \text{resp.} \quad U_t = e^{itA},\ t \in \mathbb{R}, \quad \text{with} \quad A = \int_{\mathbb{R}} p\, dP(p).$$

For $\xi \in \mathcal{H}$, we thus obtain finite positive measures $P^\xi := \langle \xi, P(\cdot)\xi\rangle$, and we define

$$\mathcal{H}^J_{\rm temp} := \{\xi \in \mathcal{H}^J : e^{\pi p}\, dP^\xi(p) \ \text{ temp.}\} = \{\xi \in \mathcal{H}^J : e^{-\pi p}\, dP^\xi(p) \ \text{ temp.}\}. \tag{33}$$

The equality of both spaces on the right follows from the symmetry of the measures $P^\xi$, which is a consequence of $J\xi = \xi$. For the positive selfadjoint operator $\Delta := e^{-2\pi A}$, we have $J\Delta J = \Delta^{-1}$, so that $J\mathcal{D}(\Delta^{1/4}) = \mathcal{D}(\Delta^{-1/4})$ implies that

$$\begin{aligned}\mathcal{D}(\Delta^{1/4}) \cap \mathcal{H}^J = \mathcal{D}(\Delta^{-1/4}) \cap \mathcal{H}^J &= \Big\{\xi \in \mathcal{H}^J : \int_{\mathbb{R}} e^{\pm\pi p}\, dP^\xi(p) < \infty\Big\} \\ &\subseteq \mathcal{H}^J_{\rm temp}.\end{aligned} \tag{34}$$

**Theorem 1.44** ([FNÓ25b, Thm. 6.1]) *For $\xi \in \mathcal{H}^J \cap \bigcap_{|t|<\pi/2} \mathcal{D}(e^{tA})$, the following are equivalent:*

(a) $\xi \in \mathcal{H}^J_{\rm temp}$.
(b) *The limits $\beta^\pm(\xi) := \lim_{t\to\pm\pi/2} e^{-tA}\xi$ exist in $\mathcal{H}^{-\infty}(U)$.*
(c) *There exist $C, N > 0$ such that $\|e^{\pm tA}\xi\|^2 \le C\big(\frac{\pi}{2} - |t|\big)^{-N}$ for $|t| < \pi/2$.*

***Proof*** (a) $\Leftrightarrow$ (b): From [FNÓ25a, Prop. 4], we recall that the temperedness of the measure $\nu_\xi$, given by $d\nu_\xi(p) := e^{\pi p}\, dP^\xi(p)$, is equivalent to the existence of $C, N > 0$ with

$$\int_{\mathbb{R}} e^{(\pi - t)p}\, dP^\xi(p) \le C t^{-N} \quad \text{for} \quad 0 \le t < \pi.$$

Further, [NÓ15, Lemma 10.7] shows that this condition is equivalent to the function $e^{\pi p/2}$ to define a distribution vector for the canonical multiplication representation on $L^2(\mathbb{R}, P^\xi)$. This representation is equivalent to the subrepresentation of $(U, \mathcal{H})$, generated by $\xi$, where the constant function 1 corresponds to $\xi$.

(b) $\Rightarrow$ (c): If $\lim_{t\to\pi/2} e^{tA}\xi$ exist in $\mathcal{H}^{-\infty}(U)$, then [NÓ15, Lemma 10.7], applied to the cyclic subrepresentation generated by $\xi$, implies that the measure $\nu_\xi$ is tempered. Then the argument from above implies the existence of $C, N > 0$ with

$$\|e^{tA}\xi\|^2 = \int_{\mathbb{R}} e^{2tx}\, dP^\xi(x) \le C\Big(\frac{\pi}{2} - |t|\Big)^{-N} \quad \text{for} \quad |t| < \pi/2. \tag{35}$$

If $\lim_{t\to-\pi/2} e^{tA}\xi$ also exists in $\mathcal{H}^{-\infty}(U)$, then the same argument applies again and we obtain (c).
(c) $\Rightarrow$ (a): With the leftmost equality in (35), we see that (c) implies that the measures $d\nu_\xi(x) := e^{\pm\pi x}\, dP^\xi(x)$ are tempered ([FNÓ25a, Prop. 4]). Here we use that the measure $P^\xi$ is symmetric because $J\xi = \xi$. $\square$

**Proposition 1.45** *The map $\beta^+$ defines a bijection $\beta^+ \colon \mathcal{H}^J_{\rm temp} \to \mathcal{H}^{-\infty}_{\rm KMS}$.*

***Proof*** (a) Let $\xi \in \mathcal{H}^J_{\rm temp}$. First we show that $\beta^+(\xi) \in \mathcal{H}^{-\infty}_{\rm KMS}$. To this end, note that, for a real-valued test function $\varphi \in C^\infty_c(\mathbb{R},\mathbb{R})$, we have $JU(\varphi) = U(\varphi)J$. For $\xi \in \mathcal{H}^J_{\rm temp}$ we therefore have $\eta := U(\varphi)\xi \in \mathcal{H}^J_{\rm temp}$.

For $\xi \in \mathcal{H}$ we also note that the spectral integral $U_t = \int_{\mathbb{R}} e^{itp}\, dP(p)$ representation yields with

$$\widehat{\varphi}(x) = \int_{\mathbb{R}} e^{itx}\varphi(t)\, dt$$

the relation

$$\begin{aligned} U(\varphi) &= \int_{\mathbb{R}} \varphi(t) U_t\, dt = \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{R}} e^{itp}\, dP(p)\, dt = \int_{\mathbb{R}} \varphi(t) \int_{\mathbb{R}} e^{itp}\, dt\; dP(p) \\ &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \varphi(t) e^{itp}\, dt\Big)\, dP(p) = \int_{\mathbb{R}} \widehat{\varphi}(p)\, dP(p). \end{aligned}$$

Therefore

$$dP^\eta(x) = |\widehat{\varphi}(x)|^2 dP^\xi(x),$$

where $\widehat{\varphi}$ is a Schwartz function. The temperedness of $P^\xi$ hence implies that the measure

$$e^{\pi x}\, dP^\eta(x) = e^{\pi x} |\widehat{\varphi}(x)|^2\, dP^\xi(x)$$

is finite, and thus $\eta \in \mathcal{D}(\Delta^{1/4}) \cap \mathcal{H}^J$. This implies with Proposition 1.42 that

$$U^{-\infty}(\varphi)\beta^+(\xi) = \beta^+(U(\varphi)\xi) = \beta^+(\eta) = \Delta^{1/4}\eta \in \mathtt{V}. \tag{36}$$

Next we claim that

$$\mathcal{H}^{-\infty}_{\rm KMS} = \{\alpha \in \mathcal{H}^{-\infty} \colon (\forall \varphi \in C^\infty_c(\mathbb{R},\mathbb{R}))\, U^{-\infty}(\varphi)\alpha \in \mathtt{V}\}. \tag{37}$$

In fact, if $\alpha \in \mathcal{H}^{-\infty}$ satisfies $U^{-\infty}(\varphi)\alpha \in \mathtt{V}$ for all $\varphi \in C^\infty_c(\mathbb{R},\mathbb{R})$, we apply this to a $\delta$-sequence $\varphi = \delta_n$ (cf. Definition 7.23) to obtain

$$\alpha = \lim_{n\to\infty} U^{-\infty}(\delta_n)\alpha \in \overline{\mathtt{V}}^{w-*} = \mathcal{H}^{-\infty}_{\mathrm{KMS}}$$

by the weak-$*$ closedness of $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ (Theorem 1.44). Conversely, $\alpha \in \mathcal{H}^{-\infty}_{\mathrm{KMS}}$ implies

$$U^{-\infty}(\varphi)\alpha \in \mathcal{H} \cap \mathcal{H}^{-\infty}_{\mathrm{KMS}} = \mathtt{V},$$

again by the closedness and $U(\mathbb{R})$-invariance of $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$. With (36) we thus obtain that $\beta^+(\xi) \in \mathcal{H}^{-\infty}_{\mathrm{KMS}}$.
(b) To see that $\beta^+$ is injective, we assume that $\beta^+(\xi) = 0$. Then the above argument implies that $U(\varphi)\xi \in \mathcal{H}^J \cap \mathcal{D}(\Delta^{1/4})$ vanishes for every $\varphi \in C^\infty_c(\mathbb{R},\mathbb{R})$ because $\Delta^{1/4}$ is injective. Using an approximate identity in this space, $\xi = 0$ follows.
(c) To see that $\beta^+$ is surjective, let $\gamma \in \mathcal{H}^{-\infty}_{\mathrm{KMS}}$. Replacing $\mathcal{H}$ by the cyclic subrepresentation generated by $\gamma$, resp., the subspace $U^{-\infty}(C^\infty_c(\mathbb{R},\mathbb{C}))\gamma \subseteq \mathcal{H}$, we may w.l.o.g. assume that $\mathcal{H} = L^2(\mathbb{R},\nu)$ for a positive Borel measure, where the constant function 1 corresponds to $\gamma$. Hence the measure $\nu$ on $\mathbb{R}$ is tempered ([NÓ15, Lemma 10.7]). Then, for $z = x + iy \in \mathcal{S}_\pi$, the analytic continuation of the orbit map of $\gamma = 1$ takes the form

$$U^\gamma \colon \overline{\mathcal{S}_\pi} \to L^2(\mathbb{R},\nu)^{-\infty}, \quad U^\gamma(z)(p) = e^{izp} = e^{ixp}e^{-yp}.$$

Therefore all measures $e^{-yp}\, d\nu(p), 0 \leq y \leq \pi$, are tempered. It follows in particular that they are actually finite for $0 < y < p$. Hence $\xi(p) := e^{-\pi p/2}$ is an $L^2$-function, and $\xi = U^\gamma(\pi i/2)$ implies that $J\xi = \xi$. As a consequence, the measure $dP^\xi(p) = e^{-\pi p}\, d\nu(p)$ is finite and $e^{\pi p}\, dP^\xi(p) = d\nu(p)$ is tempered, so that $\xi \in \mathcal{H}^J_{\mathrm{temp}}$. Therefore $\beta^+(\xi) = 1$ shows that $\beta^+$ is surjective. □

For $\xi, \eta \in \mathcal{H}^J_{\mathrm{temp}}$, we consider the complex-valued measure

$$P^{\xi,\eta}(E) := \langle \xi, P(E)\eta\rangle, \quad E \subseteq \mathbb{R}.$$

Then

$$\overline{P^{\xi,\eta}(E)} = \overline{\langle \xi, P(E)\eta\rangle} = \langle \eta, P(E)\xi\rangle = P^{\eta,\xi}(E) \tag{38}$$

and the relation $JP(E)J = P(-E)$ implies that

$$\begin{aligned} P^{\xi,\eta}(E) &= \langle J\xi, P(E)J\eta\rangle = \langle J\xi, JP(-E)\eta\rangle \\ &= \langle P(-E)\eta, \xi\rangle = P^{\eta,\xi}(-E) = \overline{P^{\xi,\eta}(-E)}. \end{aligned} \tag{39}$$

In particular, the measures $P^{\xi,\xi}$ are symmetric and positive.

We obtain on the strip $\mathcal{S}_{\pm\pi}$ the holomorphic function

$$\varphi^{\xi,\eta}(z) := \widehat{P^{\xi,\eta}}(z) = \int_{\mathbb{R}} e^{izp}\, dP^{\xi,\eta}(p),$$

and the temperedness of the measures $e^{\pm\pi p}\, dP^{\xi,\eta}(p)$ implies that this function has boundary values that are tempered distributions on $\pm\pi i + \mathbb{R}$. For $t \in \mathbb{R}$, we have $\varphi^{\xi,\eta}(t) = \langle \xi, U_t \eta \rangle$. Hence

$$\varphi^{\eta,\xi}(-t) = \overline{\varphi^{\xi,\eta}(t)} = \langle U_t\eta, \xi\rangle = \langle U_t J\eta, J\xi\rangle = \langle JU_t\eta, J\xi\rangle = \langle \xi, U_t\eta\rangle = \varphi^{\xi,\eta}(t),$$

and therefore

$$\overline{\varphi^{\xi,\eta}(z)} = \varphi^{\eta,\xi}(-\overline{z}) = \varphi^{\xi,\eta}(\overline{z}) \quad \text{for} \quad z \in \mathcal{S}_{\pm\pi}. \tag{40}$$

For $\alpha := \beta^+(\xi)$ and $\gamma := \beta^+(\eta)$ the distribution

$$D_{\alpha,\gamma}(\xi) := \gamma(U^{-\infty}(\xi)\alpha)$$

can be represented by the boundary values of a holomorphic function

$$\begin{aligned} D_{\alpha,\gamma}(x) &= \lim_{t\to\pi/2} \langle U_x e^{tA}\xi, e^{tA}\eta\rangle = \lim_{t\to\pi/2} \int_{\mathbb{R}} e^{(2t-ix)p}\, dP^{\xi,\eta}(p) \\ &= \varphi^{\xi,\eta}(-\pi i - x) = \varphi^{\eta,\xi}(\pi i + x). \end{aligned}$$

## 1.9 Exercises for Section 1

**Exercise 1.46** Let $X$ be a topological space, $\mathcal{H}$ be a Hilbert space and $\gamma\colon X \to \mathcal{H}$ be a map. Show that $\gamma$ is continuous if and only if the corresponding kernel function

$$K\colon X \times X \to \mathbb{C}, \quad K(x,y) := \langle \gamma(x), \gamma(y)\rangle$$

is continuous.

**Exercise 1.47** Let $(U_t = e^{itA})_{t\in\mathbb{R}}$ be a unitary one-parameter group on the complex Hilbert space $\mathcal{H}$ and consider on the complex Hilbert space $\widetilde{\mathcal{H}} := \mathcal{H} \oplus \overline{\mathcal{H}}$ the unitary one-parameter group

$$\widetilde{U}_t := U_t \oplus U_t.$$

Show that the flip involution $\widetilde{J}(v,w) := (w,v)$ and the positive operator $\widetilde{\Delta} := e^A \oplus e^{-A}$ form a modular pair of a standard subspace $\mathtt{V} \subseteq \widetilde{\mathcal{H}}$ (cf. Proposition 1.12).

**Exercise 1.48** If $\mathtt{V} \subseteq \mathcal{H}$ is a standard subspace, we consider the antiunitary representation of $\mathbb{R}^\times$, defined by

$$\gamma_{\mathtt{V}}(e^t) := \Delta_{\mathtt{V}}^{it}, \quad \gamma_{\mathtt{V}}(-1) := J_{\mathtt{V}}.$$

Show that we thus obtain a bijection between the set $\mathrm{Stand}(\mathcal{H})$ of standard subspaces of $\mathcal{H}$ and the set of antiunitary (strongly continuous) representations $\gamma\colon \mathbb{R}^\times \to \mathrm{AU}(\mathcal{H})$.

**Exercise 1.49** (Holomorphic extensions of orbit maps) Let $\mathcal{H}$ be a Hilbert space, let $A$ be a selfadjoint operator and let $U_t = e^{itA}$ be the corresponding one-parameter group.

(a) Show that the orbit map $U^v : \mathbb{R} \to \mathcal{H},\ t \mapsto U_t v$ extends to a holomorphic map

$$\mathcal{S}_{\pm\beta} := \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < \beta\} \to \mathcal{H}$$

if and only if $v \in \mathcal{D}(e^{tA})$, for all $|t| < \beta$. *Hint:* Consider the kernel

$$K^{v,v} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}, \quad (s,t) \mapsto \langle U^v(s), U^v(t)\rangle.$$

(b) Suppose $v \in \mathcal{D}(e^{tA})$, for all $|t| < \beta$. Show that $v \in \mathcal{D}(e^{\pm\beta A})$ if and only if the map $U^v : \mathcal{S}_{\pm\beta} \to \mathcal{H}$ has continuous boundary values on $\mathbb{R} \pm i\beta$.

**Exercise 1.50** Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra. For two unit vectors $\Omega_1, \Omega_2 \in \mathcal{H}$, the states $\omega_{\Omega_1}$ and $\omega_{\Omega_2}$ coincide if and only if there exists an $\mathcal{M}$-equivariant isometry

$$U \colon \overline{\mathcal{M}\Omega_1} \to \overline{\mathcal{M}\Omega_2} \quad \text{with} \quad U\Omega_1 = \Omega_2.$$

Conclude further that, if $\mathcal{M} \neq B(\mathcal{H})$, then there exist linearly independent unit vectors $\Omega_1$ and $\Omega_2$, defining the same state on $\mathcal{M}$. Hint: $\mathcal{M} \neq B(\mathcal{H})$ is equivalent to $\mathcal{M}'$ being non-trivial.

**Exercise 1.51** (The Brunetti–Guido–Longo (BGL) construction, [BGL02]) Let $G$ be a Lie group, $\sigma \in \mathrm{Aut}(G)$ be an involution and $G_\sigma := G \rtimes \{\mathbf{1}, \sigma\}$ the corresponding semidirect product. We consider an antiunitary representation $U \colon G_\sigma \to \mathrm{AU}(\mathcal{H})$, i.e., $U(G) \subseteq \mathrm{U}(\mathcal{H})$ and $U(\sigma)$ antilinear.

We consider the set

$$\mathcal{G}(G_\sigma) := \{(x,\tau) \in \mathfrak{g} \times G\sigma \;:\; \mathrm{Ad}(\tau)x = x, \tau^2 = e\}.$$

Show that:

(a) Each $(x,\tau)$ defines a morphism

$$\gamma \colon \mathbb{R}^\times \to G_\sigma, \quad \gamma(e^t) := \exp(tx), \quad \gamma(-1) := \tau.$$

(b) For each pair $(x,\tau)$ there exists a unique standard subspace $\mathtt{V} \subseteq \mathcal{H}$ with

$$J_{\mathtt{V}} = U(\tau) \quad \text{and} \quad \Delta_{\mathtt{V}} = e^{2\pi i \cdot \partial U(x)}.$$

**Exercise 1.52** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert space, $X$ be a set and $\gamma_j \colon X \to \mathcal{H}_j$, $j = 1,2$, be maps with total range. Then the following are equivalent:

(a) There exists a unitary operator $U \colon \mathcal{H}_1 \to \mathcal{H}_2$ with $U \circ \gamma_1 = \gamma_2$.
(b) $\langle \gamma_2(x), \gamma_2(y)\rangle = \langle \gamma_1(x), \gamma_1(y)\rangle$ for all $x, y \in X$.

**Exercise 1.53** (Polarization) Let $V$ and $W$ be $\mathbb{K}$-vector spaces, $\beta \colon V^n \to W$ be a symmetric $n$-linear map and $\gamma(v) := \beta(v, \cdots, v)$. Show that $\beta$ is completely determined by the values on the diagonal $\beta(v, \ldots, v)$, $v \in V$.
Hint: Consider

$$\gamma(t_1v_1 + \ldots + t_nv_n) = \sum_{m_1+\ldots+m_n=n} \frac{n!}{m_1! \cdots m_n!} t_1^{m_1} \cdots t_n^{m_n} \beta(v_1^{m_1}, \ldots, v_n^{m_n})$$

and recover $\beta(v_1, \ldots, v_n)$ as a suitable partial derivative. Alternatively, one can verify the following explicit formula:

$$\beta(v_1, \ldots, v_n) = \frac{1}{n!\,2^n} \sum_{\varepsilon_1,\ldots,\varepsilon_n \in \{1,-1\}} \varepsilon_1 \cdots \varepsilon_n \, \gamma(\varepsilon_1 v_1 + \cdots + \varepsilon_n v_n). \qquad (41)$$

**Exercise 1.54** Let $V$ be $\mathbb{K}$-vector space and $S^n(V) := (V^{\otimes n})^{S_n}$ be the $n$th symmetric power of $V$. Show that

$$S^n(V) = \mathrm{span}\{v^{\otimes n} : v \in V\}.$$

Hint: Use the same technique as in Exercise 1.53.

**Exercise 1.55** (Abstract uncertainty principle) Let $A$ and $B$ be bounded selfadjoint operator on $\mathcal{H}$ and $\Omega \in \mathcal{H}$ a unit vector. Then $\Omega$ defines a state whose expectation values for the observable $A$ is given by

$$c_A := \omega_\Omega(A) = \langle \Omega, A\Omega \rangle.$$

The variance of the observable $A$ in the state $\omega_\Omega$ is given by the expectation value

$$\sigma_A := \omega_\Omega((A - c_A\mathbf{1})^2)^{1/2} = \|(A - c_A\mathbf{1})\Omega\|.$$

It vanishes if and only if $A\Omega = c_A\Omega$, i.e., if $\Omega$ is an eigenvector of $A$.

Verify the abstract uncertainty principle ([Ro29]):

$$\sigma_A\sigma_B \geq \frac{1}{2}|\langle \Omega, [A,B]\Omega \rangle|. \tag{42}$$

**Exercise 1.56** Let $A\colon \mathcal{D}(A) \to \mathcal{H}$ and $B\colon \mathcal{D}(B) \to \mathcal{H}$ be densely defined unbounded operators on the real Hilbert space $\mathcal{H}$, so that their adjoints

$$A^*: \mathcal{D}(A^*) \to \mathcal{H}, \quad B^*: \mathcal{D}(B^*) \to \mathcal{H}$$

are also defined by

$$\langle A^*v, w \rangle = \langle v, Aw \rangle \quad \text{for} \quad w \in \mathcal{D}(A), v \in \mathcal{D}(A^*).$$

The product $AB$ is defined on $\mathcal{D}(AB) = B^{-1}\mathcal{D}(A) \subseteq \mathcal{D}(B)$ by composition. Show that:

(a) If $\mathcal{D}(AB)$ is dense, then $(AB)^*$ is an extension of $B^*A^*$.
(b) If $A$ is invertible, then $(AB)^* = B^*A^*$.

**Exercise 1.57** Let $\mathtt{V} \subset \mathcal{H}$ be a standard subspace and $U \in \mathrm{AU}(\mathcal{H})$ be a unitary or an antiunitary operator. Show that $U\mathtt{V}$ is also standard and $U\Delta_{\mathtt{V}}U^{-1} = \Delta_{U\mathtt{V}}$ and $UJ_{\mathtt{V}}U^{-1} = J_{U\mathtt{V}}$.

# 2 Euler elements

We have seen in Section 1 how nets of real subspaces arise from nets of algebras of local observables. In this section we turn to the underlying Lie algebraic structures. In the list of questions from the introduction, we shall answer question (Q2), asking which elements $h \in \mathfrak{g}$ appear in the Bisognano–Wichmann (BW) condition. This question has a surprisingly simple answer: $h$ has to be central or an Euler element, i.e., $\mathrm{ad}\, h$ is diagonalizable with $\mathrm{Spec}(\mathrm{ad}\, h) \subseteq \{-1, 0, 1\}$. In the physical context of the Lorentz and Poincaré group, these are suitably normalized generators of Lorentz boosts. This is a consequence of the Euler Element Theorem 2.3 that we discuss in Section 2.1. The rest of this section is devoted to the exploration of the set $\mathcal{E}(\mathfrak{g})$ of Euler elements in a finite-dimensional Lie algebra $\mathfrak{g}$. We present some key examples in Section 2.2, before we turn in Section 2.3 to the classification of (adjoint orbits of) Euler elements in simple Lie algebras. Their classification in non-simple Lie algebras can, to a large extent, be reduced to the simple case (Section 2.4).

Many phenomena concerning Euler elements alreay appear in the 3-dimensional Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, so that it is relevant to know when an Euler element is contained in a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{gl}_2(\mathbb{R})$, which is characterized in Section 2.5.

We conclude this section by a discussion of an abstract Lie theoretic construction that reflects some key features of modular pairs $(\Delta, J)$: the space $\mathcal{G}_E$ of Euler couples of a Lie group of the form $G_\sigma = G \rtimes \{e, \sigma\}$. This provides the context for a general construction, due to Brunett–Guido–Longo (BGL), that attaches to every antiunitary representation of $G_\tau$ a $G$-covariant family of standard subspaces (cf. [BGL02, MN21]).

## 2.1 The Euler Element Theorem

**Definition 2.1** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. We call $h \in \mathfrak{g}$ an *Euler element* if $\operatorname{ad} h$ is **non-zero** and diagonalizable with $\operatorname{Spec}(\operatorname{ad} h) \subseteq \{-1, 0, 1\}$, i.e., if

$$\mathfrak{g} = \mathfrak{g}_1(h) \oplus \mathfrak{g}_0(h) \oplus \mathfrak{g}_{-1}(h).$$

Then $\tau_h := e^{\pi i \operatorname{ad} h} \in \operatorname{Aut}(\mathfrak{g})$ is an involutive automorphism of $\mathfrak{g}$. We write $\mathcal{E}(\mathfrak{g})$ for the set of Euler elements in $\mathfrak{g}$. An Euler element $h$ is called *symmetric* (with respect to $\mathfrak{g}$) if $-h \in \mathcal{O}_h := \operatorname{Inn}(\mathfrak{g})h$, where $\operatorname{Inn}(\mathfrak{g}) = \langle e^{\operatorname{ad} \mathfrak{g}} \rangle$ is the group of inner automorphisms of $\mathfrak{g}$.

*Remark 2.2* We observe that $\mathcal{E}(\mathfrak{g}) + \mathfrak{z}(\mathfrak{g}) = \mathcal{E}(\mathfrak{g})$, where $\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \colon [x, \mathfrak{g}] = \{0\}\}$ is the center of $\mathfrak{g}$.

The following theorem ([MN24, Thm. 3.1]) provides a very satisfying answer to question (Q2).

**Theorem 2.3** (Euler Element Theorem) *Let $G$ be a connected finite-dimensional Lie group with Lie algebra $\mathfrak{g}$ and $h \in \mathfrak{g}$. Let $(U, \mathcal{H})$ be a unitary representation of $G$ with discrete kernel. Suppose that $\mathtt{V} \subseteq \mathcal{H}$ is a standard subspace and $N \subseteq G$ an identity neighborhood such that*

(a) $U(\exp(th)) = \Delta_{\mathtt{V}}^{-it/2\pi}$ *for* $t \in \mathbb{R}$, *i.e.,* $\Delta_{\mathtt{V}} = e^{2\pi i \, \partial U(h)}$, *and*
(b) $\mathtt{V}_N := \bigcap_{g \in N} U(g)\mathtt{V}$ *is cyclic.*

*Then $h$ is an Euler element or central, and the conjugation $J_{\mathtt{V}}$ satisfies*

$$J_{\mathtt{V}} U(\exp x) J_{\mathtt{V}} = U(\exp \tau_h(x)) \quad \text{for} \quad \tau_h = e^{\pi i \operatorname{ad} h}, x \in \mathfrak{g}. \tag{43}$$

**Corollary 2.4** *If* $\mathsf{H}(\mathcal{O})_{\mathcal{O}\subseteq M}$ *is a net of real subspaces on open subsets of* $M$ *satisfying* (Iso), (Cov), (RS) *and* (BW), *and* $U$ *has discrete kernel, then* $h \in \mathfrak{g}$ *is an Euler element or central.*

***Proof*** Let $\mathcal{O} \subseteq W$ be a non-empty open, relatively compact subset. Then $\overline{\mathcal{O}}$ is a compact subset of the open set $W$, so that

$$N := \{g \in G \colon g^{-1}.\overline{\mathcal{O}} \subseteq W\}$$

is an open $e$-neighborhood in $G$. For every $g \in N$ we have by (Cov) and (Iso),

$$g^{-1}.\mathsf{H}(\mathcal{O}) = \mathsf{H}(g^{-1}.\mathcal{O}) \subseteq \mathsf{H}(W) \overset{\text{(BW)}}{=} \mathtt{V}.$$

This implies that $\mathsf{H}(\mathcal{O}) \subseteq \mathtt{V}_N$, and (RS) entails that $\mathsf{H}(\mathcal{O})$ is cyclic. Now the assertion follows from Theorem 2.3. □

*Remark 2.5* (a) The relation (43) implies that, for the representations we are dealing with, we may replace $G$ by its simply connected covering group $\widetilde{G}$ or by the quotient group $G/\ker(U)$ to ensure that the involution $\tau_h^{\mathfrak{g}} = e^{\pi i \operatorname{ad} h}$ on $\mathfrak{g}$ integrates to an involution $\tau_h$ on $G$, so that we can form the semidirect product

$$G_{\tau_h} = G \rtimes \{\mathrm{id}_G, \tau_h\}.$$

Then (43) ensures that $U$ extends to an antiunitary representation of $G_{\tau_h}$ by $U(\tau_h) := J$.
(b) If $\mathtt{V}_N = \mathtt{V}$ holds in the Euler Element Theorem, then $U(g)\mathtt{V} \supseteq \mathtt{V}$ for all $g \in N$, hence $U(g)\mathtt{V} = \mathtt{V}$ for all $g \in N \cap N^{-1}$. If $G$ is connected, this implies that $U(G)$ fixes $\mathtt{V}$ and hence that $h$ is central in $\mathfrak{g}$.
(c) Suppose that $h$ is central in $\mathfrak{g}$ and that $\Delta_{\mathtt{V}} = e^{2\pi i \cdot \partial U(h)}$. Then all standard subspaces $U_g\mathtt{V}$ have the same modular group. If there exists an $e$-neighborhood $N \subseteq G$ for which $\mathtt{V}_N$ is cyclic, then we may assume that $N = N^{-1}$, and this subspace is invariant under the modular group $\Delta_{\mathtt{V}}^{i\mathbb{R}} = U(\exp(\mathbb{R}h))$. The Equality Lemma 1.9 thus shows that $\mathtt{V}_N = \mathtt{V}$. Hence $U_g\mathtt{V} \supseteq \mathtt{V}$ for all $g \in N$, and the symmetry of $N$ now implies that $U_g\mathtt{V} = \mathtt{V}$, so that $U_g$ commutes with $J_{\mathtt{V}}$. If $G$ is connected, hence generated by $N$, it follows that $J_{\mathtt{V}} \in U(G)'$.

**Problem 2.6** In view of the preceding discussion, the following question is fundamental: Suppose that $h \in \mathfrak{g}$ is an Euler element, $G$ is a corresponding connected Lie group, for which $G_{\tau_h}$ exists, and $M = G/H$ a homogeneous space. When does there exist an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$, a connected open subset $W \subseteq M$ and a net $\mathsf{H}(\mathcal{O})_{\mathcal{O}\subseteq M}$ on open subsets of $M$, satisfying (Iso), (Cov), (RS) and (BW)?

Below we shall see that this is the case if $G$ is reductive and $M$ is the non-compactly causal symmetric space associated to $G$ and $h$ (cf. Theorem 4.31). If $G$ is solvable, the corresponding question is open (cf. [BN25]). In this context it is also interesting to consider Theorem 5.19 that connects the existence of nets to a regularity condition.

### 2.2 First examples of Euler elements

Before we descend deeper into structures related to Euler elements, let us discuss some key examples.

**Example 2.7** If $E$ is a finite-dimensional vector space and $0 \neq D \in \mathrm{End}(E)$ a diagonal endomorphism with eigenvalues contained in $\{1, 0, -1\}$, then we form the solvable Lie algebra $\mathfrak{g} := E \rtimes_D \mathbb{R}$. Here $h := (0, 1)$ is an Euler element of $\mathfrak{g}$.

**Examples 2.8** (a) In $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ the diagonal matrix

$$h = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{44}$$

is an Euler element. Conversely, every Euler element $h' \in \mathfrak{sl}_2(\mathbb{R})$ must be diagonalizable on $\mathbb{R}^2$ (Exercise 2.21) and the difference between its eigenvalues must be 1. In view of $\mathrm{tr}(h') = 0$, it is conjugate to $h$. The set of Euler elements in $\mathfrak{sl}_2(\mathbb{R})$ is

$$\mathcal{E}(\mathfrak{sl}_2(\mathbb{R})) = \big\{x \in \mathfrak{sl}_2(\mathbb{R})\colon \det(x) = -\tfrac{1}{4}\big\} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a^2 + bc = \frac{1}{4}\right\}$$

and $\mathrm{Inn}(\mathfrak{g}) \cong \mathrm{SO}_{1,2}(\mathbb{R})_e$ acts transitively on this set. In the following, we shall also use the Euler element

$$k = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{45}$$

The element

$$z_{\mathfrak{k}} := \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = [h, k] \quad \text{satisfies} \quad [z_{\mathfrak{k}}, h] = -k,$$

so that we have

$$e^{-\frac{\pi}{2}\operatorname{ad} z_{\mathfrak{k}}} h = -[z_{\mathfrak{k}}, h] = k \quad \text{and} \quad e^{\pm\pi \operatorname{ad} z_{\mathfrak{k}}} h = -h. \tag{46}$$

(b) If $\mathcal{A}$ is a real unital associative algebra, then $h = \frac{1}{2}\mathrm{diag}(1, -1)$ is also Euler in the Lie algebra $\mathfrak{gl}_2(\mathcal{A}) = (M_2(\mathcal{A}), [\cdot, \cdot])$, of $(2 \times 2)$-matrices with entries in $\mathcal{A}$, with respect to the commutator bracket. If $\sigma \in \mathrm{Aut}(\mathcal{A})$ is an involutive automorphism, then $\sigma$ extends by

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{pmatrix}$$

to a Lie algebra automorphism of $\mathfrak{gl}_2(\mathcal{A})$, and $\mathfrak{g} = \mathfrak{gl}_2(\mathcal{A})^\sigma$ contains the Euler element $h$ with $\mathfrak{g}_1(h) \cong \mathcal{A}^\sigma$. For the involution $\tau := \sigma\tau_h$, we also find a Lie algebra with $\mathfrak{g}_1(h) \cong \mathcal{A}^{-\sigma}$.

This construction provides a rich supply of Lie algebras with Euler elements. It even works for Jordan algebras $\mathcal{A}$ (cf. [JvNW34]), hence in particular also for alternative algebras. We refer to [KSTT19], [dG18] and [Be25] for recent classification results in small dimensions.

**Examples 2.9** (a) In the simple Lie algebra $\mathfrak{g} := \mathfrak{sl}_n(\mathbb{R})$, we write $n \times n$-matrices as block $2 \times 2$-matrices according to the partition $n = k + (n-k)$. Then

$$h_k := \frac{1}{n} \begin{pmatrix} (n-k)\mathbf{1}_k & 0 \\ 0 & -k\mathbf{1}_{n-k} \end{pmatrix} \tag{47}$$

is diagonalizable with the two eigenvalues $\frac{n-k}{n} = 1 - \frac{k}{n}$ and $-\frac{k}{n}$. Therefore $h_k$ is an Euler element (Exercise 2.21) whose 3-grading is given by

$$\mathfrak{g}_0(h) = \Big\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a \in \mathfrak{gl}_k(\mathbb{R}), d \in \mathfrak{gl}_{n-k}(\mathbb{R}), \mathrm{tr}(a) + \mathrm{tr}(d) = 0 \Big\},$$

$$\mathfrak{g}_1(h) = \begin{pmatrix} 0 & M_{k,n-k}(\mathbb{R}) \\ 0 & 0 \end{pmatrix}, \quad \mathfrak{g}_{-1}(h) \cong \begin{pmatrix} 0 & 0 \\ M_{n-k,k}(\mathbb{R}) & 0 \end{pmatrix}.$$

It is easy to see that $h_1, \ldots, h_{n-1}$ represent all conjugacy classes of Euler elements in $\mathfrak{sl}_n(\mathbb{R})$, whose restricted root system is of type $A_{n-1}$, cf. the general Classification Theorem 2.12 below.

The Euler element $h_k$ is symmetric, i.e., $-h_k \in \mathrm{Inn}(\mathfrak{g})h_k$, if and only if $n = 2k$. In fact, if $h_k$ is symmetric, then its eigenvalues have to be symmetric, which is equivalent to $n = 2k$. That this condition is sufficient follows by embedding $h_k$ into an $\mathfrak{sl}_2(\mathbb{R})$-subalgebra of block matrices with entries in $M_k(\mathbb{R})$ and using Example 2.8.

(b) In the reductive Lie algebra $\mathfrak{gl}_n(\mathbb{R})$, we infer from (a) that all conjugacy classes of Euler elements are represented by elements of the form

$$h = \lambda\mathbf{1} + h_k, \quad k = 1, \ldots, n-1.$$

Then $h$ is symmetric if and only if $\lambda = 0$ and $n = 2k$.

These elements are also Euler in the semidirect sum $\mathfrak{g} := \mathbb{R}^n \rtimes \mathfrak{gl}_n(\mathbb{R})$ if and only if $\lambda = \frac{k}{n}$ or $\lambda = \frac{k}{n} - 1$, which leads to the two Euler elements

$$h' = \begin{pmatrix} \mathbf{1}_k & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad h'' = \begin{pmatrix} 0 & 0 \\ 0 & -\mathbf{1}_{n-k} \end{pmatrix}.$$

In the first case,

$$\mathfrak{g}_1 \cong \mathbb{R}^k \oplus M_{k,n-k}(\mathbb{R}), \ \ \mathfrak{g}_0 = \mathfrak{gl}_k(\mathbb{R}) \oplus (\mathbb{R}^{n-k} \rtimes \mathfrak{gl}_{n-k}(\mathbb{R})) \quad \text{and} \quad \mathfrak{g}_{-1} = M_{n-k,k}(\mathbb{R}),$$

whereas in the second case

$$\mathfrak{g}_1 \cong M_{k,n-k}(\mathbb{R}), \ \ \mathfrak{g}_0 = (\mathbb{R}^k \rtimes \mathfrak{gl}_k(\mathbb{R})) \oplus \mathfrak{gl}_{n-k}(\mathbb{R}) \quad \text{and} \quad \mathfrak{g}_{-1} = \mathbb{R}^{n-k} \oplus M_{n-k,k}(\mathbb{R}).$$

None of these Euler elements is symmetric because $\lambda \neq 0$.

**Examples 2.10** (a) In the Poincaré Lie algebra $\mathfrak{g} = \mathbb{R}^{1,d} \rtimes \mathfrak{so}_{1,d}(\mathbb{R})$, every Euler element $h$ is conjugate to the generator $h \in \mathfrak{so}_{1,d}(\mathbb{R})$ of a Lorentz boost:

$$h.\mathbf{e}_0 = \mathbf{e}_1, \quad h.\mathbf{e}_1 = \mathbf{e}_0 \quad \text{and} \quad h.\mathbf{e}_j = 0 \quad \text{for} \quad j > 1$$

and $\mathfrak{z}(\mathfrak{g}) = \{0\}$; see also Remark 1.29. In fact, Lemma 2.15 below reduces the assertion to the simple Lie algebra $\mathfrak{so}_{1,d}(\mathbb{R})$, and in this case the assertion follows from the Classification Theorem 2.12.
(b) (cf. [BN25]) The 4-dimensional split oscillator group is

$$G := \mathrm{Heis}(\mathbb{R}^2) \rtimes_\alpha \mathbb{R} \quad \text{with} \quad \alpha_t = e^{th}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that

$$\mathfrak{g} = \mathfrak{heis}(\mathbb{R}^2) \rtimes \mathbb{R}h$$

and $h$ is an Euler element in $\mathfrak{g}$. We choose a basis $p, q, z \in \mathfrak{heis}(\mathbb{R}^2)$ with

$$[q,p] = z, \quad [h,q] = 1, \quad [h,p] = -1, \quad [h,z] = 0.$$

The corresponding involution satisfies

$$\tau_h(z,q,p,t) = (z,-q,-p,t).$$

The Euler element $h$ is not symmetric, and all Euler elements of $\mathfrak{g}$ are, up to sign, conjugate to elements of the form

$$h_\lambda = \lambda z + h.$$

This Lie algebra can be realized as a subalgebra of $\mathfrak{sl}_3(\mathbb{R})$, where

$$h = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $h$ is an Euler element of $\mathfrak{sl}_3(\mathbb{R})$, i.e., $V := \mathbb{R}^3$ is a 2-graded $\mathfrak{g}$-module (cf. Example 2.9(a)):

$$V = V_{1/3} \oplus V_{-2/3}, \quad V_{1/3} = \mathbb{R}\mathbf{e}_1 + \mathbb{R}\mathbf{e}_3, \quad V_{-2/3} = \mathbb{R}\mathbf{e}_2.$$

*Remark 2.11* (a) If $V$ is a non-trivial irreducible $\mathfrak{sl}_2(\mathbb{R})$-module and

$$h \in \mathfrak{g} := V \rtimes \mathfrak{sl}_2(\mathbb{R})$$

an Euler element, then the semisimple element $h$ is conjugate to an element of $\mathfrak{sl}_2(\mathbb{R})$ (Lemma 2.15 and $\mathfrak{z}(\mathfrak{g}) = \{0\}$), so that we may assume that $h = \frac{1}{2}\operatorname{diag}(1, -1)$ (Example 2.8(a)). This leaves only the possibility that $\dim V = 3$ is the adjoint module.

We obtain more freedom if we replace $\mathfrak{sl}_2(\mathbb{R})$ by $\mathfrak{gl}_2(\mathbb{R})$. Then Example 2.9(b) also provides Euler elements in $\mathbb{R}^2 \rtimes \mathfrak{gl}_2(\mathbb{R})$ (cf. [MN22, Lemma 2.15] or Proposition 2.18). We may also consider non-trivial 1-dimensional representations of $\mathfrak{gl}_2(\mathbb{R})$, for which

$$\mathbb{R} \rtimes \mathfrak{gl}_2(\mathbb{R}) \cong (\mathbb{R} \rtimes \mathbb{R}\mathbf{1}) \oplus \mathfrak{sl}_2(\mathbb{R}).$$

This example shows already how restrictive the existence of a 3-grading is for semidirect sums.
(b) If $h \in \mathfrak{g}$ is an Euler element contained in a subalgebra $\mathfrak{s} \cong \mathfrak{sl}_2(\mathbb{R})$, then all simple $\mathfrak{s}$-submodules of $\mathfrak{g}$ must be 1 or 3-dimensional. If $h$ is contained in a subalgebra $\mathfrak{l} \cong \mathfrak{gl}_2(\mathbb{R})$, then also 2-dimensional irreducible submodules may occur (cf. Proposition 2.18 below).

## 2.3 Euler elements in simple Lie algebras

We now present a classification of Euler elements in simple real Lie algebras, following [MN21] (see also [Mo25]). As they correspond to 3-gradings, it can also be derived from [KA88]. We also reproduce the list of the 18 types from [Kan98, p. 600] and Kaneyuki's lecture notes [Kan00].

Let $\mathfrak{g}$ be a real semisimple Lie algebra. An involutive automorphism $\theta \in \operatorname{Aut}(\mathfrak{g})$ is called a *Cartan involution* if its eigenspaces

$$\mathfrak{k} := \mathfrak{g}^\theta = \{x \in \mathfrak{g} \colon \theta(x) = x\} \quad \text{and} \quad \mathfrak{p} := \mathfrak{g}^{-\theta} = \{x \in \mathfrak{g} \colon \theta(x) = -x\}$$

have the property that they are orthogonal with respect to the Cartan–Killing form

$$\kappa(x, y) = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y),$$

which is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. This implies that

$$\kappa(x, \theta x) < 0 \quad \text{for} \quad x \neq 0. \tag{48}$$

Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \tag{49}$$

is called a *Cartan decomposition*. Cartan involutions always exist and two such involutions are conjugate under the group $\operatorname{Inn}(\mathfrak{g})$ of inner automorphism, so they produce isomorphic decompositions ([HN12, Thm. 13.2.11]).

The subalgebra $\mathfrak{k}$ is maximal *compactly embedded.* [6] An element $x \in \mathfrak{g}$ is elliptic if and only if its adjoint orbit $\mathcal{O}_x = \mathrm{Inn}(\mathfrak{g})x$ intersects $\mathfrak{k}$, and $x \in \mathfrak{g}$ is hyperbolic if and only if $\mathcal{O}_x \cap \mathfrak{p} \neq \emptyset$.

For the finer structure theory, we start with a Cartan involution $\theta$ and fix a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$. As $\mathfrak{a}$ is abelian, $\mathrm{ad}\,\mathfrak{a}$ is a commuting set of diagonalizable operators, hence simultaneously diagonalizable. For a linear functional $0 \neq \alpha \in \mathfrak{a}^*$, the simultaneous eigenspaces

$$\mathfrak{g}_\alpha := \mathfrak{g}_\alpha(\mathfrak{a}) := \{y \in \mathfrak{g} \colon (\forall x \in \mathfrak{a})\ [x,y] = \alpha(x)y\}$$

are called *root spaces* and

$$\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a}) := \{\alpha \in \mathfrak{a}^* \setminus \{0\} \colon \mathfrak{g}_\alpha \neq 0\}$$

is called the set of *restricted roots* (see also (171) in the appendix).

**From now on we assume that $\mathfrak{g}$ is simple.** Then $\Sigma$ is an irreducible root system, hence of one of the following types:

$$A_n, \quad B_n, \quad C_n, \quad D_n, \quad E_6, E_7, E_8, \quad F_4, \quad G_2 \quad \text{or} \quad BC_n, n \geq 1 \tag{50}$$

(cf. [Bou90]).

The adjoint orbit of an Euler element in $\mathfrak{g}$ contains a unique $h \in \Pi^\star$. For any Euler element $h \in \Pi^\star$, we have $\alpha(h) \in \{0,1\}$ for $\alpha \in \Pi$ because the values of the roots on $h$ are the eigenvalues of $\mathrm{ad}\,h$. If such an element exists, then the irreducible root system $\Sigma$ must be reduced. Otherwise, for any root $\alpha$ with $2\alpha \in \Sigma$, we must have $\alpha(h) = 0$ because $\mathrm{ad}\,x$ has only three eigenvalues. As the set of such roots generates the same linear space as $\Sigma$, this leads to $h = 0$. This excludes the non-reduced irreducible root systems of type $BC_n$.

To see how many possibilities we have for Euler elements in $\mathfrak{a}$, we recall that $\Pi$ is a linear basis of $\mathfrak{a}^*$, so that, for each $j \in \{1, \ldots, n\}$, there exists a uniquely determined element

$$h_j \in \mathfrak{a}, \quad \text{satisfying} \quad \alpha_k(h_j) = \delta_{kj}. \tag{51}$$

The following theorem lists for each irreducible root system $\Sigma$ the possible Euler elements in the positive chamber $\Pi^\star$. Since every adjoint orbit in $\mathcal{E}(\mathfrak{g})$ has a unique representative in $\Pi^\star$, this classifies the $\mathrm{Inn}(\mathfrak{g})$-orbits in $\mathcal{E}(\mathfrak{g})$ for any non-compact simple real Lie algebra. For **semisimple** Lie algebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, an element $x = (x_1, \ldots, x_k)$ is an Euler element if and only if its non-zero components $x_j \in \mathfrak{g}_j$ are Euler elements, and its orbit is

$$\mathcal{O}_x = \mathcal{O}_{x_1} \times \cdots \times \mathcal{O}_{x_k}.$$

[6] We call a subalgebra $\mathfrak{c} \subseteq \mathfrak{g}$ compactly embedded if the closure of the subgroup generated by $e^{\mathrm{ad}\,\mathfrak{c}}$ in $\mathrm{Aut}(\mathfrak{g})$ is compact.

Therefore it suffices to consider simple Lie algebras, and for these the root system $\Sigma$ is irreducible. As every complex simple Lie algebra $\mathfrak{g}$ is also a real simple Lie algebra, our discussion also covers complex Lie algebras.

**Theorem 2.12** *Suppose that $\mathfrak{g}$ is a non-compact simple real Lie algebra, with restricted root system $\Sigma \subseteq \mathfrak{a}^*$ of type $X_n$ as in (50). We follow the conventions of the tables in [Bou90] for the classification of irreducible root systems and the enumeration of the simple roots $\alpha_1, \ldots, \alpha_n$. Then every Euler element $h \in \mathfrak{a}$ on which $\Pi$ is non-negative is one of $h_1, \ldots, h_n$, and for every irreducible root system, the Euler elements among the $h_j$ are the following:*

$$A_n : h_1, \ldots, h_n, \qquad B_n : h_1, \qquad C_n : h_n, \qquad D_n : h_1, h_{n-1}, h_n,$$
$$E_6 : h_1, h_6, \qquad E_7 : h_7. \tag{52}$$

*For the root systems $BC_n$, $E_8$, $F_4$ and $G_2$ no Euler element exists (they have no 3-grading). The Euler elements which are symmetric in the sense that $-h \in \mathcal{O}_h = \mathrm{Inn}(\mathfrak{g})h$, are*

$$A_{2n-1} : h_n, \qquad B_n : h_1, \qquad C_n : h_n, \qquad D_{2n} : h_1, h_{2n-1}, h_{2n}, \qquad D_{2n+1} : h_1,$$
$$E_7 : h_7. \tag{53}$$

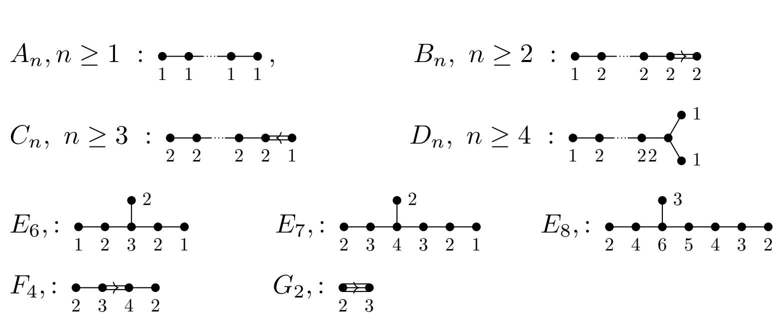



Figure 4: Dynkin diagrams with the coefficients of the highest root.

***Proof*** Writing the highest root in $\Sigma$ with respect to the simple system $\Pi$ as $\alpha_{\max} = \sum_{j=1}^n c_j \alpha_j$, we have $c_j \in \mathbb{Z}_{>0}$ for each $j$. These are the labels in the Dynkin diagrams above. If $h \in \Pi^\star$ is an Euler element, then $\Pi(h) \subseteq \{0, 1\}$, and $1 = \alpha_{\max}(h) = \sum_{j=1}^n c_j \alpha_j(h)$ implies that at most one value $\alpha_j(h)$ can be 1, and then the others are 0, i.e., $h = h_j$ for some $j \in \{1, \ldots, n\}$. Conversely, $h_j$ is an Euler element if and only if $c_j = 1$. Consulting the tables on the irreducible root systems in [Bou90], we obtain the Euler elements listed in (52).

To determine the symmetric ones, let $w_0 \in \mathcal{W}$ be the element of the Weyl group, which is uniquely determined by $w_0^* \Pi = -\Pi$ for the dual action of $\mathcal{W}$ on $\mathfrak{a}^*$. Then $h_j' := w_0(-h_j)$ is the Euler element in the positive chamber representing the orbit $\mathcal{O}_{-h_j}$. Therefore $h_j$ is symmetric if and only if $-h_j \in \mathcal{W}h_j$, which is equivalent to $h_j' = h_j$. Using the description of $w_0$ and the root systems in [Bou90], now leads to

$$A_{n-1} : h'_j = h_{n-j}, \quad B_n : h'_1 = h_1, \quad C_n : h'_n = h_n, \tag{54}$$

$$D_n : h'_1 = h_1, h'_n = \begin{cases} h_{n-1} & \text{for } n \text{ odd,} \\ h_n & \text{for } n \text{ even,} \end{cases} \tag{55}$$

$$E_6 : h'_1 = h_6, \quad E_7 : h'_7 = h_7. \tag{56}$$

Hence the symmetric Euler elements are those listed in (53). ☐

There are many types of simple 3-graded Lie algebras that are neither complex nor hermitian (cf. Proposition 2.13 below); for instance the Lorentzian algebras $\mathfrak{so}_{1,n}(\mathbb{R})$. We refer to [Kan98, p. 600] or [Kan00]. for the list of all 18 types which is reproduced below in a different order. We identify $\mathfrak{so}^*(4n)$ with the Lie algebra $\mathfrak{u}_{2r}(\mathbb{H}, \Omega)$ of the isometry group of the non-degenerate skew-hermitian form on $\mathbb{H}^{2r}$ defined by the matrix $\Omega = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$.

| | $\mathfrak{g}$ | $\Sigma(\mathfrak{g}, \mathfrak{a})$ | $h$ | $\mathfrak{g}_1(h)$ |
|---|---|---|---|---|
| | Complex Lie algebras | | | |
| 1 | $\mathfrak{sl}_n(\mathbb{C})$ | $A_{n-1}$ | $h_j, 1 \leq j \leq n-1$ | $M_{j,n-j}(\mathbb{C})$ |
| 2 | $\mathfrak{sp}_{2n}(\mathbb{C})$ | $C_n$ | $h_n$ | $\mathrm{Sym}_n(\mathbb{C})$ |
| 3a | $\mathfrak{so}_{2n+1}(\mathbb{C})$ | $B_n$ | $h_1$ | $\mathbb{C}^{2n-1}$ |
| 3b | $\mathfrak{so}_{2n}(\mathbb{C})$ | $D_n$ | $h_1$ | $\mathbb{C}^{2n-2}$ |
| 4 | $\mathfrak{so}_{2n}(\mathbb{C})$ | $D_n$ | $h_{n-1}, h_n$ | $\mathrm{Alt}_n(\mathbb{C})$ |
| 5 | $\mathfrak{e}_6(\mathbb{C})$ | $E_6$ | $h_1 = h'_6$ | $M_{1,2}(\mathbb{O})_{\mathbb{C}}$ |
| 6 | $\mathfrak{e}_7(\mathbb{C})$ | $E_7$ | $h_7$ | $\mathrm{Herm}_3(\mathbb{O})_{\mathbb{C}}$ |
| | Hermitian tube type Lie algebras | | | |
| 7 | $\mathfrak{su}_{n,n}(\mathbb{C})$ | $C_n$ | $h_n$ | $\mathrm{Herm}_n(\mathbb{C})$ |
| 8 | $\mathfrak{sp}_{2n}(\mathbb{R})$ | $C_n$ | $h_n$ | $\mathrm{Sym}_n(\mathbb{R})$ |
| 9a | $\mathfrak{so}_{2,d}(\mathbb{R})$ | $C_2$ $(2 < d)$ | $h_1$ | $\mathbb{R}^{1,d-1}$ |
| 10 | $\mathfrak{so}^*(4n) \cong \mathfrak{u}_{2r}(\mathbb{H}, \Omega)$ | $C_n$ | $h_n$ | $\mathrm{Herm}_n(\mathbb{H})$ |
| 11 | $\mathfrak{e}_{7(-25)}$ | $C_3$ | $h_3$ | $\mathrm{Herm}_3(\mathbb{O})$ |
| | Non-hermitian split forms | | | |
| 12 | $\mathfrak{sl}_n(\mathbb{R})$ | $A_{n-1}$ | $h_j, 1 \leq j \leq n-1$ | $M_{j,n-j}(\mathbb{R})$ |
| 9b | $\mathfrak{so}_{n,n+1}(\mathbb{R})$ | $B_n$ | $h_1$ | $\mathbb{R}^{2n-1}$ |
| 13 | $\mathfrak{so}_{n,n}(\mathbb{R})$ | $D_n$ | $h_{n-1}, h_n$ | $\mathrm{Alt}_n(\mathbb{R})$ |
| 14 | $\mathfrak{e}_6(\mathbb{R})$ | $E_6$ | $h_1 = h'_6$ | $M_{1,2}(\mathbb{O}_{\mathrm{split}})$ |
| 15 | $\mathfrak{e}_7(\mathbb{R})$ | $E_7$ | $h_7$ | $\mathrm{Herm}_3(\mathbb{O}_{\mathrm{split}})$ |
| | Non-hermitian non-split forms | | | |
| 16 | $\mathfrak{sl}_n(\mathbb{H})$ | $A_{n-1}$ | $h_j, 1 \leq j \leq n-1$ | $M_{j,n-j}(\mathbb{H})$ |
| 17 | $\mathfrak{u}_{n,n}(\mathbb{H})$ | $C_n$ | $h_n$ | $\mathrm{Aherm}_n(\mathbb{H})$ |
| 9c | $\mathfrak{so}_{p,q}(\mathbb{R}), 2 \neq p \neq q-1$ | $B_p$ $(p < q)$ | $h_1$ | $\mathbb{R}^{p+q-2}$ |
| | | $D_p$ $(p = q)$ | | |
| 18 | $\mathfrak{e}_{6(-26)}$ | $A_2$ | $h_1$ | $M_{1,2}(\mathbb{O})$ |

Table 1: Simple 3-graded Lie algebras

In our context, hermitian simple Lie algebras (cf. Appendix 7.2.3) are of particular interest. We collect some properties of hermitian Lie algebras in the the following proposition.

**Proposition 2.13** *For a simple real Lie algebra, the following assertions hold:*

(a) $\mathfrak{g}$ *is hermitian if and only if it contains a non-trivial closed convex* $\mathrm{Inn}(\mathfrak{g})$*-invariant cone* $C_{\mathfrak{g}}$.
(b) *A simple hermitian Lie algebra contains an Euler element if and only if it is of tube type, and in this case* $\mathrm{Inn}(\mathfrak{g})$ *acts transitively on* $\mathcal{E}(\mathfrak{g})$.

***Proof*** (a) is a consequence of the Kostant–Vinberg Theorem (cf. [HÓ97, Lem. 2.5.1]).
(b) Since the restricted root system of a hermitian simple Lie algebra is of type $C_r$ or $BC_r$ (see [MNÓ23, §3.1] or Table 2 below), and the first case characterizes the algebras of tube type, the assertion follows from Theorem 2.12 because the root system $C_r$ only permits one class of Euler elements. □

*Remark 2.14* As $h \in \mathfrak{a}$ implies $\theta(h) = -h$, the Cartan involution $\theta$ always maps $h$ into $-h$, but this only implies that $h$ is symmetric if $\theta \in \mathrm{Inn}(\mathfrak{g})$. This is the case if $\mathfrak{g}$ is hermitian, so that in these Lie algebras all Euler elements are symmetric (cf. Proposition 2.13).

The classification of Euler elements requires some interpretation. So let us first see what it says about complex simple Lie algebras $\mathfrak{g}$. In (52) we see that, only if $\mathfrak{g}$ is not of type $E_8, F_4$ or $G_2$, the Lie algebra $\mathfrak{g}$ contains an Euler element. Euler elements correspond to 3-gradings of the root system and these in turn to hermitian real forms $\mathfrak{g}^\circ$, where $ih_j \in \mathfrak{z}(\mathfrak{k}^\circ)$ generates the center of a maximal compactly embedded subalgebra $\mathfrak{k}^\circ$ ([Ne00, Thm. A.V.1]). We thus obtain the following possibilities. In Table 2, we write $\mathfrak{g}^\circ$ for the hermitian real form, $\mathfrak{g}$ for the complex Lie algebra, $\Sigma$ for its restricted root system, and $h_j$ for the corresponding Euler element.

| $\mathfrak{g}^\circ$ (hermitian) | $\Sigma(\mathfrak{g}^\circ, \mathfrak{a}^\circ)$ | $\mathfrak{g} = (\mathfrak{g}^\circ)_{\mathbb{C}}$ | $\Sigma(\mathfrak{g}, \mathfrak{a})$ | Euler element |
|---|---|---|---|---|
| $\mathfrak{su}_{p,q}(\mathbb{C}), 1 \leq p \leq q$ | $BC_p (p < q)$, $C_p (p = q)$ | $\mathfrak{sl}_{p+q}(\mathbb{C})$ | $A_{p+q-1}$ | $h_p$ |
| $\mathfrak{so}_{2,d}(\mathbb{R}), d > 2$ | $C_2$ | $\mathfrak{so}_{2+d}(\mathbb{C})$ | $B_{\frac{d+1}{2}}$, $d$ odd | $h_1$ |
| | | | $D_{1+\frac{d}{2}}$, $d$ even | |
| $\mathfrak{sp}_{2n}(\mathbb{R})$ | $C_n$ | $\mathfrak{sp}_{2n}(\mathbb{C})$ | $C_n$ | $h_n$ |
| $\mathfrak{so}^*(2n)$ | $BC_{\lfloor \frac{n}{2} \rfloor} (n \text{ odd})$, $C_{\frac{n}{2}} (n \text{ even})$ | $\mathfrak{so}_{2n}(\mathbb{C})$ | $D_n$ | $h_{n-1}, h_n$ |
| $\mathfrak{e}_{6(-14)}$ | $BC_2$ | $\mathfrak{e}_6$ | $E_6$ | $h_1 = h_6'$ |
| $\mathfrak{e}_{7(-25)}$ | $C_3$ | $\mathfrak{e}_7$ | $E_7$ | $h_7$ |

Table 2: Simple hermitian Lie algebras $\mathfrak{g}^\circ$ ($\mathfrak{g}$ as in (1)-(6) in Table 1).

They are pairwise distinct except for the exceptional isomorphisms ([HN12, §17]):

$A_1 = B_1 = C_1 : \mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{so}_{1,2}(\mathbb{R}) \cong \mathfrak{su}_{1,1}(\mathbb{C}) \cong \mathfrak{sp}_2(\mathbb{R})$
$D_2 = A_1 \cup A_1 : \mathfrak{so}_{2,2}(\mathbb{R}) \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{sl}_2(\mathbb{R}), \mathfrak{so}^*(4) \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_3(\mathbb{R})$
$B_2 = C_2 : \mathfrak{so}_{2,3}(\mathbb{R}) \cong \mathfrak{sp}_4(\mathbb{R})$.
$A_3 = D_3 : \mathfrak{so}_{2,4}(\mathbb{R}) \cong \mathfrak{su}_{2,2}(\mathbb{C})$, and $\mathfrak{so}^*(6) \cong \mathfrak{su}_{1,3}(\mathbb{C})$
$D_4 : \mathfrak{so}^*(8) \cong \mathfrak{so}_{2,6}(\mathbb{C})$.

In the correspondence of Euler elements in simplex complex Lie algebras and their hermitian real forms, those real forms corresponding to symmetric Euler elements are of particular interest. Comparing with the list of hermitian simple Lie algebras of tube type (Table 3), we see that they correspond precisely to 3-gradings specified by symmetric Euler elements, as listed in (53). Since the Euler elements $h_{n-1}$ and $h_n$ for the root system of type $D_n$ are conjugate under a diagram automorphism, they correspond to isomorphic hermitian real forms.

| $\mathfrak{g}^\circ$ (hermitian) | $\Sigma(\mathfrak{g}^\circ, \mathfrak{a}^\circ)$ | $\mathfrak{g} = (\mathfrak{g}^\circ)_\mathbb{C}$ | $\Sigma(\mathfrak{g}, \mathfrak{a})$ | symm. Euler element $h$ |
|---|---|---|---|---|
| $\mathfrak{su}_{n,n}(\mathbb{C})$ | $C_n$ | $\mathfrak{sl}_{2n}(\mathbb{C})$ | $A_{2n-1}$ | $h_n$ |
| $\mathfrak{so}_{2,d}(\mathbb{R}), d > 2$ | $C_2$ | $\mathfrak{so}_{2+d}(\mathbb{C})$ | $B_{\frac{d+1}{2}}$, $d$ odd | $h_1$ |
| | | | $D_{1+\frac{d}{2}}$, $d$ even | |
| $\mathfrak{sp}_{2n}(\mathbb{R})$ | $C_n$ | $\mathfrak{sp}_{2n}(\mathbb{C})$ | $C_n$ | $h_n$ |
| $\mathfrak{so}^*(4n)$ | $C_n$ | $\mathfrak{so}_{4n}(\mathbb{C})$ | $D_{2n}$ | $h_{2n-1}, h_{2n}$ |
| $\mathfrak{e}_{7(-25)}$ | $C_3$ | $\mathfrak{e}_7$ | $E_7$ | $h_7$ |

Table 3: Simple hermitian Lie algebras $\mathfrak{g}^\circ$ of tube type ((7)-(11) in Table 1)

## 2.4 Euler elements in general Lie algebras

To analyze Euler elements in general Lie algebra, it is instructive to consider abelian subalgebras $\mathfrak{a} \subseteq \mathfrak{g}$ which are maximal with respect to the property that $\operatorname{ad} \mathfrak{a}$ is diagonalizable. It follows from [KN96, Thm. III.3], applied to the symmetric Lie algebra $(\mathfrak{g}^{\oplus 2}, \tau_{\rm flip})$, that they are conjugate under $\operatorname{Inn}(\mathfrak{g})$. Moreover, there always exists an $\operatorname{ad} \mathfrak{a}$-invariant Levi complement $\mathfrak{s}$ ([KN96, Prop. I.2]), so that

$$\mathfrak{a} = \mathfrak{a}_\mathfrak{r} \oplus \mathfrak{a}_\mathfrak{s} \quad \text{for} \quad \mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}, \quad \mathfrak{a}_\mathfrak{r} = \mathfrak{a} \cap \mathfrak{r}, \quad \mathfrak{a}_\mathfrak{s} = \mathfrak{a} \cap \mathfrak{s}.$$

Then $[\mathfrak{a}_\mathfrak{r}, \mathfrak{s}] \subseteq \mathfrak{r} \cap \mathfrak{s} = \{0\}$. As $\mathfrak{g}$ is a nilpotent module of the ideal $[\mathfrak{g}, \mathfrak{r}]$, it further follows that

$$\mathfrak{a}_\mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}] = \mathfrak{a} \cap [\mathfrak{g}, \mathfrak{r}] \subseteq \mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}],$$

so that

$$\mathfrak{a} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{a}_\mathfrak{r}^c \oplus \mathfrak{a}_\mathfrak{s}, \tag{57}$$

where $\mathfrak{a}_\mathfrak{r}^c \subseteq \mathfrak{a}_\mathfrak{r}$ is a complement of $\mathfrak{z}(\mathfrak{g})$ in $\mathfrak{a}_\mathfrak{r}$.

**Lemma 2.15** *For an Euler element $h \in \mathfrak{g}$, the following assertions hold:*

(a) *$\mathcal{O}_h$ intersects $\mathfrak{a}$, hence also $\mathfrak{z}(\mathfrak{g})+\mathfrak{l}$, where $\mathfrak{l} = \mathfrak{a}_\mathfrak{r}^c \oplus \mathfrak{s}$ is a reductive subalgebra of $\mathfrak{g}$. Moreover, $\mathcal{O}_h \cap \mathfrak{a} = \mathcal{W}.h$, where $\mathcal{W} := \mathcal{W}(\mathfrak{s}, \mathfrak{a}_\mathfrak{s})$ is the Weyl group of the restricted root system $\Sigma(\mathfrak{s}, \mathfrak{a}_\mathfrak{s})$.*

(b) *If $h \in [\mathfrak{g}, \mathfrak{g}]$ is an Euler element contained in the commutator algebra, then $\mathcal{O}_h + \mathfrak{z}(\mathfrak{g})$ intersects every Levi complement.*

***Proof*** (a) That $\mathcal{O}_h \cap \mathfrak{a} = \mathcal{W}.h$ follows from [KN96, Thms. III.3, III.10], applied to the symmetric Lie algebra $(\mathfrak{g}^{\oplus 2}, \tau_{\rm flip})$. The rest of (a) now follows from (57) and the fact that $[\mathfrak{a}_\mathfrak{r}, \mathfrak{s}] = \{0\}$.
(b) In view of (a), we may assume that $h \in \mathfrak{a}$. Then $h \in \mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{z}(\mathfrak{g}) + \mathfrak{a}_\mathfrak{s} \subseteq \mathfrak{z}(\mathfrak{g}) + \mathfrak{s}$ implies (b). Now the assertion follows from the fact that $\mathcal{O}_h + \mathfrak{z}(\mathfrak{g})$ is invariant under $\mathrm{Inn}(\mathfrak{g})$ and that any two Levi complements are conjugate under this group. □

For refinements of the following proposition we refer to [MNÓ26a, §2.1].

**Proposition 2.16** ([MN21, Prop. 3.2]) *The following assertions hold:*

(i) *An Euler element $h \in \mathfrak{g}$ is symmetric if and only if $h$ is contained in a Levi complement $\mathfrak{s}$ and $h$ is a symmetric Euler element in $\mathfrak{s}$.*

(ii) *Let $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ be a Levi decomposition.*

  (a) *If $h \in \mathfrak{g}$ is a symmetric Euler element, then*

$$\mathcal{O}_h = \mathcal{O}_{q(h)} = \mathrm{Inn}(\mathfrak{g})(\mathcal{O}_h \cap \mathfrak{s}),$$

  *where $q \colon \mathfrak{g} \to \mathfrak{s}$ is the projection along $\mathfrak{r}$.*

  (b) *Two symmetric Euler elements are conjugate under $\mathrm{Inn}(\mathfrak{g})$ if and only if their images in $\mathfrak{s}$ are conjugate under $\mathrm{Inn}(\mathfrak{s})$.*

***Proof*** (i) As $\mathcal{O}_h \subseteq h + [\mathfrak{g}, \mathfrak{g}]$ follows from the invariance of the affine subspace $h + [\mathfrak{g}, \mathfrak{g}]$ under $\mathrm{Inn}(\mathfrak{g})$, the relation $-h \in \mathcal{O}_h$ implies $h \in [\mathfrak{g}, \mathfrak{g}]$. In view of Lemma 2.15(b), there exists a Levi complement $\mathfrak{s}$ with $h \in \mathfrak{z}(\mathfrak{g}) + \mathfrak{s}$. Then $\mathfrak{r}$ and $\mathfrak{s}$ are $\mathrm{ad}\, h$-invariant, so that the $\mathrm{ad}\, h$-eigenspaces of the restrictions satisfy

$$\mathfrak{r} = \mathfrak{r}_1(h) + \mathfrak{r}_0(h) + \mathfrak{r}_{-1}(h) \quad \text{and} \quad \mathfrak{s} = \mathfrak{s}_1(h) + \mathfrak{s}_0(h) + \mathfrak{s}_{-1}(h),$$

and define 3-gradings of $\mathfrak{r}$ and $\mathfrak{s}$. Further $\mathfrak{g}_{\pm 1}(h) \subseteq [h, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{s} = [\mathfrak{s}, \mathfrak{s}] \subseteq [\mathfrak{g}, \mathfrak{g}]$ imply that $\mathfrak{g} = \mathfrak{r}_0(h) + [\mathfrak{g}, \mathfrak{g}]$. The fact that $[\mathfrak{g}, \mathfrak{g}]$ is an ideal and $\mathfrak{r}_0(h)$ is a subalgebra of $\mathfrak{g}$ entails that the subgroup $\mathrm{Inn}_\mathfrak{g}([\mathfrak{g}, \mathfrak{g}])$ of $\mathrm{Inn}(\mathfrak{g})$ is normal, and that $\mathrm{Inn}(\mathfrak{g}) = \mathrm{Inn}_\mathfrak{g}([\mathfrak{g}, \mathfrak{g}])\, \mathrm{Inn}(\mathfrak{r}_0(h))$. As $\mathrm{Inn}(\mathfrak{r}_0(h))$ fixes $h$, this in turn shows that $\mathcal{O}_h = \mathrm{Inn}_\mathfrak{g}([\mathfrak{g}, \mathfrak{g}])h = \mathrm{Inn}_\mathfrak{g}([\mathfrak{g}, \mathfrak{r}])\, \mathrm{Inn}_\mathfrak{g}(\mathfrak{s})h$. Writing $h = h_z + h_s$ with $h_z \in \mathfrak{z}(\mathfrak{g})$ and $h_s \in \mathcal{E}(\mathfrak{s})$, we thus find $x \in [\mathfrak{g}, \mathfrak{r}]$ and $s \in \mathrm{Inn}_\mathfrak{g}(\mathfrak{s})$ such that [7]

[7] Here we use that the ideal $[\mathfrak{g}, \mathfrak{r}]$ is nilpotent, so that the exponential function of the corresponding group $\mathrm{Inn}_\mathfrak{g}([\mathfrak{g}, \mathfrak{r}])$ is surjective, see [HN12, Cor. 11.2.7].

$$-h_z - h_s = -h = e^{\operatorname{ad} x} s.h = h_z + e^{\operatorname{ad} x} s.h_s. \tag{58}$$

Applying the Lie algebra homomorphism $q\colon \mathfrak{g} \to \mathfrak{s}$ to both sides, we derive from $q(h_z) = 0$ and $q \circ e^{\operatorname{ad} x} = q$ that $-h_s = s.h_s$, and therefore by (58)

$$e^{\operatorname{ad} x} h_s = h_s + 2h_z.$$

We conclude that the unipotent linear map $e^{\operatorname{ad} x}$ preserves the plane $\mathbb{R}h_s + \mathbb{R}h_z$, and this implies that $\operatorname{ad} x = \log(e^{\operatorname{ad} x})$ also has this property. We thus arrive at

$$[h, x] = [h_s, x] \subseteq \mathbb{R}h_s + \mathbb{R}h_z \subseteq \mathfrak{g}_0(h),$$

so that $x \in \mathfrak{g}_0(h) = \mathfrak{g}_0(h_s)$, which in turn leads to $0 = e^{\operatorname{ad} x} h_s - h_s = 2h_z$, i.e., $h = h_s \in \mathfrak{s}$.

To prove the second assertion of (i), we observe that the projection $q\colon \mathfrak{g} \to \mathfrak{s} \cong \mathfrak{g}/\mathfrak{r}$ satisfies

$$q(\mathcal{O}_x) = \mathcal{O}^{\mathfrak{s}}_{q(x)} \quad \text{for} \quad x \in \mathfrak{g}. \tag{59}$$

Writing $\mathcal{E}_{\mathrm{sym}}(\mathfrak{g})$ for the set of symmetric Euler elements in $\mathfrak{g}$, we obtain $q(\mathcal{E}_{\mathrm{sym}}(\mathfrak{g})) \subseteq \mathcal{E}_{\mathrm{sym}}(\mathfrak{s})$. If, conversely, $h \in \mathcal{E}_{\mathrm{sym}}(\mathfrak{s})$, then we clearly have $-h \in \operatorname{Inn}_{\mathfrak{g}}(\mathfrak{s})h \subseteq \operatorname{Inn}(\mathfrak{g})h$, so that $h \in \mathcal{E}_{\mathrm{sym}}(\mathfrak{g})$.
(ii)(a) As $\mathcal{O}_h$ intersects $\mathfrak{s}$ by (i), $q(\mathcal{O}_h) \cap \mathcal{O}_h \neq \emptyset$, and since $\operatorname{Inn}(\mathfrak{s})$ acts transitively on $q(\mathcal{O}_h)$ by (59), we obtain $q(\mathcal{O}_h) \subseteq \mathcal{O}_h$ and thus $q(\mathcal{O}_h) = \mathcal{O}_h \cap \mathfrak{s}$. This further leads to

$$\mathcal{O}_h = \operatorname{Inn}(\mathfrak{g})(\mathcal{O}_h \cap \mathfrak{s}) = \operatorname{Inn}(\mathfrak{g})q(\mathcal{O}_h) = \operatorname{Inn}(\mathfrak{g})\mathcal{O}^{\mathfrak{s}}_{q(h)} = \mathcal{O}_{q(h)}.$$

(ii)(b) follows immediately from (a). □

Proposition 2.16 reduces, for a given Lie algebra $\mathfrak{g}$, the description of symmetric Euler elements up to conjugation by inner automorphisms to the case of simple Lie algebras.

It would be nice to have a classification of Euler elements in any Lie algebra $\mathfrak{g}$, but, due to the complexity of Levi decompositions $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$, this is not a well-posed problem. If $\mathfrak{g}$ is reductive, then the classification of Euler elements in $\mathfrak{g}$ follows immediately from the case of simple Lie algebras, which is described in Theorem 2.12. For symmetric Euler elements $h$, Proposition 2.16 reduces the classification to the semisimple case, but then one has to describe the module structure of the radical. [8]

**Example 2.17** (An example from symplectic geometry) A particularly interesting Lie algebra which is neither semisimple nor solvable is the *conformal Jacobi–Lie algebra*

[8] The role of the symmetry of $h$ for the existence of nets of real subspaces is still not completely understood. It certainly plays an important role in specifying locality conditions (cf. Section 6.2). If $h$ is not symmetric, one may be forced to also take non-connected causal manifolds $M$ into consideration, resp., to replace $G$ by a suitable non-connected group.

$$\mathfrak{g} = \mathfrak{hcsp}(V,\omega) := \mathfrak{heis}(V,\omega) \rtimes \mathfrak{csp}(V,\omega),$$

where $(V,\omega)$ is a symplectic vector space, $\mathfrak{heis}(V,\omega) = \mathbb{R} \oplus V$ is the corresponding Heisenberg algebra with the bracket $[(z,v),(z',v')] = (\omega(v,v'),0)$, and

$$\mathfrak{csp}(V,\omega) := \mathfrak{sp}(V,\omega) \oplus \mathbb{R}\,\mathrm{id}_V$$

is the *conformal symplectic Lie algebra* of $(V,\omega)$. The hyperplane ideal

$$\mathfrak{j} := \mathfrak{heis}(V,\omega) \rtimes \mathfrak{sp}(V,\omega)$$

(the *Jacobi–Lie algebra*) can be identified by the linear isomorphism

$$\varphi\colon \mathfrak{j} \to \mathrm{Pol}_{\leq 2}(V), \qquad \varphi(z,v,x)(\xi) := z + \omega(v,\xi) + \frac{1}{2}\omega(x\xi,\xi), \quad \xi \in V$$

with the Lie algebra of polynomials $\mathrm{Pol}_{\leq 2}(V)$ of degree $\leq 2$ on $V$, endowed with the Poisson bracket ([Ne00, Prop. A.IV.15]). The set

$$C_\mathfrak{g} := \{f \in \mathrm{Pol}_{\leq 2}(V) \colon f \geq 0\}$$

is a pointed generating invariant cone in $\mathfrak{j}$. The element $h_0 := \mathrm{id}_V$ defines a derivation on $\mathfrak{j}$ by $(\mathrm{ad}\, h_0)(z,v,x) = (2z,v,0)$ for $z \in \mathbb{R}, v \in V, x \in \mathfrak{sp}(V,\omega)$. Any involution $\tau_V$ on $V$ satisfying $\tau_V^*\omega = -\omega$ defines by

$$\widetilde{\tau}_V(z,v,x) := (-z, -\tau_V(v), \tau_V x \tau_V) \tag{60}$$

an involution on $\mathfrak{g}$ with $\widetilde{\tau}_V(h_0) = h_0$, and $-\widetilde{\tau}_V(C_\mathfrak{g}) = C_\mathfrak{g}$ follows from

$$\varphi(\widetilde{\tau}_V(z,v,x)) = -\varphi(z,v,x) \circ \tau_V.$$

Considering $h_\mathfrak{s} := \frac{1}{2}\tau_V$ as an element of $\mathfrak{sp}(V,\omega)$, the element

$$h := h_\mathfrak{s} + \tfrac{1}{2}\,\mathrm{id}_V \in \mathfrak{csp}(V,\omega) \tag{61}$$

is Euler in $\mathfrak{g}$. Writing $V = V_1 \oplus V_{-1}$ for the $\tau_V$-eigenspace decomposition, we have

$$\mathfrak{g}_{-1} = 0 \oplus 0 \oplus \mathfrak{sp}(V,\omega)_{-1}, \quad \mathfrak{g}_0 = 0 \oplus V_{-1} \oplus \mathfrak{sp}(V,\omega)_0 \cong V_{-1} \rtimes \mathfrak{gl}(V_{-1}),$$
$$\mathfrak{g}_1 = \mathbb{R} \oplus V_1 \oplus \mathfrak{sp}(V,\omega)_1.$$

Note that

$$\tau_h = e^{\pi i\, \mathrm{ad}\, h} = \widetilde{\tau}_V. \tag{62}$$

Here $\mathfrak{g}_1$ can be identified with the space $\mathrm{Pol}_{\leq 2}(V_{-1})$ of polynomials of degree $\leq 2$ on $V_{-1}$ and

$$C_+ = C_\mathfrak{g} \cap \mathfrak{g}_1 = \{f \in \mathrm{Pol}_{\leq 2}(V_{-1}) \colon f \geq 0\}.$$

This cone is invariant under the natural action of the affine group $G_0 \cong \mathrm{Aff}(V_{-1})_0 \cong V_{-1} \rtimes \mathrm{GL}(V_{-1})_0$ whose Lie algebra is $\mathfrak{g}_0$. We also note that

$$\mathfrak{g}_{-1} \cong \mathrm{Pol}_2(V_1) \quad \text{and} \quad C_- = -C_\mathfrak{g} \cap \mathfrak{g}_{-1} = \{f \in \mathrm{Pol}_2(V_1) \colon f \leq 0\},$$

so that $C_-$ is also pointed and generating.

The Euler element $h$ is not symmetric because $\dim \mathfrak{g}_1 \neq \dim \mathfrak{g}_{-1}$.

We also claim that the Lie algebra $\mathfrak{hsp}(V,\omega)$ contains **no Euler element**. In fact, as it is perfect,[9] and $\mathfrak{heis}(V,\omega) \rtimes \mathfrak{sp}(V,\omega)$ is a Levi decomposition, it suffices by Lemma 2.15 to show that no Euler element of $\mathfrak{g}$ is contained in $\mathbb{R} \oplus \{0\} \oplus \mathfrak{sp}(V,\omega)$. Since all Euler elements $h$ in the hermitian Lie algebra $\mathfrak{sp}(V,\omega)$ are conjugate (Proposition 2.13), it suffices to consider $h = h_\mathfrak{s} + (\lambda, 0, 0), \lambda \in \mathbb{R}$. As

$$\mathrm{Spec}(\mathrm{ad}\, h) = \mathrm{Spec}(\mathrm{ad}\, h_\mathfrak{s}) = \{\pm 1, \pm\tfrac{1}{2}, 0\},$$

$h$ is not Euler in $\mathfrak{heis}(V,\omega) \rtimes \mathfrak{sp}(V,\omega)$.

## 2.5 Euler elements in low-dimensional subalgebras

Many phenomena concerning Euler elements alreay appear in the 3-dimensional Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, so that it is relevant to know when an Euler element is contained in a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{gl}_2(\mathbb{R})$. The following result characterizes this property.

**Proposition 2.18** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $h \in \mathcal{E}(\mathfrak{g})$ an Euler element. If $h$ is not contained in the solvable radical* $\mathrm{rad}(\mathfrak{g})$*, then there exists a Lie subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ containing $h$ such that*

(a) $\mathfrak{b} \cong \mathfrak{sl}_2(\mathbb{R})$ *if and only if $h$ is symmetric, and*
(b) $\mathfrak{b} \cong \mathfrak{gl}_2(\mathbb{R})$ *if $h$ is not symmetric.*
(c) *If $h$ is symmetric, then* $\mathrm{Inn}_\mathfrak{g}(\mathfrak{b}) \cong \mathrm{PSL}_2(\mathbb{R})$.
(d) *If $h$ is not symmetric and $\mathfrak{g}$ is simple, then* $\mathrm{Inn}_\mathfrak{g}([\mathfrak{b},\mathfrak{b}]) \cong \mathrm{SL}_2(\mathbb{R})$.

***Proof*** (a) If $h \in \mathfrak{b} \cong \mathfrak{sl}_2(\mathbb{R})$, then $h$ is symmetric because all Euler elements in $\mathfrak{sl}_2(\mathbb{R})$ are symmetric by Example 2.8. If, conversely, $h$ is symmetric, then Proposition 2.16 implies that $h$ is contained in a Levi complement $\mathfrak{s}$. Therefore [MN21, Thm. 3.13] implies that $h$ is contained in an $\mathfrak{sl}_2$-subalgebra.
(b) Suppose that $h$ is not symmetric and pick a maximal abelian hyperbolic subspace $\mathfrak{a} \subseteq \mathfrak{g}$ containing $h$. With [KN96, Prop. I.2] we find an $\mathfrak{a}$-invariant Levi complement $\mathfrak{s} \subseteq \mathfrak{g}$. Then $\mathfrak{a}_\mathfrak{s} := \mathfrak{a} \cap \mathfrak{s}$ is maximal hyperbolic in $\mathfrak{s}$ and $\mathfrak{a} = \mathfrak{a}_\mathfrak{s} + \mathfrak{z}_\mathfrak{a}(\mathfrak{s})$. As $h$ is not contained in $\mathrm{rad}(\mathfrak{g})$, there exists a root $\alpha \in \Delta(\mathfrak{s},\mathfrak{a})$

[9] A Lie algebra $\mathfrak{g}$ is called *perfect* if $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$, i.e., each element is a sum of commutators.

with $\alpha(h) = 1$ and root vectors $x_\alpha \in \mathfrak{s}_\alpha$ and $y_\alpha \in \mathfrak{s}_{-\alpha}$ with $h_\alpha := [x_\alpha, y_\alpha] \neq 0$ (see Lemma 7.9). We stress that $x_\alpha \in \mathfrak{s}_1(h)$. Now

$$\mathfrak{b}_\alpha := \mathbb{R}x_\alpha + \mathbb{R}y_\alpha + \mathbb{R}h_\alpha \cong \mathfrak{sl}_2(\mathbb{R})$$

and $[h, \mathfrak{b}_\alpha] \subseteq \mathfrak{b}_\alpha$. Hence $\mathfrak{b} := \mathbb{R}h + \mathfrak{b}_\alpha$ is a Lie subalgebra of $\mathfrak{g}$. As $h$ is not symmetric, $h \notin \mathfrak{b}_\alpha$, and therefore $\mathfrak{b} \cong \mathfrak{gl}_2(\mathbb{R})$.
(c) If $h$ is symmetric and $\mathfrak{b} = [\mathfrak{b}, \mathfrak{b}] \cong \mathfrak{sl}_2(\mathbb{R})$ as in (a), then the fact that $\mathfrak{b}$ contains an Euler element of $\mathfrak{g}$ implies that all simple $\mathfrak{b}$-submodules of $\mathfrak{g}$ are either trivial of isomorphic to the adjoint representation of $\mathfrak{sl}_2(\mathbb{R})$ (consider eigenspaces of $\operatorname{ad} h$). This implies that $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{b}) \cong \mathrm{PSL}_2(\mathbb{R})$.
(d) Suppose that $\mathfrak{g}$ is simple. If $h$ is not symmetric, then the Weyl group reflection $s_\alpha$ corresponding to the root $\alpha$ from above satisfies

$$s_\alpha(h) = h - \alpha(h)\alpha^\vee = h - \alpha^\vee.$$

As $h$ is not contained in $\mathbb{R}\alpha^\vee \subseteq \mathfrak{b}_\alpha$, we have $s_\alpha(h) \notin \mathbb{R}h$.

The simplicity of $\mathfrak{g}$ ensures that the root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ is irreducible and 3-graded by $h \in \mathfrak{a}$. Therefore

$$\Delta_0 := \{\alpha \in \Delta \colon \alpha(h) = 0\}$$

spans a hyperplane in $\mathfrak{a}^*$, which coincides with $h^\perp$, and thus $\mathbb{R}h = \Delta_0^\perp$ by duality. Since $s_\alpha(h)$ is not contained in $\mathbb{R}h$, there exists a $\beta \in \Delta_0$ with $\beta(s_\alpha(h)) \neq 0$. Now $\beta(h) = 0$ implies

$$0 \neq \beta(s_\alpha(h)) = -\beta(\alpha^\vee).$$

As $s_\alpha(h)$ is an Euler element, we obtain $|\beta(\alpha^\vee)| = 1$. Therefore the central element $e^{\pi i \operatorname{ad} \alpha^\vee}$ of $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{b}_\alpha)$ acts non-trivially, and this implies that $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{b}_\alpha) \cong \mathrm{SL}_2(\mathbb{R})$ because it is a linear Lie group with non-trivial center ([HN12, Ex. 9.5.18]). □

### 2.6 Abstract Euler couples and the BGL net

We conclude Section 2 with a construction that generalizes the algebraic construction of free fields for AQFT models presented in [BGL02]. It is based on an abstract Lie theoretic construction that reflects some key features of modular pairs $(\Delta, J)$: the space $\mathcal{G}_E$ of Euler couples of a Lie group of the form $G_\sigma = G \rtimes \{e, \sigma\}$. This provides the context for a general construction, assigning to every antiunitary representation of $G_\sigma$ a $G$-covariant family of standard subspaces indexed by $\mathcal{G}_E$. We refer to [MN21] and [MNÓ26b] for details; see also Exercise 1.51.

**Definition 2.19** For an involution $\sigma \in \mathrm{Aut}(G)$, we write $G_\sigma := G \rtimes \{\mathrm{id}_G, \sigma\}$ for the corresponding group extension and $G\sigma = G \times \{\sigma\}$ for the $G$-coset of the involution $\sigma$.

The set

$$\mathcal{G} := \mathcal{G}(G_\sigma) := \{(h, \tau) \in \mathfrak{g} \times G\sigma \;:\; \tau^2 = e, \mathrm{Ad}(\tau)h = h\}$$

is called the *abstract wedge space of* $G_\sigma$. An element $(h, \tau) \in \mathcal{G}$ is called an *Euler couple* if $h \in \mathcal{E}(\mathfrak{g})$ and

$$\mathrm{Ad}(\tau) = \tau_h. \tag{63}$$

In this case $\tau$ is called an *Euler involution* on $G$. We write $\mathcal{G}_E \subseteq \mathcal{G}$ for the subset of Euler couples.

(a) Consider the homomorphism $\varepsilon \colon G_\sigma \to \{\pm 1\}$, defined by $\ker \varepsilon = G$. On $\mathfrak{g}$ we consider the *twisted adjoint action* of $G_\sigma$ which changes the sign on odd group elements:

$$\mathrm{Ad}^\varepsilon \colon G_\sigma \to \mathrm{Aut}(\mathfrak{g}), \qquad \mathrm{Ad}^\varepsilon(g) := \varepsilon(g)\,\mathrm{Ad}(g). \tag{64}$$

It extends to an action of $G_\sigma$ on $\mathcal{G}$ by

$$g.(h, \tau) := (\mathrm{Ad}^\varepsilon(g)h, g\tau g^{-1}). \tag{65}$$

(b) (Duality operation) The notion of a "causal complement" is defined on the abstract wedge space as follows: For $W = (h, \tau) \in \mathcal{G}$, we define the *dual wedge* by

$$W' := (-h, \tau) = \tau.W.$$

Note that $(W')' = W$ and $(gW)' = gW'$ for $g \in G$ by (65). This relation fits the geometric interpretation in the context of wedge domains in spacetime manifolds (see also Section 6.2).

**Definition 2.20** If $(U, \mathcal{H})$ is an antiunitary representation of $G_\sigma$, then we obtain a standard subspace $\mathsf{H}_U(W)$, determined for $W = (h, \tau) \in \mathcal{G}$ by the couple of operators (cf. Proposition 1.6):

$$J_{\mathsf{H}_U(W)} = U(\tau) \quad \text{and} \quad \Delta_{\mathsf{H}_U(W)} = e^{2\pi i \cdot \partial U(h)}, \tag{66}$$

and thus a $G$-equivariant map $\mathsf{H}_U \colon \mathcal{G} \to \mathrm{Stand}(\mathcal{H})$ (cf. Exercise 1.57). This is the so-called *Brunetti–Guido–Longo (BGL) net*

$$\mathsf{H}_U^{\mathrm{BGL}} \colon \mathcal{G}(G_\sigma) \to \mathrm{Stand}(\mathcal{H}).$$

For a detailed discussion of the properties of this net and the structures on $\mathcal{G}$, we refer to [MN21] and [MNÓ26a].

The BGL construction provides an abstract perspective on families of modular pairs $(\Delta, J)$ in terms of Lie theoretic data. This is rather close in spirit to the approach to nets of local algebras in the context of Geometric Modular Action (GMA), where concrete wedge regions in a spacetime manifold are replaced by abstract index sets for families of algebras of local observables. In this context it is natural to try to recover as much as possible of the spacetime geometry from this data. In [SW03] this is done for the action of the Poincaré group on 3-dimensional Minkowski space. The underlying idea is to consider the family of modular conjugations, which in the group theoretic context corresponds to the conjugacy class of the involution $(e, \sigma)$, also denoted $\sigma$ in the extended group $G_\sigma$:

$$M_\sigma := \{(g,e)(e,\sigma)(g,e)^{-1} \colon g \in G\} = \{(gg^\sharp, \sigma) \colon g \in G\}, \quad g^\sharp = g\sigma(g)^{-1}.$$

This set consists of involutions and carries a natural structure of a symmetric space $G/G^\sigma$ (Exercise 3.59, [NÓ17, §5.1]). It generates $G_\sigma$ as a group if and only if the identity component

$$G_e^\sharp = \{gg^\sharp \colon g \in G\} \quad \text{of the set} \quad G^\sharp = \{g \in G \colon g^\sharp = g\}$$

generates $G$. To see when this is the case, we write

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \quad \text{with} \quad \mathfrak{h} = \mathfrak{g}^\sigma \quad \text{and} \quad \mathfrak{q} = \mathfrak{g}^{-\sigma}.$$

Then $\mathfrak{h}$ is a Lie subalgebra, $[\mathfrak{h}, \mathfrak{q}] \subseteq \mathfrak{q}$, and $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{h}$, and $M_\sigma$ generates $G_\sigma$ if and only if

$$[\mathfrak{q}, \mathfrak{q}] = \mathfrak{h},$$

which is equivalent to the subspace $\mathfrak{q}$ to generate $\mathfrak{g}$ as a Lie algebra ([HN12, Thm. 9.6.1]). If this condition is satisfied, then the group $G_\sigma$ is generated by $M_\sigma$, and one can relate geometric properties to relations that can be expressed in terms of products of involutions.

## 2.7 Exercises for Section 2

**Exercise 2.21** Let $h \in \mathfrak{sl}_n(\mathbb{R})$. Show that $h$ is an Euler element if and only if $h$ is diagonalizable with 2 eigenvalues $\lambda$, $\mu$ satisfying $\lambda - \mu = 1$.

**Exercise 2.22** Describe the conjugacy classes of Euler elements in the Lie algebras $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{gl}_n(\mathbb{R})$ and $\mathfrak{so}_{1,n}(\mathbb{R})$ up to conjugation.

## 3 Causal homogeneous spaces and wedge regions

The Euler Element Theorem 2.3 provides us with the information that Euler elements are the natural candidates for the Lie algebra elements $h$ arising in the Bisognano–Wichmann property (BW), but it provides no information on how to find appropriate regions $W \subseteq M$.

Motivated by the Bisognano–Wichmann property (BW) in AQFT, the modular flow $\alpha_t^W(m) = \exp(th).m$ on $W \subseteq M$ should, in a suitable sense, correspond to the "flow of time" on the spacetime region $W$. This is based on the interpretation of the modular group in the context of the Tomita–Takesaki Theorem as the dynamics of the corresponding quantum system, the **thermal time hypothesis**, a point of view advocated by A. Connes and C. Rovelli (cf. [CR94]). References for the AQFT perspective on this issue are [BB99, BY99, BMS01, Bo98, SW03, Bo09], [CLRR22, §3]. For a perspective from non-commutative geometry, see [KG09], [Kot19] and [He25].

To formulate what it means that a vector field generates on an open domain $W \subseteq M$ a flow that qualifies as a "flow of time" requires a *causal structure on the manifold* $M$, i.e., in each tangent space $T_m(M)$, we specify a pointed, generating, closed convex cone $C_m \subseteq T_m(M)$. [10] We think of elements in the interior $C_m^\circ$ as *positive timelike*, i.e., tangent vectors to curves describing the dynamics on a region in $M$ (following the "flow of time").

**Assumption:** Unless otherwise specified, we assume in these notes that $M$ is a homogeneous space, i.e., $M \cong G/H$ for a closed subgroup $H \subseteq G$ with Lie algebra $\mathfrak{h}$ (cf. also Remark 3.6). Then the tangent space $T_{eH}(M)$ in the base point identifies naturally with the quotient space $\mathfrak{q} := \mathfrak{g}/\mathfrak{h}$. Hence the existence of a $G$-invariant causal structure on $M$ is equivalent to the existence of an $\mathrm{Ad}_{\mathfrak{q}}(H)$-invariant pointed generating cone $C_{\mathfrak{q}} \subseteq \mathfrak{q}$ (cf. [Ne91], [HN93], [HÓ97], [Se71, Se76]). Then

$$C_{gH} := g.C_{eH} = g.C_{\mathfrak{q}} \quad \text{for} \quad g \in G,$$

is the corresponding causal structure on $M = G/H$, where we write

$$G \times TM \to TM, \quad (g, v) \mapsto g.v = T(\sigma_g)v, \quad \sigma_g(m) = g.m,$$

for the induced action of $G$ on the tangent bundle $TM$ of $M$.

[10] A closed convex cone $C$ in a finite-dimensional vector space $V$ is called *pointed* if $C \cap -C = \{0\}$, and *generating* if $C - C = V$, i.e., if $C$ has interior points.

### 3.1 Causal structures and wedge regions

Coming back to the question of how to find $W$, let us fix an Euler element $h \in \mathfrak{g}$. Then we call

$$X_h^M(m) := \frac{d}{dt}\Big|_{t=0} \exp(th).m \tag{67}$$

the corresponding *modular vector field*. In view of the Thermal Time Hypothesis, the subset $W$ should be contained in the *positivity region*

$$W_M^+(h) := \{m \in M \colon X_h^M(m) \in C_m^\circ\}, \tag{68}$$

which is the largest open subset on which the flow is "future-directed", i.e., timelike in the Lorentzian context. For $m = gH \in M = G/H$ and the linear projection $p_\mathfrak{q} \colon \mathfrak{g} \to \mathfrak{q} = \mathfrak{g}/\mathfrak{h} \cong T_{eH}(M)$, we have

$$X_h^M(gH) = \frac{d}{dt}\Big|_{t=0} \exp(th).gH = \frac{d}{dt}\Big|_{t=0} gg^{-1}\exp(th).gH = g.p_\mathfrak{q}(\mathrm{Ad}(g)^{-1}h). \tag{69}$$

By $G$-invariance of the causal structure, this calculation shows that $X_h^M(gH) \in C_{gH}^\circ$ is equivalent to $p_\mathfrak{q}(\mathrm{Ad}(g)^{-1}h) \in C^\circ$, so that we obtain the Lie algebraic description

$$W_M^+(h) = \{gH \in G/H \colon \ \mathrm{Ad}(g)^{-1}h \in p_\mathfrak{q}^{-1}(C^\circ)\} \tag{70}$$

of the positivity region.

**Definition 3.1** A *wedge region* $W$ for $h$ on the causal homogeneous space $M$ is a connected component of the positivity region $W_M^+(h)$.

At this point it is not clear why to focus on connected components and not the whole positivity region. As the concrete examples, where $W_M^+(h)$ is not connected, show, the inclusions $\mathsf{H}(W) \subseteq \mathsf{H}(W_M^+(h))$ are often proper for nets on $M$ (see Theorem 5.33 for concrete situations). If this is the case and $\mathsf{H}(W) = \mathtt{V}$, then the subspace $\mathsf{H}(W_M^+(h))$ cannot be separating by the Equality Lemma 1.9. Therefore the connected components turn out to be the natural choice for wedge regions. In this context, Theorem 5.19 shows that small open $\exp(\mathbb{R}h)$-invariant subsets may already satisfy (BW).

**Example 3.2** In Minkowski space $M = \mathbb{R}^{1,d-1}$ (Remark 1.29), the causal structure is given by the constant cone field $C_x = C$ for $x \in M$ and

$$C = \{x \in \mathbb{R}^{1,d-1} \colon x_0 \geq \sqrt{\mathbf{x}^2}\}.$$

Here $M$ is a homogeneous space of the Poincaré group

$$G = \mathbb{R}^{1,d-1} \rtimes \mathrm{SO}_{1,d-1}(\mathbb{R})_e$$

with base point 0, whose stabilizer is the Lorentz group $\mathrm{SO}_{1,d-1}(\mathbb{R})_e$. For the Lorentz boost $h(x) = (x_1, x_0, 0, \cdots, 0)$, the corresponding vector field is linear, i.e.,

$$X_h^M(x) = h(x),$$

and its values are positive timelike, i.e., contained in $C^\circ$ if and only if $x_1 > |x_0|$, which specifies the Rindler wedge $W_R = \{(x_0, \mathbf{x}) \colon x_1 > |x_0|\}$.

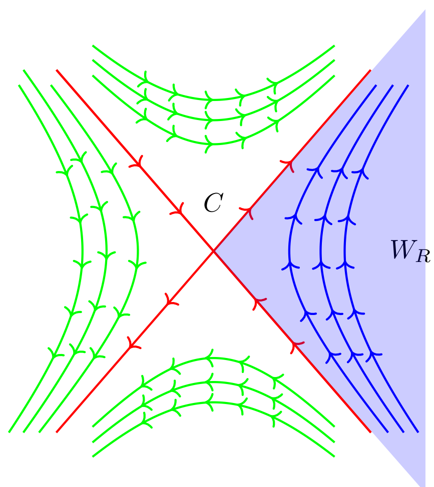



Figure 5: 2-dimensional Minkowski space with Rindler wedge $W_R$ and trajectories of the boost (positive timelike in $W_R$).

**Lemma 3.3** *Any wedge region $W \subseteq W_M^+(h)$ is invariant under the identity component $G_e^h$ of the centralizer*

$$G^h := \{g \in G \colon \mathrm{Ad}(g)h = h\}$$

*of the Euler element $h$, hence in particular under* $\exp(\mathbb{R}h)$.

The following proposition provides a sufficient criterion for the positivity region $W_M^+(h)$ on $M$ being non-empty. The condition $h \in \mathfrak{h}$ is equivalent to the base point being fixed under the modular flow.

**Proposition 3.4** (Sufficient conditions for the existence of wedge regions) *For $M = G/H$ and an Euler element $h \in \mathfrak{h}$, suppose that $\tau_h \in \mathrm{Aut}(G)$ fixes $H$ and induces an anti-causal map $\tau_h^M$ on $M$, i.e., $\tau_h^M(C_m) = -C_{\tau_h^M(m)}$ for $m \in M$. Then $W_M^+(h) \neq \emptyset$.*

***Proof*** For the action of the one-parameter group $e^{\mathbb{R}\,\mathrm{ad}\,h}$ on $\mathfrak{q} := \mathfrak{g}/\mathfrak{h}$, we write $\mathfrak{q}_j$, $j = 1, 0, -1$, for the corresponding $\mathrm{ad}\,h$-eigenspace and [11]

[11] For the linear vector field defined by $h$ on $\mathfrak{q}$, the positivity region is $W_{\mathfrak{q}}^+(h) = C_+^\circ + \mathfrak{q}_0 + C_-^\circ$ (cf. (72) below). This is why we consider these two cones.

$$C_\pm := \pm C \cap \mathfrak{q}_{\pm 1}.$$

In view of (70), it suffices to show that, for $x_{\pm 1} \in C_\pm^\circ$, there exists $t > 0$ such that

$$g_t := \exp(tx_{-1})\exp(tx_1)$$

satisfies $\mathrm{Ad}(g_t)^{-1}h \in p_{\mathfrak{q}}^{-1}(C^\circ)$. With Lemma 3.7 below, we see that $-\tau_h(C) = C$ implies that

$$C_+^\circ - C_-^\circ = (C_+ - C_-)^\circ \subseteq C^\circ.$$

For $t > 0$ we then have $e^{-t\,\mathrm{ad}\,x_{-1}}h = h - t[x_{-1}, h] = h - tx_{-1}$ because $(\mathrm{ad}\,x_{-1})^2 h \in \mathfrak{g}_{-2}(h) = \{0\}$. We thus obtain

$$\begin{aligned}\mathrm{Ad}(g_t)^{-1}h &= e^{-t\,\mathrm{ad}\,x_1}e^{-t\,\mathrm{ad}\,x_{-1}}h = e^{-t\,\mathrm{ad}\,x_1}(h - tx_{-1}) = h + tx_1 - te^{-t\,\mathrm{ad}\,x_1}x_{-1}\\ &= h + t(x_1 - x_{-1}) - t(e^{-t\,\mathrm{ad}\,x_1} - \mathbf{1})x_{-1}.\end{aligned}$$

As $p_{\mathfrak{q}}(h) = 0$, this element is contained in $p_{\mathfrak{q}}^{-1}(C^\circ)$ if and only if this is the case for

$$x_1 - x_{-1} - (e^{t\,\mathrm{ad}\,x_1} - \mathbf{1})x_{-1}.$$

For $t \to 0$, this expression tends to $x_1 - x_{-1} \in C^\circ$, so that, for some $t > 0$, we have $g_t H \in W_M^+(h)$ by (70). □

*Remark 3.5* If we start with a homogeneous space $M = G/H$ and an Euler element $h \in \mathfrak{g}$, the space $M$ carries a $G$-invariant causal structure for which the positivity region $W_M^+(h)$ is non-empty if there exists an open subset $\mathcal{O} \subseteq G$ such that $p_{\mathfrak{q}}(\mathrm{Ad}(H\mathcal{O})h) \subseteq \mathfrak{q}$ is contained in a pointed open convex cone. In fact, the minimal cone $C$ with this property is $\mathrm{Ad}_{\mathfrak{q}}(H)$-invariant, and if it is also generating, it defines a $G$-invariant causal structure on $M$ for which (70) implies that $q_M(\mathcal{O}^{-1}) \subseteq W_M^+(h)$.

This condition depends very much on the geometry of the adjoint orbit $\mathcal{O}_h$, the $H$-action on this orbit and its position with respect to $\mathfrak{h} = \ker p_{\mathfrak{q}}$.

*Remark 3.6* In these notes we consider only homogeneous spaces, which corresponds to the fact that the unitary representations of $G$ we study typically live on the "one-particle space" in QFT. However, this does not cover all geometric contexts encountered in Physics, in particular when it comes to interactions and "$n$-point functions". For instance, if $M$ is a homogeneous causal $G$-space, then $M^n$, the space of $n$-tuples $(m_1, \ldots, m_n)$ of points in $M$ carries a natural diagonal $G$-action and a natural causal structure given by

$$C_{(m_1,\ldots,m_n)} = C_{m_1} \times \cdots \times C_{m_n} \subseteq T_{(m_1,\ldots,m_n)}(M^n) \cong \prod_{j=1}^n T_{m_j}(M).$$

The $G$-action on $M^n$ has no open orbits for large $n$, it is easy to see that the positivity region of $h$ in $M^n$ is

$$W_{M^n}^+(h) = W_M^+(h)^n.$$

To understand how wedge regions look like, we first discuss some simple classes of examples.

### 3.1.1 One-parameter groups on affine causal spaces

To develop the key facts on modular flows on causal homogeneous spaces, we start in this subsection with the case of *causal affine spaces,* i.e., pairs $(E, C)$, where $E$ is a finite-dimensional vector space and $C \subseteq E$ a pointed generating closed convex cone.

Specifically, we consider the following data (cf. [NÓØ21]):

(A1) $E$ is a finite-dimensional real vector space.
(A2) $h \in \mathrm{End}(E)$ is diagonalizable with eigenvalues $\{-1, 0, 1\}$ and $\tau_h := e^{\pi i h}$.
(A3) $C \subseteq E$ is a pointed, generating closed convex cone, invariant under the one-parameter group $e^{\mathbb{R}h}$ and the involution $-\tau_h$.

Writing $E_\lambda = E_\lambda(h) := \ker(h - \lambda \mathbf{1})$ for the $h$-eigenspaces and $E^\pm := \ker(\tau_h \mp \mathbf{1})$ for the $\tau_h$-eigenspaces, (A2) implies

$$E = E_1 \oplus E_0 \oplus E_{-1}, \quad E^- = E_1 \oplus E_{-1}, \quad \text{and} \quad E^+ = E_0. \tag{71}$$

We put $C_\pm := \pm C \cap E_{\pm 1}$. For $x \in E$, we write $x = x_1 + x_0 + x_{-1}$ for the decomposition into $h$-eigenvectors.

**Lemma 3.7** *For the projections*

$$p_{\pm 1} : E \to E_{\pm 1}, x \mapsto x_{\pm 1}, \quad \textit{and} \quad p^- : E \to E_1 \oplus E_{-1} = E^-, x \mapsto x_1 + x_{-1},$$

*the following assertions hold:*

(i) $p_{\pm 1}(C) = \pm C_\pm$ *and* $p_{\pm 1}(C^\circ) = \pm C_\pm^\circ \neq \emptyset$.
(ii) $p^-(C) = C \cap E^- = C_+ \oplus -C_-$ *and* $p^-(C^\circ) = C^\circ \cap E^- = C_+^\circ \oplus -C_-^\circ$.
(iii) $C \subseteq C_+ \oplus E_0 \oplus -C_-$.

***Proof*** (i) From $\pm C_\pm \subset C$, we get $\pm C_\pm \subset p_{\pm 1}(C)$. Using the $e^{th}$-invariance of $C$ and writing $x = x_1 + x_0 + x_{-1}$ as before, $e^{th}x = e^t x_1 + x_0 + e^{-t}x_{-1}$. Now take the limit $t \to \infty$ to see that

$$C \ni e^{-t}e^{th}x = x_1 + e^{-t}x_0 + e^{-2t}x_{-1} \to x_1 \quad \text{as} \quad t \to \infty.$$

We likewise get $x_{-1} = \lim_{t \to -\infty} e^t e^{th} x \in C$. It follows that $x_\pm \in \pm C_\pm$, so that $p_{\pm 1}(C) = \pm C_\pm$. As $p_{\pm 1}$ are projections and $C^\circ \neq \emptyset$, it follows that $p_{\pm 1}(C^\circ) \subseteq \pm C_\pm^\circ$. To obtain equality, it suffices to observe that $C_+^\circ \oplus -C_-^\circ \subseteq (E^- \cap C)^\circ \subseteq C^\circ$ follows from $-\tau_h(C) = C$.
(ii) The two leftmost equalities follow from $-\tau_h(C) = C$, and the second two rightmost equalities from (i) and $p^- = p_1 + p_{-1}$.
(iii) follows from (ii). □

As the linear vector field on $E$ corresponding to $h$ is given by $X_h^E(x) = x_1 - x_{-1}$, Lemma 3.7(ii) implies that its positivity domain is the wedge region

$$W_E^+(h) = C_+^\circ \oplus E_0 \oplus C_-^\circ \quad \text{for} \quad C_\pm = \pm C \cap E_{\pm 1}. \tag{72}$$

In particular, it is not empty. Here $-\tau_h(C) = C$ in (A3) ensures that $C^\circ$ intersects $E^- = \mathrm{im}(h)$. Otherwise we would include cones of the form $C = C_1 + C_0 + C_{-1}$ with $C_j \subseteq E_j$. Any such cone is invariant under $e^{\mathbb{R}h}$, but for such cones $0 \notin C_0^\circ$ implies that $C^\circ \cap (E_{+1} + E_{-1}) = \emptyset$, so that $W_E^+(h) = \emptyset$.

**Example 3.8** (The affine group on $\mathbb{R}$) We endow $M = \mathbb{R}$ with the canonical causal structure given by $C_x = \mathbb{R}_{\geq 0}$ for $x \in \mathbb{R}$. Then the connected *affine group*

$$G = \mathrm{Aff}(\mathbb{R})_e = \mathbb{R} \rtimes \mathbb{R}_+$$

is 2-dimensional. Its elements are denoted $(b, a)$, and they act by the affine, orientation preserving maps $(b, a)x = ax + b$ on the real line.

Here $h = (0, 1) \in \mathfrak{g}$ is an Euler element whose flow is given by $\alpha_t(x) = e^t x$, generated by the vector field $X_h^{\mathbb{R}}(x) = x$. Its positivity region is

$$W_{\mathbb{R}}^+(h) = \{x \in \mathbb{R} : x > 0\} = \mathbb{R}_+$$

and the corresponding reflection is $\tau_h(x) = -x$.

All other Euler elements in $\mathfrak{g}$ are of the form $h' = (x, \pm 1)$, where $\mathcal{O}_h = \mathbb{R} \times \{1\}$ and $\mathcal{O}_{-h} = \mathbb{R} \times \{-1\}$. The corresponding positivity regions are the proper unbounded open intervals in $\mathbb{R}$.

The first example refers also to an affine causal space, but now we enlarge the linear part of the automorphism group.

**Example 3.9** (Poincaré group and Rindler wedges) (see also Example 3.2) The example arising most prominently in physics is the connected *Poincaré group*

$$G := P(d)_+^\uparrow := \mathbb{R}^{1,d-1} \rtimes \mathrm{SO}_{1,d-1}(\mathbb{R})_e.$$

It acts on $d$-dimensional Minkowski space $\mathbb{R}^{1,d-1}$ as an isometry group of the Lorentzian metric given by $\beta(x, y) = x_0 y_0 - \mathbf{x}\mathbf{y}$ for $x = (x_0, \mathbf{x}) \in \mathbb{R}^{1,d-1}$. The $G$-action preserves the constant cone field defined by the closed future light cone

$$C = \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d-1} \colon x_0 \geq \sqrt{\mathbf{x}^2}\}.$$

The generator

$$h(x_0, x_1, x_2, \ldots, x_{d-1}) = (x_1, x_0, 0, \ldots, 0)$$

of the Lorentz boost in the $(x_0, x_1)$-plane is an Euler element in $\mathfrak{so}_{1,d-1}(\mathbb{R})$, and $e^{\pi i h}$ acts by the reflection

$$\tau_h(x) = (-x_0, -x_1, x_2, \ldots, x_{d-1}),$$

for which $-\tau_h(C) = C$. In view of the Classification Theorem 2.12, the fact that the restricted root system of $\mathfrak{so}_{1,d-1}(\mathbb{R})$ is of type $A_1$ implies that there exists only one conjugacy class of Euler elements in $\mathfrak{so}_{1,d-1}(\mathbb{R})$. With Lemma 2.15 it follows that the same holds for the Poincaré–Lie algebra $\mathfrak{g}$ because its center is trivial. So Euler elements in this Lie algebra are precisely the Lorentz boosts in different affine coordinate systems.

By (72), the positivity region of $h$ is

$$W_M^+(h) = \mathbb{R}_+(\mathbf{e}_0+\mathbf{e}_1)-\mathbb{R}_+(\mathbf{e}_0-\mathbf{e}_1)+\mathrm{span}\{\mathbf{e}_2,\ldots,\mathbf{e}_{d-1}\} = \{x \in \mathbb{R}^{1,d-1} : |x_0| < x_1\}.$$

It is called the *standard right wedge* or *Rindler wedge* $W_R$ and plays a key role in AQFT as a localization region for a uniformly accelerated observer, represented by an orbit of the modular flow in $W_R$ ([BGL02, LL15]; see also Remark 1.29).

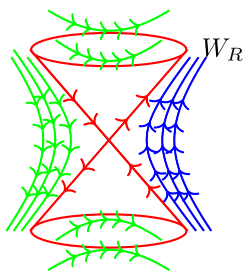



Figure 6: $\mathbb{R}^{1,2}$ with modular flow

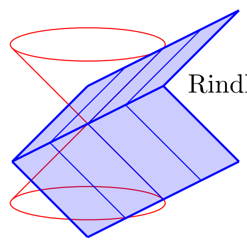



Figure 7: $\mathbb{R}^{1,2}$ with Rindler wege

### 3.1.2 More examples of wedge regions

The following example is the smallest compact one. It is a causal flag manifold. We refer to Section 3.4 for more on this class of examples.

**Example 3.10** (The action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{S}^1$) The group $G := \mathrm{SL}_2(\mathbb{R})$ acts on the one-point compactification $M = \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\} \cong \mathbb{S}^1$ of $\mathbb{R}$ by

$$g.x := \frac{ax+b}{cx+d} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \quad x \in \mathbb{R},$$

and the Lie algebra element $Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ corresponds on $\mathbb{R}$ to the vector field

$$X_Y^{\mathbb{R}}(x) = b + (a-d)x - cx^2.$$

The subgroup $\mathrm{SO}_2(\mathbb{R})$ acts transitively by

$$\rho(t).x := \begin{pmatrix} \cos(t/2) & \sin(t/2) \\ -\sin(t/2) & \cos(t/2) \end{pmatrix}.x = \frac{\cos(t/2)\cdot x + \sin(t/2)}{-\sin(t/2)\cdot x + \cos(t/2)} = \frac{x+\tan(t/2)}{1-\tan(t/2)\cdot x},$$

generated by the everywhere positive vector field (identified with a function on $\mathbb{R} \subseteq \mathbb{R}_\infty$)

$$X^{\mathbb{S}^1}_{z_\mathfrak{k}}(x) = \frac{1}{2}(1+x^2) \quad \text{for} \quad z_\mathfrak{k} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{so}_2(\mathbb{R}) \subseteq \mathfrak{g}.$$

As the period of this flow is $2\pi$, it induces a diffeomorphism $\mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}_\infty$. This shows that the natural causal structure on $\mathbb{R}$ extends to $M$ in a $G$-invariant fashion.

In $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ we consider the Euler element $h = \frac{1}{2}\operatorname{diag}(1,-1)$ (cf. Example 2.8). The flow it generates on $\mathbb{R}_\infty$ is given by $\alpha_t(x) = e^t x$, fixing 0 and $\infty$. Accordingly,

$$W_M^+(h) = \mathbb{R}_+ \subseteq \mathbb{R}_\infty.$$

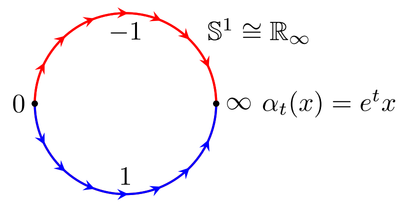



Figure 8: The hyperbolic flow on the circle $\mathbb{R}_\infty \cong \mathbb{S}^1$; $W_M^+(h)$ in blue.

As $G$ acts transitively on the set $\mathcal{O}_h = \mathcal{E}(\mathfrak{sl}_2(\mathbb{R}))$ of Euler elements in $\mathfrak{sl}_2(\mathbb{R})$ (Example 2.8), and the positivity regions satisfy the equivariance relation $W^+_{\mathbb{S}^1}(\operatorname{Ad}(g)h) = g.W^+_{\mathbb{S}^1}(h)$, their positivity regions in $\mathbb{S}^1$ are precisely the non-dense open intervals.

The Cayley transform

$$C : \mathbb{R}_\infty \to \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}, \quad C(x) := \frac{i-x}{i+x} = \frac{1+ix}{1-ix}, \qquad C(\infty) := -1,$$

is a homeomorphism, identifying $\mathbb{R}_\infty$ with the unit circle in $\mathbb{C}$. Its inverse is

$$C^{-1} : \mathbb{S}^1 \to \mathbb{R}_\infty, \quad C^{-1}(z) = i\frac{1-z}{1+z}$$

(cf. Exercise 3.54). It maps the upper semicircle $\mathbb{S}^1_+ = \{z \in \mathbb{S}^1 \colon \operatorname{Im} z > 0\}$ to the positive half-line $\mathbb{R}_+$. The Cayley transform intertwines the action of $\mathrm{SL}_2(\mathbb{R})$ with the action of $\mathrm{SU}_{1,1}(\mathbb{C})$ on the circle $\mathbb{S}^1 \subseteq \mathbb{C}$ by fractional linear transformations. This action preserves the causal structure on $\mathbb{S}^1$, specified by $C_z = i\mathbb{R}_{\geq 0} z \subseteq T_z(\mathbb{S}^1) = i\mathbb{R}z$ for $z \in \mathbb{S}^1$.

**Example 3.11** (cf. also Examples 1.28) The Lie group $G := \mathrm{SL}_2(\mathbb{R})$ has three classes of causal homogeneous spaces. In Example 3.10 we have already seen its action on the 1-dimensional circle $\mathbb{S}^1$, which is the flag manifold $\mathbb{P}(\mathbb{R}^2)$ of 1-dimensional linear subspaces of $\mathbb{R}^2$, endowed with the natural action of $\mathrm{SL}_2(\mathbb{R})$.

Observing that $\mathrm{Ad}(\mathrm{SL}_2(\mathbb{R})) \cong \mathrm{SO}_{1,2}(\mathbb{R})_e$ (Exercise 3.58), we obtain two other examples:

- Two-dimensional *de Sitter space*

$$\mathrm{dS}^2 = \{(x_0, x_1, x_2) \in \mathbb{R}^{1,2} : x_0^2 - x_1^2 - x_2^2 = -1\}$$

carries an $\mathrm{SO}_{1,2}(\mathbb{R})_e$-invariant causal structure with the positive cone in the base point $\mathbf{e}_1$ given by

$$C_{\mathbf{e}_1} := \{(x_0, 0, x_2) : x_0 \geq |x_2|\} \subseteq T_{\mathbf{e}_1}(\mathrm{dS}^2) = \mathbb{R}\mathbf{e}_0 + \mathbb{R}\mathbf{e}_2.$$

The inversion $-\mathbf{1}$ on $\mathrm{dS}^2$ is an anti-causal map. For the Euler element defined by $h(x_0, x_1, x_2) = (x_1, x_0, 0)$, we obtain the connected wedge region

$$W^+_{\mathrm{dS}^2}(h) = \{(x_0, x_1, x_2) \in \mathrm{dS}^2 : x_1 > |x_0|\}.$$

The wedge region $W := W^+_{\mathrm{dS}^2}(h)$ and the orbits of the modular flow in $W$ are displayed in Figure 9.

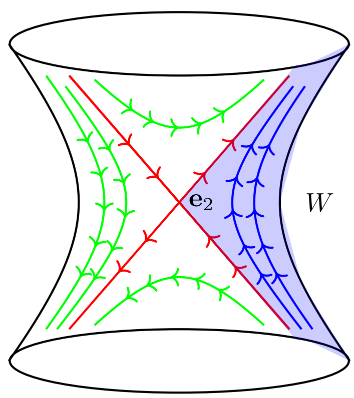



Figure 9:
2-dimensional de Sitter space $\mathrm{dS}^2$. Red lines are specified by $x_2 = 1$ and arrows indicate the modular flow. The blue ones are positively causal and lie in $W_+^{\mathrm{dS}^2}(h)$. The green ones are either negative timelike or spacelike, and the red ones lightlike.

- Two-dimensional *anti-de Sitter space*

$$\mathrm{AdS}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^{2,1} : x_1^2 + x_2^2 - x_3^2 = 1\}$$

carries an $\mathrm{SO}_{2,1}(\mathbb{R})_e$-invariant causal structure with the positive cone in the base point $\mathbf{e}_2$ given by

$$C_{\mathbf{e}_2} := \{(x_1, 0, x_3) : x_1 \geq |x_3|\} \subseteq T_{\mathbf{e}_2}(\mathrm{AdS}^2) = \mathbb{R}\mathbf{e}_1 + \mathbb{R}\mathbf{e}_3.$$

It is uniquely determined by the vector field

$$X^{\mathrm{AdS}^2}_{z_{\mathfrak{k}}}(x_1, x_2, x_3) = (x_2, -x_1, 0)$$

being everywhere positive timelike. The inversion $-\mathbf{1}$ on $\mathrm{AdS}^2$ is a causal map. For the Euler element defined by $X_h^{\mathrm{AdS}^2}(x_1,x_2,x_3)=(x_3,0,x_1)$, we obtain the positivity region

$$W^+_{\mathrm{AdS}^2}(h)=\{(x_1,x_2,x_3)\in \mathrm{AdS}^2\colon x_2x_3>0, |x_1|<|x_3|\}.$$

It has two connected components, specified by the sign of $x_2$ ([NÓ23a, Lem. 11.3]). In view of $x_2^2-1=x_3^2-x_1^2$, the relation $|x_1|<|x_3|$, which is equivalent to $X_h^{\mathrm{AdS}^2}(x)$ being timelike, is equivalent to $|x_2|>1$. This region has four connected components, and $x_2x_3>0$ selects the two on which it is positive timelike. These two components of $W^+_{\mathrm{AdS}^2}(h)$ are exchanged by the inversion $-\mathbf{1}$.

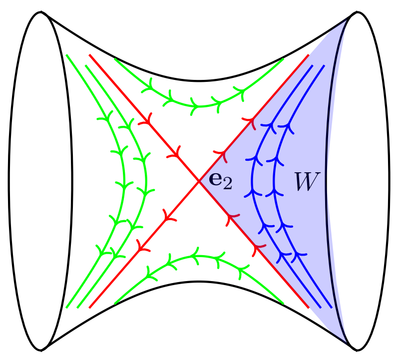



Figure 10: 2-dimensional anti-de Sitter space $\mathrm{AdS}^2$.

Red lines are specified by $x_2=1$ and arrows indicate the modular flow. The blue ones are positively causal and lie in one component of $W_+^{\mathrm{AdS}^2}(h)$. The green ones are either negative timelike or spacelike, and the red ones lightlike.

As homogeneous spaces, $\mathrm{AdS}^2$ and $\mathrm{dS}^2$ can be identified with the adjoint orbit $\mathcal{O}_h\cong G/G^h\cong \mathcal{E}(\mathfrak{sl}_2(\mathbb{R}))$, where $h=\frac{1}{2}\operatorname{diag}(1,-1)$ is an Euler element in $\mathfrak{sl}_2(\mathbb{R})$ (cf. Example 2.8). However, both carry non-isomorphic causal structures, where $\mathrm{AdS}^2$ admits closed causal curves and $\mathrm{dS}^2$ does not.

## 3.2 The compression semigroup of a wedge region

Let $M=G/H$ be a causal homogeneous space with causal structure given by the cone field $(C_m)_{m\in M}$. Writing $X_y^M(m)=\frac{d}{dt}\big|_{t=0}\exp(ty).m\in T_m(M)$, for $y\in\mathfrak{g}$, the set

$$C_M:=\{y\in\mathfrak{g}\colon (\forall m\in M)X_y^M(m)\in C_m\}\overset{(69)}{=}\bigcap_{g\in G}\mathrm{Ad}(g)p_{\mathfrak{q}}^{-1}(C) \tag{73}$$

of those Lie algebra elements whose vector fields on $M$ are everywhere positive is a closed convex $\mathrm{Ad}(G)$-invariant cone in $\mathfrak{g}$. If $G$ acts effectively on $M$, i.e., any $g\neq e$ acts non-trivially on $M$, then it is also pointed because elements in $C_M\cap -C_M$ correspond to vanishing vector fields. This cone is a geometric analog of the positive cone $C_U$ of a unitary representation of $G$ (see (151)).[12] The following observation shows that it behaves in many respects similarly (cf. [Ne22]).

**Examples 3.12** We consider the action of $G = \mathrm{SL}_2(\mathbb{R})$ on $\mathbb{S}^1$, $\mathrm{dS}^2$ and $\mathrm{AdS}^2$ and recall that every non-trivial invariant cone $C \subseteq \mathfrak{sl}_2(\mathbb{R})$ either contains $z_{\mathfrak{k}}$ or $-z_{\mathfrak{k}}$ (cf. (174) in Appendix 7.2.4). For $\mathbb{S}^1$ and $\mathrm{AdS}^2$, the flow generated by $z_{\mathfrak{k}} \in \mathfrak{so}_2(\mathbb{R})$ is everywhere positive timelike, so that the invariant cones $C_{\mathbb{S}^1}$ and $C_{\mathrm{AdS}^2}$ in $\mathfrak{sl}_2(\mathbb{R})$ are non-trivial, containing $z_{\mathfrak{k}}$. On de Sitter space $\mathrm{dS}^2$, the flow generated by $z_{\mathfrak{k}}$ is spacelike, so that $C_{\mathrm{dS}^2} = \{0\}$.

As any connected component $W \subseteq W_M^+(h) \subseteq M$ is invariant under $G_e^h \supseteq \exp(\mathbb{R}h)$,[13] the same holds for the closed convex cone

$$C_W := \{y \in \mathfrak{g} \colon (\forall m \in W)\ X_y^M(m) \in C_m\} \supseteq C_M. \qquad (74)$$

Below we show that this cone determines the tangent wedge of the compression semigroup $S_W$ of $W$.

**Proposition 3.13** *For a connected component $W \subseteq W_M^+(h)$, its compression semigroup*

$$S_W := \{g \in G \colon g.W \subseteq W\}$$

*is a closed subsemigroup of $G$ with $G_W := S_W \cap S_W^{-1} \supseteq G_e^h$ and*

$$\mathbf{L}(S_W) := \{x \in \mathfrak{g} \colon \exp(\mathbb{R}_+ x) \subseteq S_W\} = \mathfrak{g}_0(h) + C_{W,+} + C_{W,-},$$

*with*

$$C_{W,\pm} := \pm C_W \cap \mathfrak{g}_{\pm 1}(h).$$

*In particular, the convex cone $\mathbf{L}(S_W)$ has interior points if $C_M$ does.*

***Proof*** As $W \subseteq M$ is an open subset, its complement $W^c := M \setminus W$ is closed, and thus

$$S_W = \{g \in G \colon g^{-1}.W^c \subseteq W^c\}$$

is a closed subsemigroup of $G$, so that its tangent wedge $\mathbf{L}(S_W)$ is a closed convex cone in $\mathfrak{g}$ ([HHL89], [HN93, §1.4]).

Let $m = gH \in W$, so that $p_{\mathfrak{q}}(\mathrm{Ad}(g)^{-1}h) \in C^\circ$. For $x \in \mathfrak{g}_{\pm 1}(h)$ we then derive from $\mathfrak{g}_{\pm 2}(h) = \{0\}$ that

$$e^{\mathrm{ad}\, x} h = h + [x, h] = h \mp x.$$

This leads to

[12] Note that the existence of a pointed generating invariant cone in a Lie algebra $\mathfrak{g}$ has strong structural implications (cf. [Ne00]). If, f.i., $\mathfrak{g}$ is simple, then it must be hermitian.

[13] Recall that $G^h = \{g \in G \colon \mathrm{Ad}(g)h = h\}$.

$$\begin{aligned} p_{\mathfrak{q}}(\mathrm{Ad}(\exp(x)g)^{-1}h) &= p_{\mathfrak{q}}(\mathrm{Ad}(g)^{-1}e^{-\operatorname{ad} x}h) = p_{\mathfrak{q}}(\mathrm{Ad}(g)^{-1}(h \pm x)) \\ &= p_{\mathfrak{q}}(\mathrm{Ad}(g)^{-1}h) \pm p_{\mathfrak{q}}(\mathrm{Ad}(g)^{-1}x). \end{aligned}$$

For $x \in C_{W,\pm}$, we have $p_{\mathfrak{q}}(\pm \mathrm{Ad}(g)^{-1}x) \in C$, so that $p_{\mathfrak{q}}(\mathrm{Ad}(\exp(x)g)^{-1}h) \in C^\circ$, which in turn implies that $\exp(x).m \in W$ for $m \in W$. So $\exp(C_{W,\pm}) \subseteq S_W$, and thus $C_{W,\pm} \subseteq \mathbf{L}(S_W)$. The invariance of $W$ under the identity component $G_e^h$ of the centralizer of $h$ further entails $\mathfrak{g}_0(h) \subseteq \mathbf{L}(S_W)$, so that

$$C_{W,+} + \mathfrak{g}_0(h) + C_{W,-} \subseteq \mathbf{L}(S_W). \tag{75}$$

We now prove the converse inclusion. Let $x \in \mathfrak{g}_1(h)$. If $X_x^M(m) \notin C_m$ for $m = gH$, i.e., $p_{\mathfrak{q}}(\mathrm{Ad}(g)^{-1}x) \notin C$, then there exists a $t_0 > 0$ with

$$p_{\mathfrak{q}}(\mathrm{Ad}(g)^{-1}h) + t_0 \cdot p_{\mathfrak{q}}(\mathrm{Ad}(g)^{-1}x) \notin C$$

([Ne00, Prop. V.1.6]), so that $\exp(t_0 x).m \notin W$. We conclude that

$$\mathbf{L}(S_W) \cap \mathfrak{g}_1(h) = C_{W,+}.$$

Further, the invariance of the closed convex cone $\mathbf{L}(S_W)$ under $e^{\mathbb{R} \operatorname{ad} h}$ implies that, for

$$x = x_{-1} + x_0 + x_1 \in \mathbf{L}(S_W) \quad \text{and} \quad x_j \in \mathfrak{g}_j(h),$$

we have

$$x_{\pm 1} = \lim_{t \to \infty} e^{-t} e^{\pm t \operatorname{ad} h} x \in \mathbf{L}(S_W) \cap \mathfrak{g}_{\pm 1}(h) = C_{W,\pm},$$

which implies the other inclusion $\mathbf{L}(S_W) \subseteq C_{W,+} + \mathfrak{g}_0(h) + C_{W,-}$, hence equality by (75).

Let $p_\pm \colon \mathfrak{g} \to \mathfrak{g}_{\pm 1}(h)$ denote the projection along the other eigenspaces of $\operatorname{ad} h$. Then

$$C_{W,\pm} \supseteq C_{M,\pm} := \pm C_M \cap \mathfrak{g}_{\pm 1}(h) = \pm p_\pm(C_M)$$

also follows from Lemma 3.7. Therefore $C_M^\circ \neq \emptyset$ implies $C_{W,\pm}^\circ \neq \emptyset$, and this is equivalent to $\mathbf{L}(S_W)^\circ \neq \emptyset$. □

*Remark 3.14* In many situations, such as the action of $\mathrm{PSL}_2(\mathbb{R})$ on the circle $\mathbb{S}^1 \cong \mathbb{P}_1(\mathbb{R})$ (Example 3.10), the cones $C_{W,\pm} \supseteq C_{M,\pm}$ coincide, and we believe that this is probably always the case. If $x \in C_{W,+}$, then the positivity region

$$\Omega_x := \{m \in M \colon X_x^M(m) \in C_m\}$$

contains $W$ (by definition). With the notation

$$\mathfrak{n}_{\mathfrak{g}}(\mathbb{R}x) = \{y \in \mathfrak{g} \colon [y, x] \subseteq \mathbb{R}x\},$$

we see that it is also invariant under the identity component $N_x = \langle \exp \mathfrak{n}_{\mathfrak{g}}(\mathbb{R}x) \rangle$ of the normalizer $N_x = \{g \in G \colon \mathrm{Ad}(g)x \in \mathbb{R}x\}$ of $\mathbb{R}x$, so that

$$\Omega_x \supseteq N_x.W \supseteq \bigcup_{t>0} \exp(-tx).W. \qquad (76)$$

Clearly, $\Omega_x = M$ follows if $N_x.W$ is dense in $M$, and we are not aware of examples, for which this is not the case.

**The Rindler wedge in Minkowski space.** Let $G = P(d)_e$ be the identity component of the *Poincaré group*

$$P(d) := \mathbb{R}^{1,d-1} \rtimes \mathrm{O}_{1,d-1}(\mathbb{R}),$$

acting on Minkowski space $M = \mathbb{R}^{1,d}$, and $h \in \mathfrak{g}$ the Euler element corresponding to the Lorentz boost in the $(\mathbf{e}_0, \mathbf{e}_1)$-plane with wedge region

$$W = W_R = \{x \in \mathbb{R}^{1,d-1} \colon x_1 > |x_0|\}$$

(Example 3.9). The corresponding reflection is $\tau_h = \mathrm{diag}(-1, -1, 1, \ldots, 1)$. Then Lemma 3.15 below asserts that

$$S_W = \overline{W} \rtimes \big(\mathrm{SO}_{d-1}(\mathbb{R}) \times \mathrm{SO}_{1,1}(\mathbb{R})^{\uparrow}\big),$$

so that

$$C_{W,\pm} = \mathbf{L}(S_W) \cap \mathfrak{g}_1(h) = \mathbb{R}_+(\pm\mathbf{e}_0 + \mathbf{e}_1)$$

consists of constant vector fields, and thus $C_{W,\pm} = C_{M,\pm}$ holds in this case. For the constant vector field $x = \mathbf{e}_0 + \mathbf{e}_1 \in C_{W,+}$, the domain $W - \mathbb{R}_+ x$ is an open half space, hence in particular not dense in $M$, but $N_x.W \supseteq W + \mathbb{R}^{1,d} = \mathbb{R}^{1,d}$ is.

**Lemma 3.15** *The stabilizer group of $W_R$ is*

$$G_{W_R} \cong (E_R \rtimes \mathrm{SO}_{d-2}(\mathbb{R})) \times \mathrm{SO}_{1,1}(\mathbb{R})_e, \qquad (77)$$

*where $E_R \rtimes \mathrm{SO}_{d-2}(\mathbb{R})$ denotes the connected group of proper euclidean motions on the edge*

$$E_R := \mathrm{span}\{\mathbf{e}_2, \ldots, \mathbf{e}_{d-1}\} \cong \mathbb{R}^{d-2}$$

*of $W_R$, and $\mathrm{SO}_{1,1}(\mathbb{R})$ acts on $\mathrm{span}\{\mathbf{e}_0, \mathbf{e}_1\}$. The compression semigroup of $W_R$ is*

$$S_{W_R} := \{g \in P(d) \colon gW_R \subseteq W_R\} = \overline{W_R} \rtimes \mathrm{O}_{1,d-1}(\mathbb{R})_{W_R}.$$

***Proof*** The stabilizer group $P(d)_{W_R}$ contains the translation group corresponding to the edge $E_R$, and $gW_R = W_R$ implies $g(0) \in E_R$, so that

$$P(d)_{W_R} \cong E_R \rtimes \mathrm{O}_{1,d-1}(\mathbb{R})_{W_R}.$$

Further, each $g \in \mathrm{O}_{1,d-1}(\mathbb{R})$ preserving $E_R$ also preserves its orthogonal complement, so that

$$\mathrm{O}_{1,d-1}(\mathbb{R})_{W_R} = \mathrm{O}_{1,1}(\mathbb{R})_{W_R} \times \mathrm{O}_{d-2}(\mathbb{R}) = (\mathrm{SO}_{1,1}(\mathbb{R})_e\{\mathbf{1}, r_1\}) \times \mathrm{O}_{d-2}(\mathbb{R}),$$

where $r_1 = \mathrm{diag}(1, -1, 1, \ldots, 1)$.

Next we use Lemma 7.32 in Appendix 7.7 to see that

$$S_{W_R} = \overline{W_R} \rtimes \{g \in \mathrm{SO}_{1,d-1}(\mathbb{R})_e \colon gW_R \subseteq W_R\}.$$

Any $g \in \mathrm{SO}_{1,d-1}(\mathbb{R})_e$ with $gW_R \subseteq W_R$ satisfies $gE_R = E_R$ because $g$ is injective and $\dim E_R < \infty$. This in turn implies that $g$ commutes with $\tau_h = \mathrm{diag}(-1, -1, 1, \ldots, 1)$, so that $g = g_1 \oplus g_2$ with $g_1 \in \mathrm{O}_{1,1}(\mathbb{R})$ preserving the wedge region $W_R^2 \subseteq \mathbb{R}^{1,1} = \mathrm{span}\{\mathbf{e}_0, \mathbf{e}_1\}$. As $g_1 W_R^2$ is a quarter plane bounded by light rays, it cannot be strictly smaller than $W_R^2$, hence $g_1 W_R^2 = W_R^2$, and finally $gW_R = W_R$. This completes the proof. □

### 3.3 Causal Lie groups

This this subsection, we discuss the most structured examples of causal homogeneous spaces, i.e., causal groups with a biinvariant causal structure.

Let $G$ be a connected Lie group and $C_{\mathfrak{g}} \subseteq \mathfrak{g}$ be a pointed generating closed convex cone. Then $C_g := g.C_{\mathfrak{g}} \subseteq T_g(G)$ defines on $G$ a left-invariant causal structure. Here the notation $g.x = T(\lambda_g)x$ refers to the canonical left translation action of $G$ on $TG$. We likewise write $v.g \in T(G)$ for the right translation of a tangent vector $v$ by $g$. [14]

These structures become more interesting if $C_{\mathfrak{g}}$ is invariant under $\mathrm{Ad}(G)$, so that the action of $G \times G$ by $(g_1, g_2).g = g_1 g g_2^{-1}$ preserves the causal structure. [15]

If $h_0 \in \mathfrak{g}$ is an Euler element, then $h := (h_0, h_0) \in \mathfrak{g}^{\oplus 2}$ is Euler as well. It generates the flow

$$\alpha_t(g) = \exp(th_0) g \exp(-th_0).$$

The corresponding vector field is

$$X_h^G(g) = \frac{d}{dt}\Big|_{t=0} \exp(th_0) g \exp(-th_0) = h_0.g - g.h_0 = g.(\mathrm{Ad}(g)^{-1}h_0 - h_0),$$

so that

[14] Note that $T(G)$ actually carries a Lie group structure, and that the map $(\mathfrak{g}, +) \rtimes_{\mathrm{Ad}} G \to T(G), (x, g) \mapsto g.x$ is an isomorphism of Lie groups. Here $T_e(G) \cong (\mathfrak{g}, +)$ is an abelian nornal subgroup and the 0-section is a subgroup isomorphic to $G$.

[15] That a $G$ action on $M$ preserves the causal structure $(C_m)_{m \in M}$ means that $g.C_m = C_{g.m}$ for $g \in G, m \in M$.

$$W_G^+(h) = \{g \in G\colon \operatorname{Ad}(g)^{-1}h_0 - h_0 \in C_{\mathfrak{g}}^\circ\} = \{g \in G\colon \operatorname{Ad}(g)h_0 - h_0 \in -C_{\mathfrak{g}}^\circ\}. \tag{78}$$

Here the second equality follows from the invariance of the open cone $C_{\mathfrak{g}}^\circ$ under $\operatorname{Ad}(G)$. We also note that $W_G^+(h)$ is an open subsemigroup of $G$, contained in the closed subsemigroup

$$S(h_0, C_{\mathfrak{g}}) := \{g \in G\colon h_0 - \operatorname{Ad}(g)h_0 \in C_{\mathfrak{g}}\}. \tag{79}$$

The semigroup property follows from the fact that $\operatorname{Ad}(g_j)^{-1}h_0 - h_0 \in C_{\mathfrak{g}}$ for $j = 1, 2$ implies that

$$\begin{aligned} \operatorname{Ad}(g_1g_2)^{-1}h_0 - h_0 &= \operatorname{Ad}(g_2)^{-1}(\operatorname{Ad}(g_1)^{-1}h_0 - h_0) + \operatorname{Ad}(g_2)^{-1}h_0 - h_0 \\ &\in C_{\mathfrak{g}} + C_{\mathfrak{g}} \subseteq C_{\mathfrak{g}}. \end{aligned}$$

For the $G$-invariant order structure on $\mathfrak{g}$, defined by

$$x \leq_{C_{\mathfrak{g}}} y \quad \text{if} \quad y - x \in C_{\mathfrak{g}},$$

we have

$$S(h_0, C_{\mathfrak{g}}) = \{g \in G\colon \operatorname{Ad}(g)h_0 \leq_{C_{\mathfrak{g}}} h_0\}.$$

We likewise have for the strict order, defined by

$$x <_{C_{\mathfrak{g}}} y \quad \text{if} \quad y - x \in C_{\mathfrak{g}}^\circ$$

that

$$W_G^+(h) = \{g \in G\colon \operatorname{Ad}(g)h_0 <_{C_{\mathfrak{g}}} h_0\}.$$

To describe this domain, we need the two pointed generating $\operatorname{Ad}(G^{h_0})$-invariant cones

$$C_\pm = \pm C_{\mathfrak{g}} \cap \mathfrak{g}_{\pm 1} \tag{80}$$

(cf. Lemma 3.7).

We claim that, if $-\tau_g(C_{\mathfrak{g}}) = C_{\mathfrak{g}}$, then

$$\exp(C_+^\circ)G^{h_0}\exp(C_-^\circ) \subseteq W_G^+(h_0). \tag{81}$$

By passing to the closure, this implies

$$\exp(C_+)G^{h_0}\exp(C_-) \subseteq S(h_0, C_{\mathfrak{g}}). \tag{82}$$

As the centralizer $G^{h_0}$ of $h_0$ is obviously contained in $S(h_0, C_{\mathfrak{g}})$ and $\exp(C_+)G^h = G^h\exp(C_+)$, it suffices to show that $\exp(C_+^\circ)\exp(C_-^\circ) \subseteq W_G^+(h_0)$. For $x_\pm \in C_\pm^\circ$, this follows from

$$\begin{aligned}
&e^{\operatorname{ad} x_+} e^{\operatorname{ad} x_-} h_0 - h_0 = e^{\operatorname{ad} x_+}(h_0 + [x_-, h_0]) - h_0 = e^{\operatorname{ad} x_+}(h_0 + x_-) - h_0 \\
&= [x_+, h_0] + e^{\operatorname{ad} x_+} x_- = -x_+ + e^{\operatorname{ad} x_+} x_- = e^{\operatorname{ad} x_+}(x_- - x_+) \\
&\in -e^{\operatorname{ad} x_+}(C_+^\circ - C_-^\circ) \subseteq -C_{\mathfrak g}^\circ
\end{aligned}$$

(cf. the proof of Proposition 3.4). Here we used $-\tau_h(C_{\mathfrak g}) = C_{\mathfrak g}$ for the inclusion $C_+^\circ - C_-^\circ \subseteq C_{\mathfrak g}^\circ$ (Lemma 3.7(ii)).

**Complex Olshanski semigroups.** To motivate the general concept of a complex Olshanski semigroup, introduced below, we note that, if $C_{\mathfrak g}$ has interior points, then the open convex cone

$$\mathfrak g + iC_{\mathfrak g}^\circ \subseteq \mathfrak g_{\mathbb C}$$

is a *tube domain* in the complex vector space $\mathfrak g_{\mathbb C}$. Replacing $\mathfrak g_{\mathbb C}$ by a corresponding complex Lie group $G_{\mathbb C}$, containing a connected subgroup $G$ with Lie algebra $\mathfrak g$, a natural "non-commutative" variant of a tube domain is the subset

$$G\exp(iC_{\mathfrak g}^\circ),$$

which is always open. It turns out that, under mild conditions on $G$, the subset $S := G\exp(iC_{\mathfrak g})$ is a closed subsemigroup of $G_{\mathbb C}$ with tangent wedge

$$\mathbf L(S) = \{z \in \mathfrak g_{\mathbb C} \colon \exp(\mathbb R_+ z) \subseteq S\} = \mathfrak g + iC_{\mathfrak g}.$$

A concrete example is the subsemigroup

$$S = \{g \in \mathrm{GL}_n(\mathbb C) \colon \|g\| \leq 1\} = \mathrm U_n(\mathbb C)\exp(\{X \in \mathrm{Herm}_n(\mathbb C) \colon X \leq 0\})$$

of invertible contractions on $\mathbb C^n$. Here is the formal definition.

**Definition 3.16** (Complex Olshanski semigroups) Let $C_{\mathfrak g} \subseteq \mathfrak g$ be a pointed generating closed convex $\mathrm{Ad}(G)$-invariant cone. If $G$ is simply connected and $\eta_G \colon G \to G_{\mathbb C}$ its universal complexification, then $\eta_G(G)\exp(iC_{\mathfrak g})$ is a closed subsemigroup of $G_{\mathbb C}$ for which the polar mapp

$$\eta_G(G) \times C_{\mathfrak g} \to \eta_G(G)\exp(iC_{\mathfrak g}), \quad (g, x) \mapsto g\exp(ix)$$

is a homeomorphism and a diffeomorphism from $\eta_G(G) \times C_{\mathfrak g}^\circ$ onto the interior. We refer to [Ne00, §IX.1] or [HN93, 3.20] for proofs and details on such semigroups. We then define the *complex Olshanski semigroup* $S_G(iC_{\mathfrak g})$ as the simply connected covering of $\eta_G(G)\exp(iC_{\mathfrak g})$.

If $G$ is not simply connected and $q_G \colon \widetilde G \to G$ its simply connected covering group, then we put

$$S_G(iC_{\mathfrak g}) := S_{\widetilde G}(iC_{\mathfrak g})/\Gamma,$$

where $\Gamma \subseteq Z(\widetilde G)$ is the discrete kernel of the covering map $q_G$, which acts by multiplication on $S_{\widetilde G}(iC_{\mathfrak g})$.

By restricting the exponential function $\mathfrak{g}_{\mathbb{C}} \to G_{\mathbb{C}}$, we obtain by lifting to $S_{\widetilde{G}}(iC_{\mathfrak{g}})$ an exponential map

$$\mathrm{Exp}\colon \mathfrak{g} + iC_{\mathfrak{g}} \to S_G(iC_{\mathfrak{g}}),$$

corresponding in this context to the exponential function, even if $S_G(iC_{\mathfrak{g}})$ is not contained in any Lie group. It may be a covering or a quotient of a subsemigroup of a Lie group. In all cases the map

$$G \times C_{\mathfrak{g}} \to S_G(iC_{\mathfrak{g}}), \quad (g, x) \mapsto g\,\mathrm{Exp}(ix)$$

is a homeomorphism and a diffeomorphism on the interior.

**Definition 3.17** (The semigroups of KMS points) Assume that $\tau_h^{\mathfrak{g}} = e^{\pi i \,\mathrm{ad}\, h}$ integrates to an automorphism $\tau_h$ of $G$ and that $-\tau_h^{\mathfrak{g}}(C_{\mathfrak{g}}) = C_{\mathfrak{g}}$. Using the complex Olshanski semigroup $S_G(iC_{\mathfrak{g}})$, we define the subsemigroup

$$G_{\mathrm{KMS}} \subseteq G$$

*of KMS points in* $G$ as the set of those elements $g \in G$ for which the orbit map

$$\alpha^g\colon \mathbb{R} \to G, \quad \alpha^g(t) = \alpha_t(g)$$

extends analytically to a map $\overline{\mathcal{S}_\pi} \to S(iC_{\mathfrak{g}})$ on the closed strip, such that

$$\alpha^g(\mathcal{S}_\pi) \subseteq S(iC_{\mathfrak{g}}^\circ) \quad \text{and} \quad \alpha^g(\pi i) = \tau_h(g).$$

We refer to [Ne22, §2.4] for a detailed discussion and to Definition 3.49 below for a generalization. This concept is inspired by the KMS vectors from Definition 1.41.

**Theorem 3.18** *If $G$ is simply connected, $h \in \mathfrak{g}$ an Euler element, and $C_{\mathfrak{g}} \subseteq \mathfrak{g}$ a pointed closed convex invariant cone with $-\tau_h^{\mathfrak{g}}(C_{\mathfrak{g}}) = C_{\mathfrak{g}}$, then*

$$S(h, C_{\mathfrak{g}}) = \exp(C_+)G^h \exp(C_-) = G^h \exp(C_+ + C_-), \tag{83}$$

*the positivity domain*

$$W_G^+(h) = \exp(C_+^\circ)G^h \exp(C_-^\circ) = S(h, C_{\mathfrak{g}})^\circ$$

*is a subsemigroup, and*

$$G_{\mathrm{KMS}} = \exp(C_+^\circ)G_e^h \exp(C_-^\circ) = G_e^h \exp(C_+^\circ + C_-^\circ) = S(h, C_{\mathfrak{g}})_e^\circ$$

*is a connected component of $W_G^+(h)$.*

***Proof*** The first two equalities in (83) are the Decomposition Theorem [Ne22, Thm. 2.16]. Further [Ne22, Thm. 2.21] shows that $S(h, C_{\mathfrak{g}})$ coincides with the set of all $g \in G$ for which $\alpha^g$ extends to a map $\overline{\mathcal{S}_\pi} \to S(iC_{\mathfrak{g}})$.

Next we show that the additional requirement that $\alpha^g(\mathcal{S}_\pi) \subseteq S(iC_\mathfrak{g}^\circ)$ specifies the open subset $G^h \exp(C_+^\circ + C_-^\circ) = S(h, C_\mathfrak{g})^\circ$. For $g = g_0 \exp(x_1 + x_{-1})$ with $x_{\pm 1} \in C_\pm$, we have

$$\alpha^g(z) = g_0 \operatorname{Exp}(e^z x_1 + e^{-z} x_{-1}).$$

For $z = a + ib$ with $0 < b < \pi$, we have for $x_{\pm 1} \in C_\pm^\circ$

$$\operatorname{Im}(e^z x_1 + e^{-z} x_{-1}) = \sin(b)(x_1 - x_{-1}) \in (C_+ - C_-)^\circ.$$

This shows that

$$G_e^h \exp(C_+^\circ + C_-^\circ) = S(h, C_\mathfrak{g})_e^\circ = \exp(C_+^\circ) G_e^h \exp(C_-^\circ) \subseteq G_{\mathrm{KMS}}.$$

Here we used that, for $g = g_0 \exp(x_1 + x_{-1})$, we have

$$\tau_h(g) = \tau_h(g_0) \exp(\tau_h^\mathfrak{g}(x_1 + x_{-1})) = \tau_h(g_0) \exp(-x_1 - x_{-1}),$$

so that we find for $G_{\mathrm{KMS}}$ the additional condition that $g_0 \in G^{\tau_h} = G_e^h$ (cf. [Ne22, Cor. 2.22]).

If, conversely, $x_{\pm 1} \in C_\pm$ and $\alpha^g(\pi i/2) = g_0 \operatorname{Exp}(i(x_1 - x_{-1})) \in S(iC_\mathfrak{g}^\circ)$, then

$$x_1 - x_{-1} \in C_\mathfrak{g}^\circ \cap \mathfrak{g}^{-\tau_h} = C_+^\circ - C_-^\circ$$

(Lemma 3.7). □

*Remark 3.19* For the antiholomorphic extension $\overline{\tau}_h$ of $\tau_h$ to the complex semigroup $S(iC_\mathfrak{g})$, the fixed point set

$$S(iC_\mathfrak{g})^{\overline{\tau}_h} = G^{\tau_h} \operatorname{Exp}(iC_\mathfrak{g}^{-\tau_h}) = G^{\tau_h} \operatorname{Exp}(i(C_+ - C_-))$$

is a real Olshanski semigroup in the c-dual group $G^c$ (with respect to $\tau_h$) with Lie algebra $\mathfrak{g}^c = \mathfrak{g}_0 + i\mathfrak{g}^{-\tau_h} = \mathfrak{g}^{\tau_h} + i\mathfrak{g}^{-\tau_h}$ (see [HN93, §7.3] for more on real Olshanski semigroups). The invariance condition $-\tau_h^\mathfrak{g}(C_\mathfrak{g}) = C_\mathfrak{g}$ implies that $C_\mathfrak{g}^{-\tau_h} = C_+ - C_-$ has interior points (cf. Lemma 3.7).

*Remark 3.20* In the context of causal Lie groups, specified by a pair $(G, C_\mathfrak{g})$ as above, $\mathfrak{g}$ may not contain an Euler element, but there may be an *Euler derivation* $D \in \operatorname{der}(\mathfrak{g})$, i.e., $D$ is diagonalizable with eigenvalues contained in $\{-1, 0, 1\}$ (see Example 2.17, and Example 2.7 for the case where $G = E$ is a vector space). Then $\tau_D := e^{\pi i D}$ defines an involutive automorphism of $\mathfrak{g}$, and compatibility with the causal structure corresponds to the requirements

$$e^{\mathbb{R}D} C_\mathfrak{g} = C_\mathfrak{g} \quad \text{and} \quad -\tau_D C_\mathfrak{g} = C_\mathfrak{g}. \tag{84}$$

To implement a modular flow on $G$, we assume that all automorphisms $\alpha_t^\mathfrak{g} := e^{tD}$ of $\mathfrak{g}$ integrate to automorphisms $\alpha_t$ of $G$. Then $G^\flat := G \rtimes_\alpha \mathbb{R}$ is a Lie group acting by causal automorphisms on $M := G$, where $(g, 0) \in G^\flat$ acts by left translation and $(0,t)$ by $\alpha_t$. This action leaves the biinvariant cone field on $G$ invariant, and the involution $\tau_D^G$ on $G$ is *anti-causal*, i.e., flips the cone field into its negative. Now $h^\flat := (0,1) \in \mathfrak{g}^\flat$ is an Euler element, and for every $g = (g,0) \in G \subseteq G^\flat$, we have $\mathrm{Ad}(g)h^\flat - h^\flat \in \mathfrak{g}$. We may therefore consider the closed subsemigroup

$$S(h^\flat, C_\mathfrak{g}) := \{g \in G \colon h^\flat - \mathrm{Ad}(g)h^\flat \in C_\mathfrak{g}\}$$

and find the positivity domain

$$W_G^+(h^\flat) = \{g \in G \colon h^\flat - \mathrm{Ad}(g)h^\flat \in C_\mathfrak{g}^\circ\}.$$

With the same arguments as above, we also obtain with [Ne22, Thm. 2.16]

$$W_G^+(h^\flat) = \exp(C_+^\circ)G^{h^\flat}\exp(C_-^\circ) = G^{h^\flat}\exp(C_+^\circ + C_-^\circ) = S(h^\flat, C_\mathfrak{g})^\circ. \quad (85)$$

**Examples 3.21** (Euler elements with empty positivity regions)
(a) Not every Euler element has a non-trivial positivity region. If $M = G$ is a causal Lie group with biinvariant cone field corresponding to $C_\mathfrak{g} \subseteq \mathfrak{g}$, on which $G \times G$-acts, then every Euler element $h_0 \in \mathfrak{g}$ specifies an Euler element $h := (h_0, 0) \in \mathfrak{g}^{\oplus 2}$, but the corresponding modular vector field is $X_h^G(g) = g.h$, and this is never contained in $C_g = g.C_\mathfrak{g}$ because $h \notin C_\mathfrak{g}$. This follows from the fact that $h$ is hyperbolic and the semisimple Jordan components of elements in $C_\mathfrak{g}$ are elliptic (cf. [NOe22, Cor. B.2] and Appendix 7.2.5 for the real Jordan decomposition). We also note that $\tau_h = \tau_{h_0} \oplus \mathrm{id}_\mathfrak{g}$ does not commute with the flip, hence cannot be implemented on the symmetric space $G$ in a natural way.
(b) For a **left** invariant causal structure on a Lie group $G$, the cone $C \subseteq \mathfrak{g} \cong T_e(G)$ can be any pointed generating closed convex cone. Then $W_G^+(h) \neq \emptyset$ is equivalent to $h \in C^\circ$, and in this case $W_G^+(h) = G$, so that the situation is quite degenerate.

### 3.4 Causal flag manifolds

In view of Section 2, Euler elements $h \in \mathfrak{g}$ play a key role in AQFT, and we have to understand causal homogeneous spaces $M = G/H$ for which the positivity region $W_M^+(h)$ is non-empty. Otherwise we have no wedge regions for the Bisognano–Wichmann property (BW). As the "most structured" homogeneous spaces are symmetric spaces and flag manifolds, we discuss causal flag manifolds in this subsection and causal symmetric spaces in the next one. In Physics, the most prominent example of a causal flag manifold is the conformal compactification $(\mathbb{S}^1 \times \mathbb{S}^{d-1})/\{\pm\mathbf{1}\}$ of $d$-dimensional Minkowski space (Example 3.33), which, for $d = 1$, reduces to the circle $\mathbb{S}^1$.

**Definition 3.22** To define flag manifolds for a connected semisimple Lie group, consider $x \in \mathfrak{g}$ such that $\mathrm{ad}\, x$ is diagonalizable, put

$$\mathfrak{q}_x = \sum_{\lambda \leq 0} \mathfrak{g}_\lambda(x) \quad \text{and} \quad Q_x := \{g \in G\colon \mathrm{Ad}(g)\mathfrak{q}_x = \mathfrak{q}_x\}.$$

Then $Q_x$ is called a *parabolic subgroup* of $G$ and $G/Q_x$ the corresponding flag manifold. [16]

For the description of the causal flag manifolds, we also need hermitian Lie algebras (see Table 2 in Section 2.3 and Appendix 7.2.3).

**Definition 3.23** A simple Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (see Appendix 7.2.2) is called *hermitian* if the center $\mathfrak{z}(\mathfrak{k})$ of a maximal compactly embedded subalgebra $\mathfrak{k}$ is non-zero. For hermitian Lie algebras, the restricted root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$, with respect to a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$, is either of type $C_r$ or $BC_r$ (cf. Harish-Chandra's Theorem [Ne00, Thm. XII.1.14]), and we say that $\mathfrak{g}$ is *of tube type* if the restricted root system is of type $C_r$. The terminology comes from the fact that the corresponding hermitian symmetric space $G/K$ is a tube domain, i.e., biholomorphic to $\mathrm{V}_+ + i\mathrm{V} \subseteq \mathrm{V}_\mathbb{C}$ for a real vector space V and an open convex cone $\mathrm{V}_+ \subseteq \mathrm{V}$.

**Example 3.24** The Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ is the smallest example of a hermitian Lie algebra and it is in particular of tube type. The corresponding Lie group $G = \mathrm{SL}_2(\mathbb{R})$ acts by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+b}{cz+d}$$

on the upper half plane $\mathbb{C}_+ = \mathbb{R} + i\mathbb{R}_+ \subseteq \mathbb{C}$. This is the smallest example of a tube domain. This action of $\mathbb{C}_+$ is transitive, and the stabilizer group of $i \in \mathbb{C}_+$ is $K = \mathrm{SO}_2(\mathbb{R})$, so that $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \cong \mathbb{C}_+$. In the boundary $\partial\mathbb{C}_+ = \mathbb{R}$ we recover the action on the open dense subset $\mathbb{R} \subseteq \mathbb{R}_\infty \cong \mathbb{S}^1$.

**Theorem 3.25** (Classification of causal flag manifolds, [Ne26, Thm. 2.5]) *Let $G$ be a connected semisimple Lie group and $Q \subseteq G$ be a parabolic subgroup such that $\mathfrak{q}$ contains no non-zero ideals of $\mathfrak{g}$. Suppose that the corresponding flag manifold $G/Q$ carries a $G$-invariant causal structure. Then $\mathfrak{g}$ is a direct sum of hermitian simple ideals and there exists an Euler element $h \in \mathfrak{g}$ such that*

$$\mathfrak{q} = \mathfrak{q}_h := \mathfrak{g}_0(h) + \mathfrak{g}_{-1}(h).$$

*If, conversely, this is the case, then $G/Q_h$ is a causal flag manifold.*

[16] The name *flag manifold* comes from the special case where $G = \mathrm{SL}_n(\mathbb{R})$ and $Q$ is the stabilizer subgroup of a flag $\mathcal{F} = (F_1, \ldots, F_k)$ of linear subspaces of $\mathbb{R}^n$, i.e., $F_i$ is a proper subspace of $F_{i+1}$ for $i = 1, \ldots, k-1$. Then $G/Q$ is the manifold of all flags $(E_1, \ldots, E_k)$ with $\dim E_i = \dim F_i$ for $i = 1, \ldots, k$. General flag manifolds always have embeddings into flag manifolds of $\mathrm{SL}_n(\mathbb{R})$.

If $\mathfrak{g}$ is simple hermitian, then an Euler element $h$ exists in $\mathfrak{g}$ if and only if $\mathfrak{g}$ is of tube type, and then they are all conjugate and $h$ is symmetric (Proposition 2.13). We fix one and consider the corresponding causal flag manifold $M = G/Q_h$. The tangent space in the base point is

$$\mathfrak{g}/\mathfrak{q}_h \cong \mathfrak{g}_1(h), \tag{86}$$

and the causal structure on $M$ is specified by the cone

$$C_+ = C_{\mathfrak{g}} \cap \mathfrak{g}_1(h), \tag{87}$$

where $C_{\mathfrak{g}}$ is a pointed generating closed convex $\mathrm{Ad}(G)$-invariant cone in $\mathfrak{g}$. We thus obtain an (up to sign unique) causal structure on $M$, i.e., any other cone $C'_{\mathfrak{g}}$ satisfies $C'_{\mathfrak{g}} \cap \mathfrak{g}_1(h) = C_+$ or $C'_{\mathfrak{g}} \cap \mathfrak{g}_1(h) = -C_+$ ([MNÓ23, §3.5]). This also follows from the fact that $\mathfrak{g}_1(h)$ contains only two $e^{\mathrm{ad}\,\mathfrak{g}_0}$-invariant non-trivial closed convex cones ([HNO96, Prop. A.I.5]).

On the open dense subset of $M$ obtained by embedding

$$\eta\colon \mathfrak{g}_1 \to M, \quad \eta(x) := \exp(x)Q_h,$$

the vector field $X_h^M$ is the **Euler vector field** on $\mathfrak{g}_1$, so that $\eta(C_+^\circ) \subseteq W_M^+(h)$, and we actually have that

$$W := W_M^+(h) = \eta(C_+^\circ) \tag{88}$$

([MN26, Lemma 2.7]).

**Proposition 3.26** *The compression semigroup of $W \subseteq M = G/Q_h$ is*

$$S_W = \{g \in G\colon g.W \subseteq W\} = \exp(C_+)G^h \exp(C_-), \tag{89}$$

*where $C_\pm = \pm C_{\mathfrak{g}} \cap \mathfrak{g}_{\pm 1}(h)$.*

***Proof*** As $G_1 = \exp(\mathfrak{g}_1)$ is abelian, the inclusions $\exp(C_+) \subseteq S_W$ and $G^h \subseteq S_W$ are obvious.

All elements $x \in C_+$ correspond to constant vector fields on the open subset $\eta(\mathfrak{g}_1) \subseteq M$, and since this subset is dense (Bruhat decomposition), we obtain

$$C_+ \subseteq C_M = \{y \in \mathfrak{g}\colon (\forall m \in M)\ X_y^M(m) \in C_m\}$$

(cf. (73)). The group $G$ acts on $M$ with discrete kernel ($\mathfrak{q}$ contains no non-zero ideals of $\mathfrak{g}$), so that the closed convex $\mathrm{Ad}(G)$-invariant cone $C_M \subseteq \mathfrak{g}$ is pointed, and the preceding argument yields

$$C_{M,+} := C_M \cap \mathfrak{g}_1 = C_+.$$

The linear subspace $C_M - C_M$ generated by $C_M$ is $\mathrm{Ad}(G)$-invariant, hence an ideal of $\mathfrak{g}$. Further, the cone $C_+$ is not contained in a proper ideal, so that $C_M$

is also generating, and this leads to $C_- = -C_M \cap \mathfrak{g}_{-1}$ by the discussion preceding the proposition. Thus $\exp(C_-) \subseteq S_W$ follows from Proposition 3.13. Putting everything together, we get

$$S_W \supseteq \exp(C_+)G^h \exp(C_-). \tag{90}$$

The hard part is to verify equality in (89). This involves showing that the product set on the right is a subsemigroup (which is not obvious because the factors do not commute; see [Ko95]) and that it actually coincides with $S_W$, by showing that it is maximal ([HN95]), hence equal to $S_W$. We refer to [Ne18, Lemma 3.7, Thm. 3.8] for more details and references. □

**Problem 3.27** Theorem 3.25 describes all causal flag manifolds $M = G/Q_h$ for semisimple Lie groups, but it makes good sense to ask for a result on non-semisimple groups:

(C1) Let $x \in \mathfrak{g}$ be such that $\operatorname{ad} x$ is diagonalizable, put

$$\mathfrak{q}_x = \sum_{\lambda \leq 0} \mathfrak{g}_\lambda(x) \quad \text{and} \quad Q_x := \{g \in G \colon \operatorname{Ad}(g)\mathfrak{q}_x = \mathfrak{q}_x\}.$$

Show that, if $M = G/Q_x$ is causal, then $x$ must be an Euler element (cf. [Ne26, Thm. 2.5] for similar arguments). Note that $x \in \mathfrak{q}_x$ implies that $\mathfrak{q}_x$ is self-normalizing, so that $\mathbf{L}(Q_x) = \mathfrak{q}_x$.

(C2) Assume that $h \in \mathfrak{g}$ is an Euler element. Determine those manifolds $M = Q/Q_h$ with an invariant causal structure on which $G$ acts effectively.

*Remark 3.28* (The affine case) Particular examples arise for Euler elements with $\mathfrak{g}_{-1} = \{0\}$. Then $M = G/Q_h = \eta(\mathfrak{g}_1) \cong \mathfrak{g}_1$ and we may assume that $G \cong \mathfrak{g}_1 \rtimes G_0$.

This covers the action of $\operatorname{Aff}(\mathbb{R})_e$ on $\mathbb{R}$ and of the Poincaré group on Minkowski space. More generally, we may start with a finite-dimensional real linear space $E$ and a pointed generating convex cone $C \subseteq E$. We write $\operatorname{Aut}(C) \subseteq \operatorname{GL}(E)$ for its linear automorphism group, which is a closed subgroup. Then $G := E \rtimes \operatorname{Aut}(C)$ acts transitively on the affine causal manifold $M := E$, endowed with the constant cone field $C_m = C$ for $m \in M$. Further, $h := (0, \operatorname{id}_E)$ is an Euler element with $\mathfrak{g}_{-1} = \{0\}$, $\operatorname{Aut}(C) = G^h$ and $\mathfrak{g}_1 \cong E$. The corresponding positivity region is

$$W := W_M^+(h) = C^\circ,$$

and its compression semigroup is readily identified with

$$S_W = C \rtimes \operatorname{Aut}(C)$$

because $\operatorname{Aut}(C) \subseteq S_W$ (cf. also Lemma 7.32).

Lie algebra elements $(b, a) \in \mathfrak{g} = \mathfrak{g}_1 \rtimes \mathfrak{g}_0$ correspond to affine vector fields $X(x) = b + ax$, and such a vector field is positive on all of $E$ if and only if

$b + aE \subseteq C$, which is equivalent to $a = 0$ and $b \in C$. Therefore the invariant cone $C_M \subseteq \mathfrak{g}$ coincides with $C \subseteq \mathfrak{g}_1$.

**Appendix: Euclidean Jordan algebras.** The causal flag manifolds of simple Lie groups are precisely the conformal compactifications of simple euclidean Jordan algebras ([Ne26], [Be96]).

**Definition 3.29** (a) A *Jordan algebra* is a commutative algebra V, in which the multiplication $(x, y) \mapsto L_x y = xy$ satisfies the *Jordan identity*:

$$[L_x, L_{x^2}] = 0 \quad \text{i.e.} \quad x(x^2 y) = x^2(xy) \quad \text{for} \quad x, y \in \mathrm{V}.$$

(b) A Jordan algebra is called *euclidian* or *formally real*, if, for $x_1, \ldots, x_n \in \mathrm{V}$ the relation $x_1^2 + \cdots + x_n^2 = 0$ implies $x_1 = \cdots = x_n = 0$.

**Examples 3.30** (a) The Jordan algebra structure on the spaces $\mathrm{Herm}_r(\mathbb{K}), \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, is given by the symmetrized matrix product

$$A * B := \frac{1}{2}(AB + BA). \tag{91}$$

As $A^2$ is positive semidefinite, this Jordan algebra is euclidean.
(b) On Minkowski space $\mathbb{R}^{1,d-1}$ we have a Jordan algebra structure given by

$$(x_0, \mathbf{x}) * (y_0, \mathbf{y}) = (x_0 y_0 + \mathbf{x}\mathbf{y}, x_0 \mathbf{y} + y_0 \mathbf{x}) \tag{92}$$

([FK94, pp. 25, 31]). In this case squares are contained in the closed upper light cone, and this implies that it is euclidean as well.

This Jordan algebra embeds into the Clifford algebra $\mathrm{Cl}(\mathbb{R}^d)$ generated by anticommuting elements $\mathbf{e}_1, \ldots, \mathbf{e}_d$ with $\mathbf{e}_j^2 = \mathbf{1}$, via

$$\iota \colon \mathbb{R}^{1,d} \to \mathrm{Cl}(\mathbb{R}^d), \quad x = (x_0, \mathbf{x}) \mapsto x_0 \mathbf{1} + \sum_{j=1}^d x_j \mathbf{e}_j$$

and the product (91). It is also called *the spin factor*.

If $\mathfrak{g}$ is simple hermitian and $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ its restricted root system, then either $\Sigma$ is of type $C_r$ or $BC_r$ (Table 2 in Section 2.3). Moreover, $\mathfrak{g}$ contains an Euler element if and only if $\Sigma$ is of type $C_r$, i.e., $\mathfrak{g}$ is of tube type, and in this case all Euler elements are $\mathrm{Inn}(\mathfrak{g})$-conjugate (Proposition 2.13(b)). In particular $h$ is *symmetric* in the sense that $-h \in \mathcal{O}_h := \mathrm{Inn}(\mathfrak{g})h$. Then there exist $e \in \mathfrak{g}_1(h)$ and $f = -\theta(e)/2 \in \mathfrak{g}_{-1}(h)$ such that

$$[e, f] = h, \quad \text{and also} \quad [h, e] = e, \quad [h, f] = -f \tag{93}$$

(cf. Appendix 7.2.4 for notation related to $\mathfrak{sl}_2(\mathbb{R})$). We then consider on $\mathrm{V} := \mathfrak{g}_1(h)$ the bilinear product

$$x * y := [[x, f], y], \tag{94}$$

which defines a unital euclidean Jordan algebra $(\mathrm{V}, *, e)$. In particular, the set of squares is a closed pointed generating convex cone (Koecher–Vinberg Theorem; see [FK94, Thm. III.2.1]). Its interior $\mathrm{V}_+$ is the cone of invertible squares. We refer to Table 4 below for a list of simple hermitian Lie algebras of tube type and the corresponding euclidean Jordan algebras. The only non-simple Lie algebra listed is $\mathfrak{so}_{2,2}(\mathbb{R}) \cong \mathfrak{so}_{1,2}(\mathbb{R})^{\oplus 2}$, corresponding to the non-simple Jordan algebra $\mathrm{V} = \mathbb{R}^{1,1} \cong \mathbb{R} \oplus \mathbb{R}$ (the Minkowski plane, decomposing in lightray coordinates).

| Hermitian Lie algebra | $\mathfrak{g}$ | $\mathfrak{sp}_{2r}(\mathbb{R})$ | $\mathfrak{su}_{r,r}(\mathbb{C})$ | $\mathfrak{so}^*(4r)$ | $\mathfrak{e}_{7(-25)}$ | $\mathfrak{so}_{2,d}(\mathbb{R})$ |
|---|---|---|---|---|---|---|
| Euclidean Jordan algebra | V | $\mathrm{Sym}_r(\mathbb{R})$ | $\mathrm{Herm}_r(\mathbb{C})$ | $\mathrm{Herm}_r(\mathbb{H})$ | $\mathrm{Herm}_3(\mathbb{O})$ | $\mathbb{R}^{1,d-1}$ |
| rank of V | | $r$ | $r$ | $r$ | 3 | 2 |

Table 4: Hermitian Lie algebras of tube type and euclidean Jordan algebras

**Example 3.31** The corresponding flag manifolds $M$ have interesting geometric interpretations. For

$$\Omega := \Omega_{2r} := \begin{pmatrix} 0 & \mathbf{1}_r \\ -\mathbf{1}_r & 0 \end{pmatrix} \in M_{2r}(\mathbb{K}), \quad \mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H},$$

we obtain a uniform realization of the Lie algebras $\mathfrak{sp}_{2r}(\mathbb{R}), \mathfrak{u}_{r,r}(\mathbb{C})$ and $\mathfrak{so}^*(4r)$ as

$$\mathfrak{u}(\Omega, \mathbb{K}^{2r}) := \{x \in \mathfrak{gl}_{2r}(\mathbb{K}) \colon x^*\Omega + \Omega x = 0\}. \tag{95}$$

Then $M$ is the space of maximal isotropic subspaces $L \subseteq \mathbb{K}^{2r}$ with respect to the skew-hermitian form $\beta(z, w) := z^*\Omega w$ on $\mathbb{K}^{2r}$

**Example 3.32** A particularly interesting example arises from the unitary group $\mathrm{U}_r(\mathbb{C})$, which can be realized as a causal flag manifold, endowed with the biinvariant cone field defined by

$$C_{\mathfrak{g}} = \{x \in \mathfrak{u}_r(\mathbb{C}) \colon -ix \geq 0\}.$$

Then the hermitian Lie group $\mathrm{SU}_{r,r}(\mathbb{C})$ acts transitively by causal automorphisms via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.u = (au + b)(cu + d)^{-1}$$

(see [Ne26, §8.2]).

To match this with the discussion of the preceding example, we associate to $u \in \mathrm{U}_r(\mathbb{C})$ its graph

$$\Gamma(u) = \{(z, uz) \colon z \in \mathbb{C}^r\} \subseteq \mathbb{C}^{2r}\},$$

where we thus obtain all isotropic subspaces of the hermitian form

$$h((z_1, w_1), (z_2, w_2)) := z_1^* z_2 - w_1^* w_2,$$

which is equivalent to the form specified by the hermitian matrix $i\Omega_{2r}$. As the isometry group $\mathrm{U}(\mathbb{C}^{2r}, h)$ of this form acts transitively on the set of maximal isotropic subspaces by Witt's Theorem ([Br85]), this provides a conceptional approach to the transitive action of $\mathrm{SU}_{r,r}(\mathbb{C})$ on $\mathrm{U}_r(\mathbb{C})$. For $r = 1$, we recover the action of $\mathrm{SU}_{1,1}(\mathbb{C}) \cong \mathrm{SL}_2(\mathbb{R})$ on $\mathrm{U}_1(\mathbb{C}) \cong \mathbb{S}^1$.

**Example 3.33** (The Lorentzian case) For $d$-dimensional Minkowski space $\mathrm{V} = \mathbb{R}^{1,d-1}$, we realize the conformal completion $M$ of V as the quadric

$$Q := Q(\mathbb{R}^{2,d}) := \{[(x_1, x_2, \mathbf{x})] \in \mathbb{P}(\mathbb{R}^{2,d}) \colon x_1^2 + x_2^2 - \mathbf{x}^2 = 0\}, \tag{96}$$

on which $G = \mathrm{SO}_{2,d}(\mathbb{R})_e$ acts transitively. The natural dense open embedding $\mathrm{V} \to Q$ is given by

$$\eta \colon \mathrm{V} \to Q, \quad \eta(x_0, \mathbf{x}) := \Big[\frac{1 - x_0^2 + \mathbf{x}^2}{2} : x_0 : \mathbf{x} : -\frac{1 + x_0^2 - \mathbf{x}^2}{2}\Big], \tag{97}$$

corresponding to the action of the translation group $(\mathrm{V}, +) \cong \mathfrak{g}_1(h)$ on $Q$ (cf. [HN12, §17.4], [Ne26]).

### 3.5 Causal symmetric spaces

In the preceding subsection we discussed causal flag manifolds, the "most symmetric" class of causal homogeneous space in the sense that the dimension of the acting group is rather large compared to the dimension of $M$. They represent "conformal geometries". Metric geometries, which are more rigid, correspond in this context to causal symmetric spaces.

We start with some terminology and observations concerning symmetric spaces and symmetric Lie algebras (cf. [HÓ97], [CW70], [KO08]):

- A *symmetric Lie algebra* is a pair $(\mathfrak{g}, \tau)$, where $\mathfrak{g}$ is a finite-dimensional real Lie algebra and $\tau$ is an involutive automorphism of $\mathfrak{g}$. We write

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} \quad \text{with} \quad \mathfrak{h} = \mathfrak{g}^\tau = \ker(\tau - \mathbf{1}) \quad \text{and} \quad \mathfrak{q} = \mathfrak{g}^{-\tau} = \ker(\tau + \mathbf{1}). \tag{98}$$

  We call $(\mathfrak{g}, \tau)$ *irreducible* if the representation of $\mathfrak{h}$ on $\mathfrak{q}$ is irreducible.
- A *symmetric space* is a homogeneous space of the form $M = G/H$, where $H \subseteq G^\tau$ is an open subgroup and $\tau \in \mathrm{Aut}(G)$ an involution. Then $H$ contains the identity component $G_e^\tau := (G^\tau)_e$. We call the triple $(G, \tau, H)$ a *symmetric Lie group* because this triple specifies the symmetric space $M$. For a more intrinsic approach to symmetric spaces as "reflection spaces", we refer to [Lo69].

- A *causal symmetric Lie algebra* is a triple $(\mathfrak{g}, \tau, C)$, where $(\mathfrak{g}, \tau)$ is a symmetric Lie algebra and $C \subseteq \mathfrak{q}$ is a pointed generating closed convex cone, invariant under the group $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}) := \langle e^{\mathrm{ad}\,\mathfrak{h}} \rangle \subseteq \mathrm{Aut}(\mathfrak{g})$. We call $(\mathfrak{g}, \tau, C)$
  - *compactly causal (cc)* if $C$ is *elliptic* in the sense that, for $x \in C^\circ$ (the interior of $C$), the operator $\mathrm{ad}\,x$ is semisimple with purely imaginary spectrum (cf. Appendix 7.2.5).
  - *non-compactly causal (ncc)* if $C$ is *hyperbolic* in the sense that, for $x \in C^\circ$, the operator $\mathrm{ad}\,x$ is diagonalizable over $\mathbb{R}$ (cf. Appendix 7.2.5).
- For a symmetric Lie algebra $(\mathfrak{g}, \tau)$, the pair $(\mathfrak{g}^c, \tau^c)$ with $\mathfrak{g}^c := \mathfrak{h} + i\mathfrak{q}$ and $\tau^c(x + iy) = x - iy$ is called the *c-dual symmetric Lie algebra*.
- A *modular causal symmetric Lie algebra* is a quadruple $(\mathfrak{g}, \tau, C, h)$, where $(\mathfrak{g}, \tau, C)$ is a causal symmetric Lie algebra, $h \in \mathfrak{g}^\tau$ is an Euler element, and the involution $\tau_h$ satisfies $\tau_h(C) = -C$.

*Remark 3.34* (a) $(\mathfrak{g}, \tau, C)$ is non-compactly causal if and only if $(\mathfrak{g}^c, \tau^c, iC)$ is compactly causal.
(b) $(\mathfrak{g}, \tau, C, h)$ is modular if and only if the $c$-dual quadruple $(\mathfrak{g}^c, \tau^c, iC, h)$ is modular.

*Remark 3.35* If $C_{\mathfrak{g}} \subseteq \mathfrak{g}$ is a pointed generating invariant cone in $\mathfrak{g}$ and $h \in \mathfrak{g}$ an Euler element satisfying $-\tau_h(C_{\mathfrak{g}}) = C_{\mathfrak{g}}$, then there is a variety of associated causal symmetric Lie algebras:

(a) $(\mathfrak{g}^{\oplus 2}, \tau_{\mathrm{flip}}, C, (h, h))$ with $C = \{(x, -x) \colon x \in C_{\mathfrak{g}}\}$ is a modular causal symmetric Lie algebra of group type (cf. Section 3.3).
(b) $(\mathfrak{g}_{\mathbb{C}}, \sigma, iC_{\mathfrak{g}}, h)$, with $\sigma(x + iy) = x - iy$, is a modular non-compactly causal symmetric Lie algebra of complex type.
(c) $(\mathfrak{g}, \tau_h, C_+ - C_-, h)$ is a modular compactly causal symmetric Lie algebra, called of *Cayley type* if $\mathfrak{g}$ is simple. Note that $C_{\mathfrak{g}}^{-\tau_h} = C_+ - C_-$ by Lemma 3.7(ii).
(d) $(\mathfrak{g}, \tau_h, C_+ + C_-, h)$ is a modular non-compactly causal symmetric Lie algebra of Cayley type.

In view of

$$\kappa_h = e^{\frac{\pi i}{2}\,\mathrm{ad}\,h} \colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}} \quad \text{satisfies} \quad \kappa_h(\mathfrak{g}) = \mathfrak{g}^c, \tag{99}$$

it follows that $(\mathfrak{g}, \tau_h) \cong (\mathfrak{g}^c, \tau_h)$ as symmetric Lie algebras. Moreover,

$$\kappa_h(C_{\mathfrak{g}}^{-\tau_h}) = \kappa_h(C_+ - C_-) = i(C_+ + C_-)$$

implies

$$(\mathfrak{g}, \tau_h, C_+ - C_-, h) \cong (\mathfrak{g}^c, \tau_h, i(C_+ + C_-), h) \cong (\mathfrak{g}, \tau_h, C_+ + C_-, h)^c \tag{100}$$

as modular causal symmetric Lie algebras.

*Remark 3.36* (Cartan motion groups) If $(G, \tau, H)$ is a connected symmetric Lie group corresponding to the causal symmetric Lie algebra $(\mathfrak{g}, \tau, C)$, then we obtain on $\mathfrak{q}$ a constant causal structure defined by $C$, which is invariant under the action of the semidirect product group $\mathfrak{q} \rtimes H$ (a so-called *Cartan motion group*), where $H := G^\tau_e$ (cf. Remark 3.28). If, in addition, $h \in \mathfrak{h}$ is an Euler element with $\tau_h(C) = -C$, then the pair $(\mathfrak{q}, C)$ satisfies the assumptions (A1)-(A3) from Section 3.1.1. So $\mathfrak{q}$ is an affine causal symmetric space, and (72) in Section 3.1.1 implies that

$$W^+_{\mathfrak{q}}(h) = C^\circ_+ \oplus \mathfrak{q}_0(h) \oplus C^\circ_- \quad \text{for} \quad C_\pm := \pm C \cap \mathfrak{q}_{\pm 1}(h).$$

*Remark 3.37* (Lorentzian symmetric spaces) Important examples of causal symmetric spaces are those where the causal structure comes from a Lorentzian form, for instance Minkowski space, de Sitter space $\mathrm{dS}^d$ and anti-de Sitter space $\mathrm{AdS}^d$ (see Examples 1.28 and 3.11).

If $M_1 = G_1/H_1$ is a Lorentzian symmetric space and $M_2 = G_2/H_2$ is a Riemannian symmetric space, then the product $M = M_1 \times M_2$ is also Lorentzian. Natural examples are

$$\mathrm{AdS}^p \times \mathbb{S}^q \quad \text{and} \quad \mathrm{dS}^p \times \mathrm{Hyp}^q.$$

The compact group $\mathrm{U}_n(\mathbb{C})$ carries a one-parameter family of biinvariant Lorentzian structures. We refer to [Ne26, §7] for more details and conformal embeddings of these spaces for $p + q = d$ into $Q(\mathbb{R}^{2,d})$.

### 3.5.1 Causal symmetric spaces of group type

We assume first that $\mathfrak{g}$ is simple hermitian and that $h_0 \in \mathfrak{g}$ is an Euler element. Then any $\mathrm{Ad}(G)$-invariant closed convex pointed generating cone $C_{\mathfrak{g}} \subseteq \mathfrak{g}$ specifies a biinvariant causal structure on the group $G$, considered as a symmetric space, on which $G \times G$ acts transitively. Then the Euler elements $h \in \mathfrak{g}^{\oplus 2}$ for which $W^+_G(h) \neq \emptyset$ are conjugate to $h = (h_0, h_0)$ for some Euler element $h_0 \in \mathfrak{g}$, and in this case

$$W^+_G(h) = \exp(C^\circ_+) G^h \exp(C^\circ_-) = S(h, C_{\mathfrak{g}})^\circ \tag{101}$$

follows from Theorem 3.18, cf. also (78) and (79). Note that $W^+_G(h)$ only depends on the cones $C_\pm$, hence is unique up to sign if $\mathfrak{g}$ is simple ([MNÓ23, §3.5]).

**Examples 3.38** (a) The invariant cone $C \subseteq \mathfrak{sl}_2(\mathbb{R})$ specifies a biinvariant causal structure on the group $\mathrm{SL}_2(\mathbb{R})$, for which $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ acts by left and right translations as a group of causal automorphisms.

On the other hand, we have on the 4-dimensional matrix space $M_2(\mathbb{R})$ a quadratic form, given by the determinant, and this form is invariant under the action of $G$ by

$$(g, h).A := gAh^{-1},$$

for which $\mathrm{SL}_2(\mathbb{R})$ is the orbit of the identity matrix $\mathbf{1}$. From

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

it follows that

$$\gamma\colon \mathbb{R}^{2,2} \to M_2(\mathbb{R}),$$
$$\gamma(\mathbf{e}_1) = \mathbf{1}, \quad \gamma(\mathbf{e}_2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma(\mathbf{e}_3) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma(\mathbf{e}_4) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

defines an isometry, so that $M_2(\mathbb{R}) \cong \mathbb{R}^{2,2}$ as a quadratic space. This easily implies that

$$M_2(\mathbb{R}) \supseteq \mathrm{SL}_2(\mathbb{R}) \cong \mathrm{AdS}^3 \subseteq \mathbb{R}^{2,2},$$

i.e., that 3-dimensional anti-de Sitter space is isomorphic to $\mathrm{SL}_2(\mathbb{R})$, as a causal Lie group. Actually this isomorphism also yields an isomorphism

$$\mathrm{SO}_{2,4}(\mathbb{R})_e \cong (\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}))/\{(\mathbf{1},\mathbf{1}), (-\mathbf{1},-\mathbf{1})\}.$$

(b) Another interesting family of causal groups arises from the biinvariant causal structure on $\mathrm{U}_n(\mathbb{C})$, specified by the cone

$$C = i\,\mathrm{Herm}_n(\mathbb{C}) = \{x \in \mathfrak{u}_n(\mathbb{C})\colon -ix \text{ is positive semidefinite}\}.$$

Then $G = \mathrm{U}_n(\mathbb{C}) \times \mathrm{U}_n(\mathbb{C})$ acts by left and right translations, but we have also seen in Example 3.32 that this action extends to the larger group $\mathrm{U}_{n,n}(\mathbb{C})$ by fraction linear transformations. This example is Lorentzian if and only if $n = 2$, and in this case

$$\mathrm{U}_2(\mathbb{C}) \cong (\mathbb{S}^1 \times \mathrm{SU}_2(\mathbb{C}))/\{\pm\mathbf{1}\}$$

is the causal compactification of the Minkowski space $\mathbb{R}^{1,3} \cong \mathrm{Herm}_2(\mathbb{C})$, and the group $\mathrm{SU}_{2,2}(\mathbb{C})$, a 2-fold covering of $\mathrm{SO}_{2,4}(\mathbb{R})_e$, acts by fractional linear transformations.

### 3.5.2 Modular compactly causal symmetric spaces

If $(\mathfrak{g}, \tau, C)$ is an irreducible compactly causal symmetric Lie algebra, which is not of group type, then $\mathfrak{g}$ is simple hermitian. This follows from [NÓ23b, Prop. 2.13] and $c$-duality[17]. If $\mathfrak{g}$ contains an Euler element, then $\mathfrak{g}$ is of tube type, $\mathrm{Ad}(G)$ acts transitively on $\mathcal{E}(\mathfrak{g})$ (Proposition 2.13) and there exist $\tau$-

[17] The dual symmetric Lie algebra $(\mathfrak{g}^c, \tau^c, iC)$ is irreducible, non-complex and non-compactly causal. Hence $\mathfrak{g}^c$ is simple. Moreover $\tau^c = \tau_h \theta^c$ for a causal Euler element $h \in i\mathfrak{q} = \mathfrak{q}^c$ and a Cartan involution $\theta^c$ of $\mathfrak{g}^c$. Then $\mathfrak{g}^c_0 = \mathfrak{z}_{\mathfrak{g}^c}(h) = \mathfrak{h}^c_{\mathfrak{k}} \oplus \mathfrak{q}^c_{\mathfrak{p}} = \mathfrak{h}_{\mathfrak{k}} \oplus i\mathfrak{q}_{\mathfrak{k}}$ implies that $\mathfrak{z}_{\mathfrak{g}}(ih) = \mathfrak{k}$. So $ih \in \mathfrak{z}(\mathfrak{k})$ implies that $\mathfrak{g}$ is hermitian.

fixed Euler elements (Corollary 3.47 in Appendix 3.7). Now the embedding

$$(\mathfrak{g}, \tau, C) \hookrightarrow (\mathfrak{g}^{\oplus 2}, \tau_{\rm flip}, \widetilde{C}), \quad x \mapsto (x, \tau(x)) \tag{102}$$

can be used to determine the positivity region $W_M^+(h)$ by using the results for spaces of group type.

On the global side, we consider the action of $G$ on itself by $g.x := gx\tau(g)^{-1}$. Then $M := G.e \cong G/G^\tau$ is the identity component in the fixed point set of the involution $g^\sharp := \tau(g)^{-1}$ and a symmetric space with symmetric Lie algebra $(\mathfrak{g}, \tau)$ (cf. Exercise 3.59). If $C = C_\mathfrak{g} \cap \mathfrak{q}$, then we even have an embedding of causal symmetric spaces which is equivariant for the modular flow generated by the Euler element. This easily implies with (101) that

$$W_M^+(h) = W_G^+(h) \cap M \overset{(101)}{=} S(C_\mathfrak{g}, h)^\circ \cap M, \tag{103}$$

and

$$W = G_e^h.\exp(C_+^\circ + C_-^\circ) \quad \text{for} \quad C_\pm = \pm C_\mathfrak{g}^{-\tau} \cap \mathfrak{g}_{\pm 1}(h).$$

The compression semigroup of $W$ is

$$S_W = G_W \exp(C_+ + C_-) \quad \text{with} \quad G_W = G_e^h H^h. \tag{104}$$

Furthermore, $G_W$ is open in $G^h$ ([NÓ23a, Thm. 9.1]). We refer to [NÓ23a] for details.

### 3.5.3 Non-compactly causal symmetric spaces

Irreducible non-compactly causal symmetric Lie algebras $(\mathfrak{g}, \tau, C)$ are $c$-dual to irreducible compactly causal ones. The dual $(\mathfrak{g}^c, \tau^c)$ is of group type if and only if $\mathfrak{g}$ is a complex simple Lie algebra (considered as a real one) and $\tau$ is antilinear, so that $\mathfrak{h} = \mathfrak{g}^\tau$ is a real form and $\mathfrak{g} \cong \mathfrak{h}_\mathbb{C}$. Then $(\mathfrak{g}^c, \tau^c) \cong (\mathfrak{h}^{\oplus 2}, \tau_{\rm flip})$. The existence of the causal structure implies that $\mathfrak{h}$ is hermitian, but these real forms are precisely those for which the corresponding conjugation $\tau$ is of the form $\theta\tau_h$, where $h \in \mathfrak{g}$ is an Euler element ([MNÓ23, Thm. 4.21]). So Euler elements in complex simple Lie algebras (see Table 2 in Section 2.3) determine causal symmetric Lie algebras of complex type.

This picture prevails for general simple Lie algebras $\mathfrak{g}$. Whenever $h \in \mathfrak{g}$ is an Euler element and $\theta$ a Cartan involution with $\theta(h) = -h$, then $\tau := \theta\tau_h$ is an involution of $\mathfrak{g}$. Further $h$ is also Euler in the complexification $\mathfrak{g}_\mathbb{C}$, on which the antilinear extension $\overline{\theta}$ of $\theta$ to $\mathfrak{g}_\mathbb{C}$ defines a Cartan involution. Then $\overline{\tau} := \overline{\theta}\tau_h$ is an antilinear extension of the involution $\tau = \theta\tau_h$ on $\mathfrak{g}$, and $\mathfrak{g}^c := (\mathfrak{g}_\mathbb{C})^{\overline{\tau}} = \mathfrak{h} + i\mathfrak{q}$ is a hermitian real form of $\mathfrak{g}_\mathbb{C}$ with $\mathfrak{z}(\mathfrak{k}^c) = \mathbb{R}ih$. For any invariant cone $C_{\mathfrak{g}^c} \subseteq \mathfrak{g}^c$ containing $-ih$, we then obtain by

$$C := iC_{\mathfrak{g}^c} \cap \mathfrak{q}$$

an $e^{\operatorname{ad}\mathfrak{h}}$-invariant cone in $\mathfrak{q}$ with $h \in C^\circ$. This specifies an embedding

$$(\mathfrak{g}, \tau, C) \hookrightarrow (\mathfrak{g}_{\mathbb{C}}, \overline{\tau}, iC_{\mathfrak{g}^c})$$

of causal symmetric Lie algebras of non-compact type, and we thus obtain a parametrization of irreducible non-compactly causal symmetric Lie algebras in terms of Euler elements ([MNÓ23, Thm. 4.21]). Actually this reference works with the minimal $\operatorname{Inn}(\mathfrak{h})$-invariant cones in $\mathfrak{q}$, but these are in one-to-one correspondence with the maximal ones.

**Theorem 3.39** (Classification of irreducible non-compactly causal symmetric Lie algebras) *Let $\mathfrak{g}$ be a simple real Lie algebra and pick a Cartan involution $\theta$ with $\theta(h) = -h$. Then the assignment*

$$h \mapsto (\mathfrak{g}, \tau_h\theta, C)$$

*described above defines a bijection from the set $\mathcal{E}(\mathfrak{g})/\operatorname{Inn}(\mathfrak{g})$ of conjugacy classes of Euler elements to the isomorphism classes of irreducible non-compactly causal symmetric Lie algebras with maximal $\operatorname{Inn}(\mathfrak{h})$-invariant cone.*

The following result provides an interesting perspective on how to determine the Euler element, if we are given the causal symmetric space $M$. It is characterized by the requirement that its positivity domain is non-empty.

**Theorem 3.40** ([MNÓ24, Cor. 6.3]) *For an irreducible non-compactly causal symmetric space $M = G/H$, there exists a unique conjugacy class of Euler elements $\mathcal{O}_h \subseteq \mathfrak{g}$ for which $W_M^+(h) \neq \emptyset$. In particular $W_M^+(-h) = \emptyset$ if $h$ is not symmetric.*

The preceding discussion dealt with irreducible ncc spaces, but sometimes it is also convenient to attach to a pair $(\mathfrak{g}, h)$ of a reductive Lie algebra and an Euler element $h \in \mathfrak{g}$ an ncc space in a natural fashion. We describe this construction in the following definition.

**Definition 3.41** (The canonical ncc symmetric space associated to a **reductive** Lie group) Assume that $\mathfrak{g}$ is reductive and that $G$ is a corresponding connected Lie group. We choose an involution $\theta$ on $\mathfrak{g}$ in such a way that it fixes the center pointwise and restricts to a Cartan involution on the commutator algebra $[\mathfrak{g}, \mathfrak{g}]$. Then the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ satisfies $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{k}$. We write $K := G^\theta$ for the subgroup of $\theta$-fixed points in $G$.

For an Euler element $h \in \mathfrak{g}$, we write $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{g}_1$, $h = h_z + h_2 \in \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_2$, and $\mathfrak{g}_2$ is the ideal generated by the projection $h_2$ of $h$ to the commutator algebra. We consider the involution $\tau$ on $\mathfrak{g}$ with

$$\tau|_{\mathfrak{g}_1} = \operatorname{id}_{\mathfrak{g}_1} \quad \text{and} \quad \tau|_{\mathfrak{g}_2} = \tau_h\theta.$$

We **assume** that $\tau$ integrates to an involutive automorphism $\tau^G$ of $G$. We write $\mathfrak{h} := \mathfrak{g}^\tau$ and $\mathfrak{q} := \mathfrak{g}^{-\tau} \subseteq \mathfrak{g}_2$ for the $\tau$-eigenspaces in $\mathfrak{g}$. Then there exists in $\mathfrak{q}$ a unique maximal pointed generating $e^{\mathrm{ad}\,\mathfrak{h}}$-invariant cone $C$ containing $h_2$ in its interior (Theorem 3.39 above deals with minimal cones, but the minimal and the maximal cone determine each other by duality). We choose an open $\theta$-invariant subgroup $H \subseteq G^\tau$, satisfying $\mathrm{Ad}(H)C = C$. This is always the case for $H_{\min} = G^\tau_e$ (the minimal choice of an open subgroup $H \subseteq G^\tau$). By [MNÓ23, Cor. 4.6], $\mathrm{Ad}(H)C = C$ is equivalent to $H_K = H \cap K$ fixing $h$, so that we also have a maximal choice $H_{\max} = K^{\tau_h, h} \exp(\mathfrak{h}_\mathfrak{p})$, which leads to a minimal causal symmetric space $G/H_{\max}$. We call

$$M = G/H \cong G_2/H_{2,\max} \tag{105}$$

the *(minimal) non-compactly causal symmetric space* corresponding to $\mathfrak{g}$, resp., to $G$.

### 3.5.4 Non-triviality of wedge regions

Wedge regions have been studied in detail for compactly and non-compactly causal symmetric spaces in [NÓ23a] and [NÓ23b, MNÓ24], respectively. For causal flag manifolds (Section 3.4), we refer to [MN26] and [Ne26] for a discussion of wedge regions and to [Be96, Be98, Be00, BN04] for Jordan theoretic aspects. The case of non-reductive Lie groups is still poorly understood; but see [BN25] and [Oeh22a, Oeh23]. We shall return to this topic below.

**Problem 3.42** Let $h \in \mathfrak{g}$ be an Euler element and $M = G/H$ a causal homogeneous space.

(a) How can we determine effectively if $W_M^+(h) \neq \emptyset$? A sufficient condition is given in Proposition 3.4.
(b) If $h$ is symmetric, i.e., $-h = \mathrm{Ad}(g)h$ for some $g \in G$, then $W_M^+(-h) = g.W_M^+(h)$ is nonempty if $W_M^+(h) \neq \emptyset$. The converse is not true by Example 2.17, where $W_M^+(\pm h) \neq \emptyset$ but $h$ is not symmetric. However, for irreducible non-compactly causal symmetric spaces the converse is true (Theorem 3.40). Is there a natural characterization of those cases where $W_M^+(\pm h) \neq \emptyset$?
(c) How are these conditions related to the existence of fixed points of the modular vector field $X_h^M$, i.e., to $\mathcal{O}_h \cap \mathfrak{h} \neq \emptyset$?

**Example 3.43** In this context, the Euler element

$$h_1 := \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathfrak{sl}_3(\mathbb{R})$$

from Example 2.9 is instructive. It is not symmetric; note that $-h_1 \in \mathcal{O}_{h_2} \neq \mathcal{O}_{h_1}$. The corresponding non-compactly causal symmetric space is

$$M = \{gI_{1,2}g^\top : g \in \mathrm{SL}_3(\mathbb{R})\} \subseteq \mathrm{Sym}_3(\mathbb{R}), \qquad I_{1,2} = \mathrm{diag}(1,-1,-1).$$

Then $I_{1,2} \in W_M^+(h_1) \neq \emptyset$, but $W_M^+(-h_1) = \emptyset$ and the vector field

$$X^M_{h_1}(x) = h_1 x + x h_1$$

has no zeros on $M \subseteq \mathrm{Sym}_3(\mathbb{R})$. In fact, if $X^M_{h_1}(x) = 0$, then $x$ anticommutes with $h_1$. If $v \in \mathbb{R}^3$ is an $h_1$-eigenvector with $h_1 v = \lambda v$, it follows that $h_1 x v = -\lambda x v$; contradicting the fact that the eigenvalues of $h_1$ are $\frac{2}{3}$ and $-\frac{1}{3}$. We refer to [Ne26, Prop. 5.7] for a detailed discussion of this class of spaces and their modular flows.

### 3.6 Wedge regions in non-compactly causal symmetric spaces

Having introduced several classes of causal homogeneous spaces, we now turn to wedge regions in irreducible non-compactly causal symmetric spaces, such as de Sitter space. We mainly put some of the results from [MNÓ24] into the context in which they are used below.

Here $G$ denotes a connected simple Lie group, $h \in \mathfrak{g}$ is an Euler element, $\tau = \theta\tau_h$ for a Cartan involution $\theta$ satisfying $\theta(h) = -h$, and $M = G/H$ is a corresponding non-compactly causal symmetric space, where the causal structure is specified by a maximal $\mathrm{Ad}(H)$-invariant closed convex cone $C \subseteq \mathfrak{q}$ satisfying $h \in C^\circ$ (Theorem 3.39).

First we consider the "minimal" space associated to the triple $(\mathfrak{g}, \tau, C)$. It is obtained as

$$M_{\mathrm{ad}} := G_{\mathrm{ad}}/H_{\mathrm{ad}},$$

where

$$G_{\mathrm{ad}} := \mathrm{Ad}(G) = \mathrm{Inn}(\mathfrak{g}) \quad \text{and} \quad H_{\mathrm{ad}} := K^h_{\mathrm{ad}} \exp(\mathfrak{h}_{\mathfrak{p}}) \subseteq G^\tau_{\mathrm{ad}}$$

(see [MNÓ23, Rem. 4.20(b)] for more details). Then all other causal symmetric spaces associated to the same triple $(\mathfrak{g}, \tau, C)$ are coverings of $M_{\mathrm{ad}}$. In addition, we assume that the causal structure is maximal, i.e., that $C \subseteq \mathfrak{q}$ is a maximal proper $\mathrm{Inn}(\mathfrak{h})$-invariant convex cone in $\mathfrak{q}$. Then $h \in \mathfrak{q}_{\mathfrak{k}} = \mathfrak{q} \cap \mathfrak{k}$ and we consider the domain

$$\Omega_{\mathfrak{q}_{\mathfrak{k}}} = \Big\{ x \in \mathfrak{q}_{\mathfrak{k}} : r_{\mathrm{Spec}}(\mathrm{ad}\, x) < \frac{\pi}{2} \Big\},$$

where $r_{\mathrm{Spec}}(\mathrm{ad}\, x)$ is the spectral radius of $\mathrm{ad}\, x$.

**Theorem 3.44** (Wedge regions in minimal ncc symmetric spaces) *If $M_{\mathrm{ad}} \cong G/H$ is a minimal ncc symmetric space, endowed with the maximal causal structure, and $h \in \mathfrak{q} \subseteq \mathfrak{g}$ is a corresponding Euler element, then the positivity* *domain in $M_{\rm ad}$ is connected and of the form*

$$W^+_{M_{\rm ad}}(h) = \mathrm{Ad}(G)^h_e \exp(\Omega_{\mathfrak{q}_\mathfrak{k}}).eH \tag{106}$$

For proofs we refer to [MNÓ24, Thms. 3.6] for the form of the domain, and to [MNÓ24, Thm 7.1] for its connectedness. The semigroup $S_W$ of $W$ in (106) actually is a group, as we shall see in Theorem 5.24 below.

By [MNÓ23, Rem. 4.20(a)], we have $H = H_K \exp(\mathfrak{h}_\mathfrak{p})$ with $H_K \subseteq K^h$, so that $\mathrm{Ad}(H) \subseteq H_{\rm ad}$. Therefore

$$q\colon M \to M_{\rm ad}, \quad gH \mapsto \mathrm{Ad}(g)H_{\rm ad}$$

defines a covering of causal symmetric spaces. The stabilizer in $G$ of the base point in $M_{\rm ad}$ is the subgroup

$$H^\sharp := \mathrm{Ad}^{-1}(H_{\rm ad}) = K^h \exp(\mathfrak{h}_\mathfrak{p}) \tag{107}$$

because $Z(G) = \ker(\mathrm{Ad}) \subseteq K^h$. Note that $H^\sharp$ need not be contained in $G^\tau$ because $\tau$ may act non-trivially on $K^h$ (cf. Remark 3.45). So we may consider $M_{\rm ad}$ as the homogeneous $G$-space

$$M_{\rm ad} \cong G/H^\sharp.$$

As $q$ is a $G$-equivariant covering of causal manifolds,

$$\begin{aligned} W^+_M(h) &= q^{-1}(W^+_{M_{\rm ad}}(h)) = q^{-1}(G^h_e \exp(\Omega_{\mathfrak{q}_\mathfrak{k}}).eH_{\rm ad}) = G^h_e \exp(\Omega_{\mathfrak{q}_\mathfrak{k}})H^\sharp.eH \\ &= G^h_e \exp(\Omega_{\mathfrak{q}_\mathfrak{k}})K^h.eH = G^h_e K^h \exp(\Omega_{\mathfrak{q}_\mathfrak{k}}).eH = G^h \exp(\Omega_{\mathfrak{q}_\mathfrak{k}}).eH, \end{aligned}$$

where the last equality follows from the polar decomposition $G^h = K^h \exp(\mathfrak{p}^{\tau_h})$. The inverse image under the map $q_M\colon G \to G/H = M$ is therefore given by

$$\begin{aligned} q_M^{-1}(W^+_M(h)) &= G^h \exp(\Omega_{\mathfrak{q}_\mathfrak{k}})H^\sharp = G^h \exp(\Omega_{\mathfrak{q}_\mathfrak{k}})K^h \exp(\mathfrak{h}_\mathfrak{p}) \\ &= G^h K^h \exp(\Omega_{\mathfrak{q}_\mathfrak{k}}) \exp(\mathfrak{h}_\mathfrak{p}) = G^h \exp(\Omega_{\mathfrak{q}_\mathfrak{k}}) \exp(\mathfrak{h}_\mathfrak{p}). \end{aligned}$$

Next we recall from [MNÓ24, Cor. 8.4] that the map

$$G^h_e \times_{K^h_e} \Omega_{\mathfrak{q}_\mathfrak{k}} \to W^+_{M_{\rm ad}}(h), \quad [g,x] \mapsto g\exp(x)H_{\rm ad} \tag{108}$$

is a diffeomorphism. Therefore $W^+_{M_{\rm ad}}(h)$ is an affine bundle over the Riemannian symmetric space $G^h_e/K^h_e$, hence contractible and therefore simply connected. So its inverse image $W^+_M(h)$ in $M$ is a union of open connected components, all of which are mapped diffeomorphically onto $W^+_{M_{\rm ad}}(h)$ by $q_M$, and the group $\pi_0(K^h) \cong K^h/K^h_e$ acts transitively on the set of connected components. It follows in particular that the diffeomorphism (108) lifts to a diffeomorphism

$$G_e^h \times_{K_e^h} \Omega_{\mathfrak{q}_\mathfrak{k}} \to W := W_M^+(h)_{eH}, \quad [g,x] \mapsto g\exp(x)H. \tag{109}$$

*Remark 3.45* (The possibilities for $H$) For $m \in \mathbb{N} \cup \{\infty\}$, let $G_m$ be a connected Lie group with Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ and $|Z(G_m)| = m$. For $m \in \mathbb{N}$ this means that $Z(G_m) \cong \mathbb{Z}/m\mathbb{Z}$ and $G_m$ is an $m$-fold covering of $\mathrm{Ad}(G_m) \cong \mathrm{PSL}_2(\mathbb{R}) \cong G_1$. Note that $G_2 \cong \mathrm{SL}_2(\mathbb{R})$. Further $G_\infty \cong \widetilde{\mathrm{SL}}_2(\mathbb{R})$ is simply connected with $Z(G_\infty) \cong \mathbb{Z}$.

We consider the Cartan involution $\theta(x) = -x^\top$, the Euler element

$$h = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad z_\mathfrak{k} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{k} = \mathfrak{so}_2(\mathbb{R}), \tag{110}$$

which satisfies $e^{2\pi z_\mathfrak{k}} = -\mathbf{1}$. Then

$$K_m = \exp(\mathbb{R} z_\mathfrak{k}), \quad Z(G_m) = \exp(2\pi\mathbb{Z} z_\mathfrak{k}),$$

and

$$\tau_h(\exp t z_\mathfrak{k}) = \tau(\exp t z_\mathfrak{k}) = \exp(-t z_\mathfrak{k})$$

because $\tau = \theta\tau_h$. We conclude that

$$K_m^\tau = \begin{cases} \{e\} & \text{if } \ m = \infty, \\ \{e, \exp(m\pi z_\mathfrak{k})\} & \text{otherwise.} \end{cases}$$

For $m = \infty$, $H = G_m^\tau$ is connected. For $m \in \mathbb{N}$, the group $G_m^\tau = K_m^\tau \exp(\mathfrak{h})$ has two connected components, but if $m$ is odd, then $K_m^\tau$ does not fix the Euler element $h \in C^\circ$. Therefore only $H := \exp(\mathfrak{h})$ leads to a causal symmetric space $G_m/H$. If $m$ is even, then $H$ can either be $(G_m)_e^\tau$ or $G_m^\tau$.

In $G_1 \cong \mathrm{PSL}_2(\mathbb{R})$, the subgroup $H$ corresponds to $\mathrm{SO}_{1,1}(\mathbb{R})_e$ and the non-compactly causal symmetric space $G_1/H \cong \mathrm{dS}^2$ is the 2-dimensional de Sitter space.

The universal covering $\widetilde{\mathrm{dS}}^2$ is obtained for $m = \infty$, $G_\infty = \widetilde{\mathrm{SL}}_2(\mathbb{R})$, and then $H = \exp(\mathfrak{h})$ is connected. All other coverings of $\mathrm{dS}^2$ are obtained as $G_m/H$ for $H = \exp(\mathfrak{h})$.

## 3.7 Modular reductive compactly causal symmetric spaces

To specify positivity regions and a modular flow on a symmetric $M = G/H$, we need an Euler element $h \in \mathfrak{g}$. For **compactly causal** symmetric spaces $G/H$, the existence of an Euler element already implies the existence of a modular structure (Corollary 3.47). In particular, the corresponding modular flow has a fixed point, which we then choose as base point. The corresponding positivity regions have been investigated in detail in [NÓ23a]. Here we present a quite general result on the “automatic” existence of modular structures for reductive compactly causal symmetric spaces (Proposition 3.48).

We shall need the following observation, which is a consequence of [Oeh22b, Prop. 3.12].

**Proposition 3.46** (Modular structures via Jordan involutions) *Let $\mathfrak{g}$ be simple hermitian, $h \in \mathfrak{g}$ an Euler element, and $\mathrm{V} := \mathfrak{g}_1(h)$ the corresponding euclidean Jordan algebra. For every involutive automorphism $\alpha \in \mathrm{Aut}(\mathrm{V})$, there exists a unique automorphism $\sigma_\alpha \in \mathrm{Aut}(\mathfrak{g})$ with $\sigma_\alpha|_\mathrm{V} = \alpha$, and then $(\mathfrak{g}, \tau_h\sigma_\alpha, C_\mathfrak{g}^{-\tau_h\sigma_\alpha}, h)$ is modular compactly causal. Conversely, every simple modular compactly causal Lie algebra is of this form.*

Since the modular flow on $G/H$ generated by an Euler element $h \in \mathfrak{g}$ has a fixed point if and only if $\mathcal{O}_h$ intersects $\mathfrak{h}$, the following corollary implies the existence of fixed points for irreducible compactly causal symmetric spaces.

**Corollary 3.47** (Fixed points of modular flows) *Let $(\mathfrak{g}, \tau, C)$ be simple compactly causal and $h \in \mathfrak{g}$ an Euler element. Then $\mathcal{O}_h \cap \mathfrak{h} \neq \emptyset$.*

***Proof*** Since $\mathcal{E}(\mathfrak{g}) = \mathcal{O}_h$, the assertion follows from Proposition 3.46, which asserts that $\tau$ fixes some Euler element $k$ with $\tau = \tau_k\sigma_\alpha$. ☐

**Proposition 3.48** (Modular structures on reductive compactly causal symmetric Lie algebras) *Let $(\mathfrak{g}, \tau, C)$ be an effective*[18] *reductive compactly causal symmetric Lie algebra with $C^\circ \cap [\mathfrak{g}, \mathfrak{g}] \neq \emptyset$. If $\mathfrak{g}$ contains an Euler element, then there exist an Euler element $h' \in \mathfrak{h} = \mathfrak{g}^\tau$ and a cone $C' \subseteq C$ such that $(\mathfrak{g}, \tau, C', h')$ is a modular causal symmetric Lie algebra.*

***Proof*** (cf. [Ne26, Lemma 3.3]) (a) First we use the Extension Theorem [NÓ23a, Thm. 2.4] to find a pointed generating $\mathrm{Inn}(\mathfrak{g})$-invariant cone $C_\mathfrak{g}$ in $\mathfrak{g}$ with $-\tau(C_\mathfrak{g}) = C_\mathfrak{g}$ and $C = C_\mathfrak{g} \cap \mathfrak{q}$. It follows in particular that $\mathfrak{g}$ is quasihermitian, i.e., its simple ideals are either compact or hermitian ([Ne00, Def. VII.2.15]). We write

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_h \oplus \mathfrak{u}$$

with $\mathfrak{u}$ compact semisimple and $\mathfrak{g}_h$ a sum of hermitian simple ideals. Projecting along the compact semisimple ideal $p_\mathfrak{u} \colon \mathfrak{g} \to \mathfrak{z}(\mathfrak{g}) + \mathfrak{g}_h$ (the fixed point projection of the compact group $\mathrm{Inn}(\mathfrak{u})$), it follows that

$$C_\mathfrak{g}^\circ \cap (\mathfrak{z}(\mathfrak{g}) + \mathfrak{g}_h) = p_\mathfrak{u}(C_\mathfrak{g}^\circ) \neq \emptyset$$

(cf. Lemma 7.30 in Appendix 7.7) and likewise

$$C_\mathfrak{g}^\circ \cap \mathfrak{g}_h = p_\mathfrak{u}(C_\mathfrak{g}^\circ \cap [\mathfrak{g}, \mathfrak{g}]) \neq \emptyset.$$

Here we use that our assumption implies that

[18] This is equivalent to $\mathfrak{h}$ containing no non-zero ideal of $\mathfrak{g}$.

$$\emptyset \neq C^\circ \cap [\mathfrak{g},\mathfrak{g}] = C_\mathfrak{g}^\circ \cap \mathfrak{q} \cap [\mathfrak{g},\mathfrak{g}]. \tag{111}$$

(b) Let $h_1 \in \mathfrak{g}$ be an Euler element. Then the ideal $\mathfrak{g}_1 \trianglelefteq \mathfrak{g}$ generated by $[h_1, \mathfrak{g}]$ has trivial center and contains no compact ideal, hence only simple hermitian ideals with an Euler element appear, so that they are all of tube type. The $\tau$-invariant ideal $\mathfrak{g}_2 := \mathfrak{g}_1 + \tau(\mathfrak{g}_1)$ also has only simple hermitian tube type ideals. We may thus replace $h_1$ by an Euler element $h_2 \in [\mathfrak{g},\mathfrak{g}]$ generating the ideal $\mathfrak{g}_2$.
(c) Let $\mathfrak{j} \trianglelefteq \mathfrak{g}_2$ be a minimal $\tau$-invariant ideal. Then either $\mathfrak{j}$ is simple or a sum of two simple ideals exchanged by $\tau$. In the latter case, $\mathfrak{j} \cong \mathfrak{b} \oplus \mathfrak{b}$ with $\tau$ acting by $\tau(a,b) = (b,a)$. Any generating Euler element in $\mathfrak{j}$ has non-zero components, and all these are conjugate under inner automorphisms (Proposition 2.16). So the projection of $h_2$ to $\mathfrak{j}$ is conjugate to an element of the form $(x,x) \in \mathfrak{j}^\tau$. If $\mathfrak{j}$ is simple, then $\mathfrak{h} = \mathfrak{g}^\tau$ contains an Euler element by Proposition 3.46. Putting these results on minimal invariant ideals together, we see that $h_2$ is conjugate to an element of $\mathfrak{g}^\tau$, i.e., $\mathfrak{g}^\tau$ contains an Euler element $h_3$ generating $\mathfrak{g}_2$.
(d) The involution $\tau_3 := \tau_{h_3}$ commutes with $\tau$. Next we observe that $\mathfrak{g}^{-\tau_3} \subseteq \mathfrak{g}_2$ is contained in a sum of hermitian simple ideals. Therefore [NÓ23a, Prop. 2.7(d)] implies that the cones $C_\mathfrak{g}^{\min}$ and $C_\mathfrak{g}^{\max}$ are $-\tau_3$-invariant and

$$(C_\mathfrak{g}^{\max})^{-\tau_3} = (C_\mathfrak{g}^{\min})^{-\tau_3} = C_\mathfrak{g}^{-\tau_3}.$$

As $\mathfrak{g}_2$ intersects the interior of $C_\mathfrak{g}$ by (111), and the cone $C_\mathfrak{g}^{\min} \subseteq \mathfrak{g}_2$ is generating, it follows with Lemma 7.30 in Appendix 7.7 that

$$\emptyset \neq (C \cap \mathfrak{g}_2^{-\tau_3})^\circ = (C_\mathfrak{g} \cap \mathfrak{g}_2^{-\tau_3})^\circ = C_\mathfrak{g}^\circ \cap \mathfrak{g}_2^{-\tau_3}.$$

Now

$$C' := C \cap (-\tau_3(C)) \subseteq \mathfrak{q}$$

is an $\mathrm{Inn}(\mathfrak{h})$-invariant pointed cone in $\mathfrak{q}$. As it contains $C_\mathfrak{g} \cap \mathfrak{g}_2^{-\tau_3} \cap \mathfrak{q} = C \cap \mathfrak{g}_2^{-\tau_3}$, hence interior points of $C_\mathfrak{g}$, it has non-trivial interior. Therefore $(\mathfrak{g}, \tau, C', h_3)$ is modular. □

## 3.8 The geometric KMS condition

In Definition 1.41 we have seen how to define the subspace $\mathcal{Y}_{\mathrm{KMS}} \subseteq \mathcal{Y}$ of KMS vectors for a continuous $\mathbb{R}$-action on the complex topological vector space $\mathcal{Y}$ and a commuting antilinear involution. On the non-linear, geometric side, KMS conditions can be modeled as follows. We consider a connected complex manifold $\Xi$, endowed with a smooth $\mathbb{R}$-action $(\sigma_t)_{t\in\mathbb{R}}$ by holomorphic maps and an antiholomorphic involution $\tau_\Xi$ commuting with each $\sigma_t$. We further assume that $\Xi$ is an open domain in a larger complex manifold and that the boundary $\partial \Xi$ contains a real submanifold $M$ with the property that, for every $\tau_\Xi$-fixed point $m$ in an open subset of $\Xi^{\tau_\Xi}$, the orbit map $\mathbb{R} \to \Xi, t \mapsto \sigma^m(t)$ extends to a holomorphic map $\sigma^m \colon \mathcal{S}_{\pm\pi/2} \to \Xi$, which further extends to a continuous map

$$\sigma^m \colon \overline{\mathcal{S}_{\pm\pi/2}} \to \Xi \cup M \quad \text{with} \quad \sigma^m(\pm i\pi/2) \in M. \tag{112}$$

We have already seen these structures for $M = G$ in Theorem 3.18, so that our setting generalizes the special case of KMS points in Lie groups (Definition 3.17).

**Definition 3.49** We write

$$M_{\rm KMS} \subseteq M$$

for the set of all points $m \in M$, whose orbit map $\sigma^m \colon \mathbb{R} \to M$ extends analytically to a continuous map $\overline{\mathcal{S}_\pi} \to \Xi \cup M$, analytic on $\mathcal{S}_\pi$, such that

$$\sigma^m(\pi i) = \tau_\Xi(m).$$

The elements of $M_{\rm KMS}$ are called *KMS points of* $M$. Note that this implies that $\sigma^m(t + \pi i) = \tau_\Xi(\sigma^m(t))$ for $t \in \mathbb{R}$, and hence, by antiholomorphic continuation, that $\sigma^m(\overline{z} + \pi i) = \tau_\Xi(\sigma^m(z))$ for $z \in \mathcal{S}_\pi$, so that

$$p := \sigma^m\Big(\frac{\pi i}{2}\Big) \in \Xi^{\tau_\Xi} \quad \text{and} \quad m = \sigma^p\Big(-\frac{\pi i}{2}\Big). \tag{113}$$

The following discussion sheds some new light on Examples 1.14, 1.15 and 1.16.

**Examples 3.50** (Domains in $\mathbb{C}$) In one-dimension we have the following standard examples of simply connected proper domains in $\mathbb{C}$ with their natural actions of $\mathbb{R} \times \{\pm 1\}$.
(a) (Strips) On the strip $\mathcal{S}_\pi = \{z \in \mathbb{C} \colon 0 < \operatorname{Im} z < \pi\}$ we have the antiholomorphic involution $\tau_{\mathcal{S}_\pi}(z) = \pi i + \overline{z}$ with fixed point set

$$\mathcal{S}_\pi^{\tau_{\mathcal{S}_\pi}} = \Big\{z \in \mathcal{S}_\pi \colon \operatorname{Im} z = \frac{\pi}{2}\Big\}.$$

The group $\mathbb{R}$ acts by translations commuting with $\tau_{\mathcal{S}_\pi}$ via $\sigma_t(z) = z + t$, $M := \mathbb{R} \cup (\pi i + \mathbb{R}) = \partial \mathcal{S}_\pi$ is a real submanifold, and for $\operatorname{Im} z = \pi/2$, the orbit map $\sigma^z(t)$ extends to the closure of the strip $\mathcal{S}_{\pm\frac{\pi}{2}}$ with $\sigma^z\big(\pm\frac{\pi i}{2}\big) = z \pm \frac{\pi i}{2} \in M$. For the strip we have $M_{\rm KMS} = \mathbb{R}$.

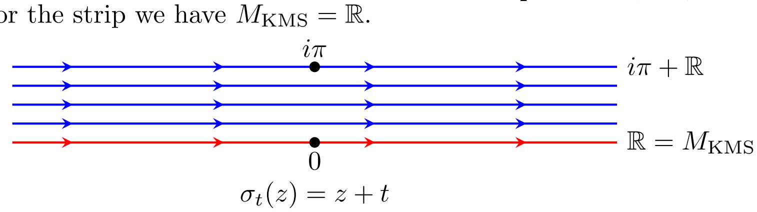


Figure 11: The KMS points for translations on the strip.

(b) (Upper half-plane) On the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$, we have the antiholomorphic involution $\tau_{\mathbb{C}_+}(z) = -\overline{z}$ and the action of $\mathbb{R}$ by dilations $\sigma_t(z) = e^t z$. Here $M := \mathbb{R} = \partial\mathbb{C}_+$ is a real submanifold, and for $z = iy$, $y > 0$, the orbit map $\sigma^z(t) = e^t z$ extends to the closure of the strip $\mathcal{S}_{\pm\frac{\pi}{2}}$ with $\sigma^z\big(\pm\frac{\pi i}{2}\big) = \pm i(iy) = \mp y$. In this case $M_{\rm KMS} = \mathbb{R}_+$.

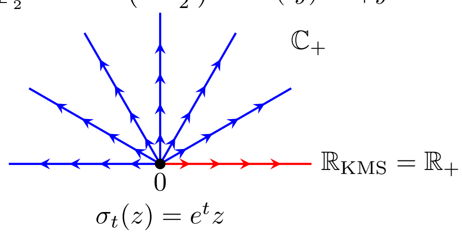



Figure 12: The dilation flow on the upper half plane $\mathbb{C}_+$.

(c) (Unit disc) On the unit disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ we have the antiholomorphic involution $\tau_{\mathbb{D}}(z) = \overline{z}$ and the action of $\mathbb{R} \cong \mathrm{SO}_{1,1}(\mathbb{R})_e$ by the fractional linear maps

$$\sigma_t(z) = \frac{\cosh(t/2)z + \sinh(t/2)}{\sinh(t/2)z + \cosh(t/2)}. \tag{114}$$

Here $M := \mathbb{S}^1 = \partial\mathbb{D}$ is a real submanifold, and for $z \in (-1,1) = \mathbb{D}\cap\mathbb{R} = \mathbb{D}^{\tau_{\mathbb{D}}}$, the orbit map $\sigma^z(t)$ extends to the closure of the strip $\mathcal{S}_{\pm\pi/2}$ with

$$\sigma^z(\pm\pi i/2) = \frac{\cos(\pi/4)z \pm i\sin(\pi/4)}{\pm i\sin(\pi/4)z + \cos(\pi/4)} = \frac{z \pm i}{\pm iz + 1} = \mp i \cdot \frac{z \pm i}{z \mp i}$$

and

$$M_{\rm KMS} = \big\{x - i\sqrt{1-x^2} \colon -1 < x < 1\big\}$$

is the lower half circle.

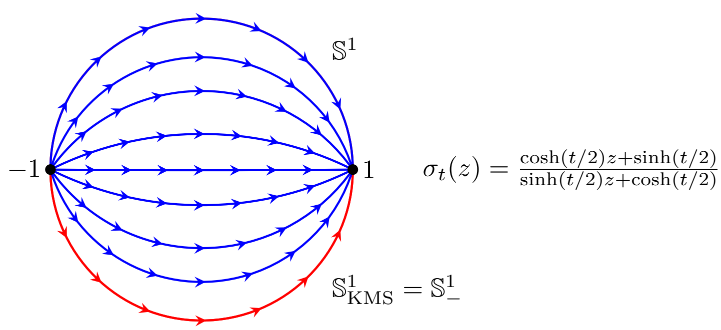



Figure 13: KMS points for the hyperbolic flow on the circle.

The biholomorphic maps

$$\mathrm{Exp}\colon \mathcal{S}_\pi \to \mathbb{C}_+, \quad z \mapsto e^z \quad \text{and} \quad \mathrm{Cay}\colon \mathbb{C}_+ \to \mathbb{D}, \quad \mathrm{Cay}(z) := \frac{z-i}{z+i} \tag{115}$$

are equivariant for the described actions of $\mathbb{R} \times \{\pm 1\}$ on the respective domains (cf. Examples 1.15 and 1.16 in Section 1.3).

**Lemma 3.51** *For a proper simply connected domain $\Omega \subseteq \mathbb{C}$, any two antiholomorphic involutions on $\Omega$ are conjugate under the group* $\mathrm{Aut}(\Omega)$ *of biholomorphic automorphisms. In particular, they have fixed points.*

***Proof*** ([ANS22, Lemma B.1]) By the Riemann Mapping Theorem, we may assume that $\Omega = \mathbb{D}$ is the unit disc. Let $\sigma\colon \mathbb{D} \to \mathbb{D}$ be an antiholomorphic involution. Then $\sigma$ is an isometry for the hyperbolic metric. Therefore the unique midpoint of the unique geodesic, connecting 0 and $\sigma(0)$, is fixed by $\sigma$. Conjugating by a suitable automorphism of $\mathbb{D}$, we may therefore assume that $\sigma(0) = 0$. Then $\psi(z) := \sigma(\overline{z})$ is a holomorphic automorphism fixing 0, hence of the form $\psi_\theta(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$, so that $\sigma(z) = e^{i\theta}\overline{z} = \gamma(\overline{\gamma^{-1}(z)})$ for $\gamma(z) = e^{i\theta/2} z$. Thus $\sigma$ is conjugate to complex conjugation. □

**Proposition 3.52** *Up to automorphisms of* $\mathbb{R} \times \{\pm 1\}$*, any antiholomorphic action of this group on a proper simply connected domain $\Omega \subseteq \mathbb{C}$ is equivalent to the one in* Examples 3.50(a)-(c).

***Proof*** Up to conjugation with biholomorphic maps, we may assume that $\tau(z) = \overline{z}$ on $\Omega = \mathbb{D}$ (Lemma 3.51). Now we simply observe that the centralizer of $\sigma_{-1}$ in the group $\mathrm{PSU}_{1,1}(\mathbb{C}) \cong \mathrm{Aut}(\mathbb{D})$ is $\mathrm{PSO}_{1,1}(\mathbb{R})$, and, up to automorphisms of $\mathbb{R} \times \{\pm 1\}$, this leads to the action in (114). [19] □

**Examples 3.53** (Examples of KMS domains)
(a) If $G = E \rtimes_\alpha \mathbb{R}$ as in Example 2.7, then $\Xi := E + iC^\circ$ is a tube domain in the complex vector space $E_\mathbb{C}$ with $E \subseteq \partial \Xi$, and Theorem 3.18 implies that

$$E_{\mathrm{KMS}} = C_+^\circ \oplus E_0 \oplus C_-^\circ,$$

which in this concrete case can be verified easily.
(b) For a causal Lie group $G$ and the complex semigroup $\Xi = S(iC_\mathfrak{g}^\circ)$, we obtain from Theorem 3.18 that

$$G_{\mathrm{KMS}} = \exp(C_+^\circ) G_e^h \exp(C_-^\circ) \quad \text{for} \quad C_\pm = \pm C_\mathfrak{g} \cap \mathfrak{g}_{\pm 1}(h).$$

(c) For a non-compactly causal symmetric space $M = G/H$, realized in the boundary of a complex crown domain $\Xi \subseteq G_\mathbb{C}/K_\mathbb{C}$ as the $G$-orbit of $o_M := \exp\big(-\frac{\pi i}{2} h\big).K_\mathbb{C}$ ([GK02, GKÓ03, GKÓ04, NÓ26]) we also have

$$M_{\mathrm{KMS}} = W_M^+(h)_{eH} = G_e^h \exp(\Omega_{\mathfrak{q}_\mathfrak{k}}).eH$$

[19] The automorphisms of the group $\mathbb{R}^\times$ have the form $\varphi(x) = \mathrm{sgn}(x)|x|^\lambda$, $\lambda \in \mathbb{R}$.

(cf. Section 4.6 and [MNÓ24, Thm. 8.2]).

## 3.9 Exercises for Section 3

**Exercise 3.54** The Cayley transform $C\colon \mathbb{R} \to \mathbb{S}^1, C(x) = \frac{i-x}{i+x}$ has a natural interpretation in terms of the stereographic projection. Show that, projecting the point $1 + 2ix$ on the tangent line through $1 \in \mathbb{S}^1 \subseteq \mathbb{C}$ with the center $-1 \in \mathbb{S}^1$ onto the circle yields $C(x)$.

**Exercise 3.55** (Positivity region of a Lorentz boost) Consider the two-dimensional Minkowski space $\mathbb{R}^{1,1} = \{(x_0, x_1) \mid x_0, x_1 \in \mathbb{R}\}$ and the $2d$-Poincaré group $G := \mathbb{R}^{1,1} \rtimes \mathrm{SO}_{1,1}(\mathbb{R})_e$.

(i) Show that
$$h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{so}_{1,1}(\mathbb{R})$$
is an Euler element in $\mathfrak{g} = \mathbb{R}^{1,1} \rtimes \mathfrak{so}_{1,1}(\mathbb{R})$.

(ii) Show that $M = \mathbb{R}^{1,1} \cong G/H$, for $H = \mathrm{SO}_{1,1}(\mathbb{R})_e$, and determine the positivity region $W_M^+(h)$ for the causal structure on $M$, specified by the cone
$$C := \{(x_0, x_1) \mid |x_1| \leq x_0\} \subseteq \mathbb{R}^{1,1} \cong T_0 M.$$

**Exercise 3.56** (Minkowski space) For the Minkowski space $E := \mathbb{R}^{1,d}$, let $\eta(x, y) := x_0 y_0 - \langle \mathbf{x}, \mathbf{y} \rangle$ and consider the closed positive light cone
$$C = \{x \in E \mid x_0 \geq 0, \quad \eta(x, x) \geq 0\},$$
so that $E$ is a causal manifold, for $C_m := C$ and $m \in E$. Show that:

(i) $G := \mathrm{SO}_{1,d}(\mathbb{R})_e$ acts by causal automorphisms on $E$ and classify $G$-orbits in $E$. Hint: Witt's Theorem, which asserts that any $\eta$-isometry between subspaces of $E$ extends to an isometry of the whole space $(E, \eta)$.

(ii) Show that, for $0 \neq m \in E$, the orbit $\mathcal{O}_m := G.m$ is a causal manifold if $T_m(\mathcal{O}_m) = m^{\perp_\eta}$ intersects $C^\circ$, i.e., $T_m(\mathcal{O}_m)$ contains timelike vectors.

(iii) Show that conditon (ii) is satisfied for $\mathbf{e}_d = (0, \ldots, 0, 1)$. If $d > 1$, then its orbit is the de Sitter space
$$\mathrm{dS}^d = \{(x_0, \mathbf{x})\colon x_0^2 - \mathbf{x}^2 = -1\}$$
and its stabilizer is $H := \mathrm{SO}_{1,d-1}(\mathbb{R})_e$, i.e. $\mathrm{dS}^d \cong \mathrm{SO}_{1,d}(\mathbb{R})_e / \mathrm{SO}_{1,d-1}(\mathbb{R})_e$.

(iv) Show that $\mathfrak{g} = \mathfrak{so}_{1,d}(\mathbb{R})$ is a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ for the stabilizer subalgebra
$$\mathfrak{h} = \mathfrak{g}_{\mathbf{e}_d} = \{A \in \mathfrak{so}_{1,d}(\mathbb{R}) : \ A\mathbf{e}_d = 0\} \cong \mathfrak{so}_{1,d-1}(\mathbb{R}),$$
and
$$\mathfrak{q} = \left\{L(x_0, \mathbf{x}) : \ (x_0, x) \in \mathbb{R}^{1,d-1}\right\} \quad \text{for} \quad L(x_0, \mathbf{x}) := \begin{pmatrix} 0 & 0 & x_0 \\ 0 & 0 & -\mathbf{x} \\ x_0 & \mathbf{x}^\top & 0 \end{pmatrix}.$$

(v) For $x \in \mathbf{e}_d^\perp$, we have $\mathrm{Exp}_{\mathbf{e}_d}(x) := \exp(L(x))\mathbf{e}_d = C(\eta(x, x))\mathbf{e}_d + S(\eta(x, x))x$, for
$$C(z) = \sum_{k=0}^\infty \frac{z^k}{(2k)!} \quad \text{and} \quad S(z) = \sum_{k=0}^\infty \frac{z^k}{(2k+1)!}.$$

Show that $\mathrm{Exp}_{\mathbf{e}_d}(x) \in \mathrm{dS}^d$ and $\frac{d}{dt}\big|_{t=0} \mathrm{Exp}_{\mathbf{e}_d}(tx) \in C$ if and only if $x \in C$.

(vi) Let $\mathcal{E}(\mathfrak{g}) \subseteq \mathfrak{g}$ be the subset of Euler elements. Show that

$$h = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0_{d-1} & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{E}(\mathfrak{g}) \quad \text{and} \quad W_M^+(h) = \mathrm{dS}^d \cap W_R,$$

for $M = \mathrm{dS}^d$ and the right Rindler wedge $W_R := \{(x_0, \mathbf{x}) \in E \colon |x_0| < x_d\}$.

(vii) For $G_e^h := \langle \exp(\mathfrak{g}_0(h)) \rangle$, show that

$$W_M^+(h) = G_e^h \exp(\Omega_{\mathfrak{q}_\mathfrak{k}}).\mathbf{e}_d,$$

where $\Omega_{\mathfrak{q}_\mathfrak{k}} := \big\{R(x) \,:\, \|x\| < \frac{\pi}{2}\big\}$, $\mathfrak{q}_\mathfrak{k} := \big\{R(x) \,:\, x \in \mathbb{R}^{d-1}\big\}$ and $R(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -x \\ 0 & x^\top & 0 \end{pmatrix}$.

**Exercise 3.57** (Causal symmetric submanifolds of $\mathrm{Sym}_n(\mathbb{R})$) In the linear space $E := \mathrm{Sym}_n(\mathbb{R})$, we consider the closed convex cone $C$ of positive semidefinite matrices, so that $E$, endowed with the constant cone field $C_m = C$, for $m \in E$, is a causal manifold. Show that:

(i) $\mathrm{GL}_n(\mathbb{R})$ acts via $g.A = gAg^\top$ by causal automorphisms on $E$.

(ii) For $G \subseteq \mathrm{GL}_n(\mathbb{R})$, any orbit $M := \{gAg^\top : g \in \mathrm{SL}_n(\mathbb{R})\}$ for which

$$T_A(M) \cap C^\circ \neq \emptyset$$

inherits the structure of a causal manifold.

(iii) Let $I_{p,q} = \mathbf{1}_p \oplus -\mathbf{1}_q$. When is the orbit $M$ of $I_{p,q}$ under $\mathrm{SL}_n(\mathbb{R})$ a causal manifold? If this is the case, find an Euler element $h \in \mathfrak{sl}_n(\mathbb{R})$ for which $W_M^+(h) \neq \emptyset$.

**Exercise 3.58** We consider the following linear bijection

$$\varphi \colon \mathbb{R}^3 \to \mathfrak{sl}_2(\mathbb{R}), \quad x = (x_0, x_1, x_2) \mapsto \widetilde{x} := \frac{1}{2}\begin{pmatrix} x_1 & -x_0 - x_2 \\ x_0 - x_2 & -x_1 \end{pmatrix},$$

and

$$\sigma_0 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that

(a) $\varphi^{-1}(X) = (-2\,\mathrm{Tr}(X\sigma_0), 2\,\mathrm{Tr}(X\sigma_2), -2\,\mathrm{Tr}(X\sigma_1))$.

(b) The Lorentz form $x^2 = x_0^2 - x_1^2 - x_2^2$ on $\mathbb{R}^3$ corresponds to the determinant by $x^2 = 4 \det \widetilde{x}$. In particular, $x \in \mathrm{dS}^2$ if and only if $\det \widetilde{x} = -\frac{1}{4}$.

(c) Show that

$$\Lambda \colon \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SO}_{1,2}(\mathbb{R})_e, \quad \Lambda(g) = \varphi^{-1} \circ \mathrm{Ad}(g) \circ \varphi$$

defines a 2-fold covering with kernel $Z(\mathrm{SL}_2(\mathbb{R})) = \{\pm\mathbf{1}\}$.

(d) The one-parameter groups $\lambda_{\sigma_i}(t) = \exp(\sigma_i t) \in \mathrm{SL}_2(\mathbb{R}), i = 1, 2$, are lifts of Lorentz boosts and $r(\theta) = \exp(-\sigma_0\theta)$ is the one-parameter group lifting the space rotations

$$\Lambda(r(\theta)) = R(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}. \tag{116}$$

**Exercise 3.59** Let $G$ be a connected Lie group and $\tau \in \mathrm{Aut}(G)$ an involutive automorphism. We consider on $G$ the involutive antiautomorphism defined by $g^\sharp := \tau(g)^{-1}$. Show that the identity compoentn of the set

$$G^\sharp := \{g \in G \colon g^\sharp = g\}$$

is a symmetric space and that the group $G$ acts transitively on this space by $g.x := gx\tau(g)^{-1}$.

## 4 Analytic continuation of orbit maps and crown domains

So far, our exploration of questions (Q1-4) led us to the insight that the natural structure to be required on $M$ is a causal $G$-invariant structure, and that the elements $h \in \mathfrak{g}$ appearing in (BW) are Euler elements. These in turn determine the corresponding regions $W \subseteq M$ as connected components of their positivity sets. In this section we now turn to question (Q4), asking for which unitary representations of $G$ nets of real subspaces satisfying (Iso), (Cov), (RS) and (BW) exist. From the Euler Element Theorem 2.3 we know that a necessary condition for the existence of such a net is not only that $h \in \mathfrak{g}$ is an Euler element, but also that we may assume that $\tau_h^{\mathfrak{g}} := e^{\pi i \operatorname{ad} h}$ integrates to an involutive automorphism $\tau_h$ on $G$, so that we can form the group

$$G_{\tau_h} = G \rtimes \{\mathbf{1}, \tau_h\}$$

and extend the representation of $G$ to an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ (cf. Section 2).

Such a representation $U$ specifies a standard subspace $\mathtt{V} = \mathtt{V}(h, U)$ by

$$\Delta_{\mathtt{V}} := e^{2\pi i \cdot \partial U(h)} \quad \text{and} \quad J_{\mathtt{V}} := U(\tau_h).$$

In view of Proposition 1.42, this standard subspace also has a description in terms of a condition resembling the KMS (Kubo–Martin–Schwinger) condition for states of operator algebras (cf. [BR96]). It consists of all vectors $\xi \in \mathcal{H}$ for which the orbit map $U_h^\xi(t) := U_h(t)\xi$ extends analytically to a continuous map $U^\xi \colon \overline{\mathcal{S}_\pi} \to \mathcal{H}$ satisfying $U_h^\xi(\pi i) = J_{\mathtt{V}}\xi$. Then $U^\xi(\pi i/2) \in \mathcal{H}^{J_{\mathtt{V}}}$ is a $J_{\mathtt{V}}$-fixed vector whose orbit map extends to the strip $\mathcal{S}_{\pm\pi/2}$ (Proposition 1.42 and Section 3.8). We now take this description as our guiding philosophy to connect standard subspaces with domains in Lie groups and homogeneous spaces.

To extend the one-dimensional strip-picture to higher dimensional Lie groups $G$, one has to specify complex manifolds $\Xi$ (crown domains), containing $G$ as a totally real submanifold,[20] to which orbit maps of $J_{\mathtt{V}}$-fixed analytic vectors extend. These domains $\Xi$ generalize the strip $\mathcal{S}_{\pm\pi/2}$, corresponding to the one-dimensional Lie group $G = \mathbb{R}$. For semisimple Lie groups,

[20] A real submanifold $N$ of a complex manifold $M$ is called *totally real* if, for every $n \in N$, the tangent space $T_n(N)$ satisfies $T_n(N) + iT_n(N) = T_n(M)$ and $T_n(N) \cap iT_n(N) = \{0\}$.

complex crown domains are obtained from crowns $\Xi_{G/K} \subseteq G_{\mathbb{C}}/K_{\mathbb{C}}$ of Riemannian symmetric spaces $G/K$ of non-compact type as their inverse images $\Xi_{G_{\mathbb{C}}}$ in the complexified group $G_{\mathbb{C}}$. These are particular well-known examples that have been used in harmonic analysis for some time (cf. [AG90], [KSt04], [FNÓ25a]), but they have never been studied systematically for general Lie groups, as outlined in Section 4.1 (cf. [BN25]).

For a crown domain $\Xi$, containing $G$, and an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$, we write

$$\mathcal{H}^\omega(\Xi) \subseteq \mathcal{H}$$

for the subspace of those analytic vectors $v$ whose orbit map $U^v\colon G \to \mathcal{H}$ extends analytically to $\Xi$. To specify the necessary boundary behavior of the extended orbit maps on $\Xi$, we put $J := U(\tau_h)$ and recall from Section 1.8.4 the dense subspace

$$\mathcal{H}^J_{\rm temp} \subseteq \mathcal{H}^J = \mathrm{Fix}(J)$$

of those $J$-fixed vectors $v$, for which the orbit map $U^v_h(t) = U(\exp th)v$ of the one-parameter group $U_h(t) = U(\exp th)$ extends to a holomorphic map $\mathcal{S}_{\pm\pi/2} \to \mathcal{H}$, and the limit

$$\beta^+(v) := \lim_{t \to -\pi/2} U^v_h(it)$$

exists in the subspace $\mathcal{H}^{-\infty}_{U_h}$ of distribution vectors of the one-parameter group $U_h$ in the weak-$*$ topology. For any real linear subspace

$$\mathtt{F} \subseteq \mathcal{H}^\omega(\Xi) \cap \mathcal{H}^J_{\rm temp},$$

we then obtain a real subspace

$$\mathtt{E} := \beta^+(\mathtt{F}) \subseteq \mathcal{H}^{-\infty}_{U_h} \subseteq \mathcal{H}^{-\infty}, \tag{117}$$

and from this space we construct a net of real subspaces indexed by open subsets $\mathcal{O} \subseteq G$ via

$$\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) := \overline{\mathrm{span}_{\mathbb{R}} U^{-\infty}(C^\infty_c(\mathcal{O}, \mathbb{R}))\mathtt{E}}. \tag{118}$$

The operators $U^{-\infty}(\varphi)$, $\varphi \in C^\infty_c(G, \mathbb{R})$, map $\mathcal{H}^{-\infty}$ into $\mathcal{H}$ because they are adjoints of continuous operators $U(\varphi) = \int_G \varphi(g)U(g)\,dg\colon \mathcal{H} \to \mathcal{H}^\infty$ (see Appendix 7.5 for details on distribution vectors). Accordingly, the closure in (118) is taken with respect to the topology of $\mathcal{H}$. It is easy to see that the net $\mathsf{H}^G_{\mathtt{E}}$ satisfies (Iso) and (Cov), and it is a key result that, if $\mathtt{F}$ is $G$-cyclic in the sense that $U(G)\mathtt{F}$ spans a dense subspace, then the net $\mathsf{H}^G_{\mathtt{E}}$ also satisfies (RS) and (BW) (Theorem 4.16).

There are many ways to construct nets of standard subspaces, but we think that the geometric approach through analytic extension of orbit maps to complex manifolds is particularly natural. Accordingly, a key assumption of our Construction Theorem 4.16 is that the dense subspace $\mathcal{H}^J_{\rm temp}$ of $\mathcal{H}^J$

contains non-trivial analytic vectors whose orbit map extends to $\Xi$. This requirement in turn is a serious restriction on the crown domain $\Xi$.

In Section 4.3 we prove the Construction Theorem that constructs for an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ natural nets of real subspaces on the group $G$ itself. In Section 4.4 we briefly discuss how such nets can be pushed forward to homogeneous spaces $M = G/H$. In the subsequent sections we describe how the Construction Theorem can be used in three important contexts:

- for representations satisfying a spectral condition (positive energy representations), where the crown domains are derived from complex Olshanski semigroups (Section 4.5).
- for general representations of semisimple Lie group, where the crown domains are derived from crowns of Riemannian symmetric spaces $G/K$ (Section 4.6), and
- for positive energy representations of the Poincaré group, where the crown domain in complexified Minkowski space is obtained from a tube domain (the complex Olshanski semigroup corresponding to the future light cone) (Section 4.7).

## 4.1 Crown domains for Lie groups

We consider the following setting:

- $G$ is a connected Lie group whose universal complexification $\eta_G \colon G \to G_{\mathbb{C}}$ has discrete kernel. If $G$ is simply connected, then $G_{\mathbb{C}}$ is the simply connected group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$, and this condition is satisfied (cf. [HN12, Thm. 15.1.4(i)]).
- $h \in \mathfrak{g}$ is an Euler element, for which the associated involution $\tau_h^{\mathfrak{g}} = e^{\pi i \,\mathrm{ad}\, h}$ of $\mathfrak{g}$ integrates to an involutive automorphism $\tau_h$ of $G$. This is always the case if $G$ is simply connected. We write

$$G_{\tau_h} = G \rtimes \{\mathbf{1}, \tau_h\} := G \rtimes_{\tau_h} \{\pm 1\} \tag{119}$$

  for the corresponding semidirect product and abbreviate $\tau_h = (e, -1)$ for the corresponding element of $G_{\tau_h}$. The universality of $G_{\mathbb{C}}$ implies the existence of a unique antiholomorphic involution $\overline{\tau}_h$ on $G_{\mathbb{C}}$ satisfying $\overline{\tau}_h \circ \eta_G = \eta_G \circ \tau_h$.

We now present an axiomatic specification of crown domains for $G$, to which orbit maps of $J$-fixed vectors in antiunitary representations may extend in such a way that boundary values lead to nets of real subspaces on $G$ as in (118).

**Definition 4.1** (a) A $(G, h)$-*crown domain* is a connected complex manifold $\Xi$ containing $G$ as a closed totally real submanifold, such that the following conditions are satisfied:

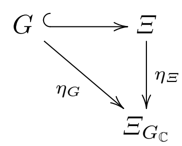
(Cr1) The natural action of $G_{\tau_h}$ on $G$ by $(g, 1).x = gx$ and $(e, -1).x = \tau_h(x)$ for $g, x \in G$, extends to an action on $\Xi$, such that $G$ acts by holomorphic maps and $\tau_h$ by an antiholomorphic involution, denoted $\overline{\tau}_h$. These extensions are unique because $G$ is totally real in $\Xi$.

(Cr2) There exists an $e$-neighborhood $W^{\rm cr} \subseteq \Xi^{\overline{\tau}_h}$ (the set of $\overline{\tau}_h$-fixed points in $\Xi$) such that, for every $p \in W^{\rm cr}$, the orbit map

$$\alpha^p : \mathbb{R} \to \Xi, \quad \alpha^p(t) := \exp(th).p$$

extends to a holomorphic map $\mathcal{S}_{\pm\pi/2} \to \Xi$.

(Cr3) The map $\eta_G : G \to G_{\mathbb{C}}$ extends to a holomorphic ($G_{\tau_h}$-equivariant) map $\eta_\Xi : \Xi \to G_{\mathbb{C}}$, which is a covering of the open subset $\Xi_{G_{\mathbb{C}}} := \eta_\Xi(\Xi)$, so that we have the commutative diagram




(b) We call the triple $(G, h, \Xi)$ a *crowned Lie group*. For crowned Lie groups $(G_j, h_j, \Xi_j)$, $j = 1, 2$, a *morphism of crowned Lie groups* is a holomorphic map $\varphi : \Xi_1 \to \Xi_2$, restricting to a Lie group morphism $\varphi_G : G_1 \to G_2$ such that

$$\mathbf{L}(\varphi_G)h_1 = h_2.$$

This implies that $\varphi_G \circ \tau_{h_1} = \tau_{h_2} \circ \varphi_G$, and, by analytic continuation, $\varphi$ intertwines the $G_{1,\tau_{h_1}}$-action on $\Xi_1$ with the $G_{2,\tau_{h_2}}$-action on $\Xi_2$.

*Remark 4.2* Note that, if $\eta_G : G \to G_{\mathbb{C}}$ is injective, so that $G$ is a subgroup of the complex Lie group $G_{\mathbb{C}}$, then (Cr1-3) are trivially satisfied for $\Xi = G_{\mathbb{C}}$. But this domain is much too large for the Construction Theorem to apply, as the example of the affine group $G = \mathrm{Aff}(\mathbb{R})_e$ already shows (Example 4.7).

*Remark 4.3* (On the condition (Cr2)) Let us denote by $\mathcal{W}$ the set of all points $p \in \Xi^{\overline{\tau}_h}$ whose orbit map $\alpha^p$ has the holomorphic extension property referred to in (Cr2). The fixed point set $\Xi^{\overline{\tau}_h}$ is invariant under the connected subgroup $G_e^h = \langle \exp \mathfrak{g}_0(h) \rangle$, which commutes with the involution $\overline{\tau}_h$ on $\Xi$. As it also commutes with $\exp(\mathbb{R}h)$, the set $\mathcal{W}$ is also $G_e^h$-invariant. This shows that, if $e$ belongs to the interior of $\mathcal{W}$, then there exists a $G_e^h$-invariant connected open subset $W^{\rm cr} \subseteq \mathcal{W}$ with $e \in W^{\rm cr}$.

If $\Omega' \subseteq \mathfrak{g}^{-\tau_h^{\mathfrak{g}}}$ is a convex open 0-neighborhood with $\exp(i\Omega') \subseteq \eta_\Xi(\Xi)$, then the exponential function $i\Omega' \to \eta_\Xi(\Xi)$ lifts uniquely to an analytic map

$$\exp : i\Omega' \to \Xi \quad \text{with} \quad \exp(0) = e, \quad \text{the unit element of } G \subseteq \Xi. \tag{120}$$

Since $i\mathfrak{g}^{-\tau_h^{\mathfrak{g}}} \subseteq \mathfrak{g}_{\mathbb{C}}^{\overline{\tau}_h^{\mathfrak{g}}}$, we have $\exp(i\Omega') \subseteq \Xi^{\overline{\tau}_h}$, so that any $G_e^h$-invariant domain $W^{\rm cr}$ contains an open subset of the form $G_e^h.\exp(i\Omega')$. For this reason, we may assume that $W^{\rm cr}$ is of this form for some $\Omega$.

For any element $x = x_1 + x_{-1} \in \mathfrak{g}^{-\tau_h^{\mathfrak{g}}}$ with $x_{\pm 1} \in \mathfrak{g}_{\pm 1}(h)$, we have

$$\zeta(ix) = x_1 - x_{-1} \in \mathfrak{g}^{-\tau_h^{\mathfrak{g}}} \quad \text{for} \quad \zeta := e^{-\frac{\pi i}{2}\operatorname{ad} h} \in \operatorname{Aut}(\mathfrak{g}_{\mathbb{C}}).$$

We thus associate to $W^{\rm cr} = G_e^h.\exp(i\Omega')$ the open subset

$$W^G := G_e^h.\exp(\Omega) \subseteq G \quad \text{for} \quad \Omega := \zeta(i\Omega') \subseteq \mathfrak{g}^{-\tau_h^{\mathfrak{g}}}. \tag{121}$$

*Remark 4.4* (On the existence of crown domains) If $\eta_G$ is injective, we may identify $G$ with a closed subgroup of $G_{\mathbb{C}}$. Then the covering required by (Cr3) is trivial, so that we may view $\Xi$ as an open subset of $G_{\mathbb{C}}$, invariant under the $G_{\tau_h}$-action. Domains with this property are easily constructed as products $\Xi := G\exp(i\Omega)$, where $\Omega \subseteq \mathfrak{g}$ is an open convex neighborhood of $[-\frac{\pi}{2}, \frac{\pi}{2}]h$, invariant under $-\tau_h^{\mathfrak{g}}$, for which the polar map

$$G \times \Omega \to G_{\mathbb{C}}, \quad (g, x) \mapsto g\exp(ix)$$

is a diffeomorphism onto an open subset. With the results from Appendix 7.3, concretely, Lemma 7.15, Section 7.3.3 and Lemma 7.18, it follows that this is the case if $G$ is simply connected and

$$\Omega = \{x \in \mathfrak{g} \colon \operatorname{Spec}(\operatorname{ad} x) \subseteq (-\pi, \pi) + i\mathbb{R}\}.$$

Then

$$\Xi^{\overline{\tau}_h} = G^{\tau_h}.\exp(i\Omega^{-\tau_h^{\mathfrak{g}}}) \supseteq G_e^h.\exp(i\Omega^{-\tau_h^{\mathfrak{g}}}),$$

so that any sufficiently small open convex 0-neighborhood $\Omega' \subseteq \Omega^{-\tau_h^{\mathfrak{g}}}$ specifies an open neighborhood $W^{\rm cr} := G_e^h.\exp(i\Omega')$ of $e$ in $\Xi^{\overline{\tau}_h}$. Since $\exp(ith) \in \Xi$ for $|t| \leq \pi/2$, an easy compactness argument shows that, if $\Omega'$ is small enough, then $\exp(ith)\exp(i\Omega') \subseteq \Xi$ for $|t| \leq \pi/2$, so that (Cr2) is satisfied. Therefore crown domains $\Xi$ satisfying (Cr1–3) exist in abundance, but, as the example of the affine group $G = \operatorname{Aff}(\mathbb{R})_e$ already shows (Example 4.7), most of them are too large.

**Example 4.5** For $G = \mathbb{R} \subseteq \mathbb{C} = G_{\mathbb{C}}$ and $h = 1$ (a basis element in $\mathfrak{g} = \mathbb{R}$), any strip

$$\Xi = \mathcal{S}_{\pm r} = \{z \in \mathbb{C} \colon |\operatorname{Im} z| < r\} \subseteq \mathbb{C} = G_{\mathbb{C}}, \qquad r \geq \pi/2,$$

is a crown domain for $G = \mathbb{R}$ with $W^{\rm cr} = G$. In this case $\tau_h = \mathrm{id}$, $\overline{\tau}_h(z) = \overline{z}$ and $G_{\tau_h} \cong (\mathbb{R}^\times, \cdot)$.

Below we shall encounter various kinds of non-abelian examples.

## 4.2 Constructions of crown domains

After the discussion of the axiomatic setup of crowned Lie groups, we now turn to more concrete tools to construct such domains in various geometric environments. The following lemma is very useful in this regard.

**Lemma 4.6** *Suppose that $\eta_G$ is injective and that $G_{\mathbb{C},\overline{\tau}_h} = G_{\mathbb{C}} \rtimes \{\mathbf{1}, \overline{\tau}_h\}$ acts on the complex manifold $M$ in such a way that $G_{\mathbb{C}}$ acts by holomorphic maps and $\overline{\tau}_h$ acts by an antiholomorphic involution $\tau_M$. Let $\Xi_M \subseteq M$ be a $G_{\tau_h}$-invariant connected open subset, for which there exists an open subset $W^{M,c} \subseteq \Xi_M^{\tau_M}$, satisfying*

$$\exp(\mathcal{S}_{\pm\pi/2}h).W^{M,c} \subseteq \Xi_M.$$

*Then, for every $m_0 \in W^{M,c}$, the open subset*

$$\Xi := \{g \in G_{\mathbb{C}} \colon g.m_0 \in \Xi_M\}$$

*satisfies* (Cr1-3) *with the open subset*

$$W^{\rm cr} := \{g \in G_{\mathbb{C}}^{\overline{\tau}_h} \colon g.m_0 \in W^{M,c}\} \subseteq \Xi^{\overline{\tau}_h}.$$

***Proof*** (Cr1): Since $\overline{\tau}_h(m_0) = \tau_M(m_0) = m_0$, $\overline{\tau}_h(\Xi) = \Xi$ follows from the antiholomorphic action of $G_{\tau_h}$ on $\Xi_M$.
(Cr2): The inclusion $\exp(\mathcal{S}_{\pm\pi/2}h)W^{\rm cr} \subseteq \Xi$ follows from the the fact that

$$\exp(\mathcal{S}_{\pm\pi/2}h)W^{M,c} \subseteq \Xi_M.$$

(Cr3) is redundant because $G \subseteq G_{\mathbb{C}}$. □

**Example 4.7** (The affine group of the real line) We consider the 2-dimensional affine group of the real line

$$G = \mathrm{Aff}(\mathbb{R})_e \cong \mathbb{R} \rtimes \mathbb{R}_+ \quad \text{with} \quad \mathfrak{g} = \mathbb{R}x \rtimes \mathbb{R}h, \quad x = (1,0), \quad h = (0,1),$$

so that

$$[h,x] = x \quad \text{and} \quad \tau_h(b,a) = (-b,a).$$

A pair $(b,a) \in G$ acts on $\mathbb{R}$ by the affine map $(b,a).x = b + ax$ and so does the complex affine group $\mathrm{Aff}(\mathbb{C}) \cong \mathbb{C} \rtimes \mathbb{C}^\times$ on the complex line $\mathbb{C}$. The antiholomorphic involution $\sigma(b,a) = (\overline{b}, \overline{a})$ satisfies $\mathrm{Aff}(\mathbb{C})^\sigma = \mathrm{Aff}(\mathbb{R}) \cong \mathbb{R} \rtimes \mathbb{R}^\times$, and $G$ is the identity component of this group. Note that $G_{\mathbb{C}} \cong \widetilde{\mathrm{Aff}}(\mathbb{C}) \cong \mathbb{C} \rtimes \mathbb{C}$, with the universal map

$$\eta_G \colon G \to G_{\mathbb{C}}, \quad \eta_G(b,a) = (b, \log a).$$

The antiholomorphic extension of $\tau_h$ to $\mathrm{Aff}(\mathbb{C})$, is given by

$$\overline{\tau}_h(b,a) = (-\overline{b}, \overline{a}) \quad \text{with} \quad \mathrm{Aff}(\mathbb{C})^{\overline{\tau}_h} = i\mathbb{R} \rtimes \mathbb{R}^\times.$$

The subset

$$\exp(\mathcal{S}_{\pm\pi/2}h) = \{0\} \times \mathbb{C}_r = \{(0,z) \in \mathbb{C} \colon \operatorname{Re} z > 0\} \subseteq \{0\} \times \mathbb{C}^\times$$

is isomorphic to the right half-plane in the complex dilation group.

First, we consider in $\mathrm{Aff}(\mathbb{C})$ the domain

$$\Xi_1 := \mathbb{C} \times \mathbb{C}_r \quad \text{with} \quad \Xi_1^{\overline{\tau}_h} = i\mathbb{R} \times \mathbb{R}_+ = (\mathrm{Aff}(\mathbb{C})^{\overline{\tau}_h})_e. \tag{122}$$

Then $W_1^c := \Xi_1^{\overline{\tau}_h}$ satisfies $\exp(\mathcal{S}_{\pm\pi/2}h)W_1^c \subseteq \Xi_1$, so that (Cr2) is satisfied. Further $\eta_G$ extends to a biholomorphic map

$$\eta_{\Xi_1} \colon \Xi_1 \to \mathbb{C} \times \mathcal{S}_{\pm\pi/2} \subseteq G_{\mathbb{C}} \cong \mathbb{C} \rtimes \mathbb{C}, \quad \eta_G(b,a) \mapsto (b, \log a).$$

In [BN25, Prop. 3.2] it is shown that $\Xi_1$ is too large for our purposes because $\mathcal{H}^\omega(\Xi_1) \cap \mathcal{H}^J_{\mathrm{temp}} = \{0\}$ holds for infinite-dimensional irreducible representations. A natural strategy to find better crown domains is inspired by Riemannian symmetric spaces (cf. Theorem 4.26 below).

The group $\mathrm{Aff}(\mathbb{C})_{\overline{\tau}_h}$ acts naturally on $M := \mathbb{C}$, where $\overline{\tau}_h.z = -\overline{z}$. The two domains $\mathbb{C}_\pm$ (upper and lower half plane) are invariant under the real group $G_{\tau_h}$, and the half-lines

$$(\mathbb{C}_\pm)^{\overline{\tau}_h} = \pm i\mathbb{R}_+$$

satisfy $\exp(\mathcal{S}_{\pm\pi/2}h).(\pm i\mathbb{R}_+) = \mathbb{C}_\pm$. For $m_0 = \pm ri$, $r > 0$, we thus obtain with Lemma 4.6 a crown domain

$$\begin{aligned}\Xi_{\pm,r} &:= \{(b,a) \in \mathrm{Aff}(\mathbb{C}) \colon b \pm rai \in \mathbb{C}_\pm\} \\ &= \{(b,a) \in \mathrm{Aff}(\mathbb{C}) \colon \pm r^{-1}b + ai \in \mathbb{C}_+\} \\ &= \{(b,a) \in \mathrm{Aff}(\mathbb{C}) \colon \pm r^{-1} \operatorname{Im} b + \operatorname{Re} a > 0\}.\end{aligned} \tag{123}$$

Conjugation with $G$ dilates the parameter $r$, so that it suffices to consider the domains $\Xi_{\pm,1}$.

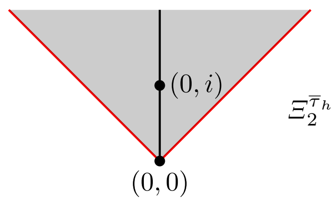
To connect with crowns of symmetric spaces, we consider $\mathbb{C}_+$ as a real 2-dimensional homogeneous space of $G$ via the orbit map $(b,a) \mapsto (b,a).i = b + ai$. It has a "complexification"

$$\eta_{\mathbb{C}_+} \colon \mathbb{C}_+ \to \mathbb{C}_+ \times \mathbb{C}_- \subseteq \mathbb{C}^2, \quad \eta_{\mathbb{C}_+}(z) = (z, \overline{z}).$$

The complex Lie group $\mathrm{Aff}(\mathbb{C})$ acts naturally on $\mathbb{C} \times \mathbb{C}$ by the diagonal action with respect to the canonical action on $\mathbb{C}$ by affine maps. We consider the complex manifold $\Xi_{\mathbb{C}\times\mathbb{C}} := \mathbb{C}_+ \times \mathbb{C}_-$ as a crown domain of the upper half plane $\mathbb{C}_+ \cong \eta_{\mathbb{C}_+}(\mathbb{C}_+)$, which is a Riemannian symmetric space of $\mathrm{SL}_2(\mathbb{R})$, acting by Möbius transformations (see [Kr08, §2.1] and [Kr09]). It carries the antiholomorphic involution $\tau(z,w) := (-\overline{z}, -\overline{w})$ and it is invariant under the real affine group $G = \mathbb{R} \rtimes \mathbb{R}_+$, so that we obtain with $\tau$ an action of $G_{\tau_h}$ on

$\Xi_{\mathbb{C}\times\mathbb{C}}$. As $\mathbb{C}_+ = G.i$, the corresponding crown domain in $\mathrm{Aff}(\mathbb{C})$, in the sense of Lemma 4.6 with $m_0 = i$, is

$$\begin{aligned}\Xi_2 &:= \{g \in \mathrm{Aff}(\mathbb{C})\colon g.\eta_{\mathbb{C}_+}(i) \in \mathbb{C}_+ \times \mathbb{C}_-\} = \{(b,a) \in \mathrm{Aff}(\mathbb{C})\colon b \pm ai \in \mathbb{C}_\pm\} \\ &= \Xi_{+,1} \cap \Xi_{-,1} = \{(b,a) \in \mathrm{Aff}(\mathbb{C})\colon |\operatorname{Im} b| < \operatorname{Re} a\}.\end{aligned}$$




For this domain

$$\Xi_2^{\overline{\tau}_h} = \{(ic,a) \in i\mathbb{R} \times \mathbb{R}_+\colon |c| < a\},$$

and, for $|t| < \pi/2$ and $(ic,a) \in \Xi_2^{\overline{\tau}_h}$, we have $e^{ith}(ic,a) = (e^{it}ic, e^{it}a)$ with

$$|\operatorname{Im}(e^{it}ic)| = \cos(t)|c| < \cos(t)a = \operatorname{Re}(e^{it}a).$$

This proves (Cr2) for $\Xi_2$ and $W_2^{\mathrm{cr}} := \Xi_2^{\overline{\tau}_h}$. We have already argued above that the domain $\Xi_1$ is too large. The smaller domain $\Xi_2$ behaves much better than $\Xi_1$. On the other hand, the even larger domains $\Xi_{\pm,1}$ from (123) work for representations satisfying the spectral condition $\mp i\partial U(x) \geq 0$ for the translation group.

**Example 4.8** (Complex Olshanski semigroups) Let $G$ be a connected Lie group for which $\eta_G$ is injective and $G_{\mathbb{C}}$ is simply connected, so that we may assume that $G \subseteq G_{\mathbb{C}}$, and let $h \in \mathfrak{g}$ be an Euler element. We assume that $C \subseteq \mathfrak{g}$ is a pointed (not necessarily generating) closed convex $\mathrm{Ad}(G)$-invariant cone, satisfying $-\tau_h(C) = C$, and that the ideal $\mathfrak{m} := C - C \trianglelefteq \mathfrak{g}$ satisfies $\mathfrak{g} = \mathfrak{m} + \mathbb{R}h$. If $\mathfrak{m} = \mathfrak{g}$, we replace $\mathfrak{g}$ by $\mathfrak{m} \rtimes_{\mathrm{ad}\,h} \mathbb{R}$, so that we may assume that $\mathfrak{g} = \mathfrak{m} \rtimes \mathbb{R}h$.

Let $M \trianglelefteq G$ and $M_{\mathbb{C}} \trianglelefteq G_{\mathbb{C}}$ be the normal integral subgroups corresponding to $\mathfrak{m}$ and $\mathfrak{m}_{\mathbb{C}}$, respectively. Since $G_{\mathbb{C}}$ is simply connected, it follows from [HN12, Thm. 11.1.21] that the subgroup $M_{\mathbb{C}} \subseteq G_{\mathbb{C}}$ is closed and simply connected, and this implies that $M \subseteq M_{\mathbb{C}}$ is closed in $G$, as the identity component of the group of fixed points of the complex conjugation on $M_{\mathbb{C}}$ with respect to $M$. Then the construction in Step 1 of the proof of [HN12, Thm. 15.1.4] shows that the inclusion map $M \hookrightarrow M_{\mathbb{C}}$ is the universal complexification of $M$. Using again that $G_{\mathbb{C}}$ is simply connected, it follows from [HN12, Prop. 11.1.19] that $G_{\mathbb{C}} \cong M_{\mathbb{C}} \rtimes_\alpha \mathbb{C}$ with $\alpha_z(g) = \exp(zh)g\exp(-zh)$, and thus

$$G \cong M \rtimes_\alpha \mathbb{R} \quad \text{with} \quad \alpha_t(g) = \exp(th)g\exp(-th).$$

We consider the complex Olshanski semigroup

$$S := M\exp(iC^\circ) \subseteq M_{\mathbb{C}},$$

for which the multiplication map $M \times C^\circ \to S, (g,x) \mapsto g\exp(ix)$ is a diffeomorphism (Definition 3.16, [Ne00, §XI.1]). Note that $S$ is invariant under $\overline{\tau}_h$. Recall from Lemma 3.7 that

$$C \subseteq C_+ + \mathfrak{g}_0(h) - C_- \quad \text{and} \quad C_+^\circ - C_-^\circ = C^\circ \cap \mathfrak{g}^{-\tau_h^{\mathfrak{g}}} \quad \text{for} \quad C_\pm := \pm C \cap \mathfrak{g}_{\pm 1}(h)$$

follow from the invariance of $C$ under $e^{\mathbb{R}\operatorname{ad} h}$ and $-\tau_h$. On the complex manifold $S$, the group $G_{\tau_h}$ acts by

$$(g,t).m = g\alpha_t(m) \quad \text{and} \quad \tau_h.m = \overline{\tau}_h(m)$$

where we use the notation $\tau_h = (e,-1) \in G_{\tau_h}$, introduced after (119). The fixed points of $\overline{\tau}_h$ form the subsemigroup $S^{\overline{\tau}_h} = M^{\tau_h}\exp(i(C_+^\circ - C_-^\circ))$.[21] For $x_{\pm 1} \in C_\pm^\circ$, we consider the element $s_0 := \exp(i(x_1 - x_{-1})) \in S^{\overline{\tau}_h}$, and in $G_{\mathbb{C}}$ the crown domain

$$\Xi = \{(g,z) \in M_{\mathbb{C}} \times \mathcal{S}_{\pm\pi/2} \colon (g,z).s_0 = g\alpha_z(s_0) \in S\}$$

(Lemma 4.6). We now verify (Cr1-2); (Cr3) is redundant because $\Xi \subseteq G_{\mathbb{C}}$.
(Cr1): First we observe that $\overline{\tau}_h(s_0) = s_0$, and that $\overline{\tau}_h(S) = S$ follows from $-\tau_h^{\mathfrak{g}}(C) = C$. Therefore

$$\overline{\tau}_h((g,z).s_0) = \overline{\tau}_h(g,z).s_0$$

implies that the domain $\Xi$ is invariant under $\overline{\tau}_h(g,z) = (\overline{\tau}_h(g), \overline{z})$.
(Cr2): We recall from [Ne22, Thms. 2.16, 2.21] that, for $y = y_1 - y_{-1} \in C_+^\circ - C_-^\circ$, we have

$$\alpha_{it}(\exp(iy)) = \exp(i(e^{it}y_1 - e^{-it}y_{-1})) \in S \quad \text{for} \quad |t| < \pi/2.$$

So we find with

$$W_M^{\mathrm{cr}} := M_e^h \exp(i(C_+^\circ + C_-^\circ))s_0^{-1} \subseteq M_{\mathbb{C}}^{\overline{\tau}_h}$$

that

$$W^{\mathrm{cr}} := G_e^h(W_M^{\mathrm{cr}} \times \{0\}) \subseteq \Xi^{\overline{\tau}_h} \quad \text{satisfies} \quad \alpha_{it}W^{\mathrm{cr}} \subseteq \Xi \quad \text{for} \quad |t| < \pi/2. \tag{124}$$

This proves (Cr2).

*Remark 4.9* The construction in Example 4.7 can also be viewed as a special case of Example 4.8. To see this, write

[21] We refere to [HN93, §7.3] for more details on such “real Olshanski semigroups”.

$$G = \mathrm{Aff}(\mathbb{R})_e = \mathbb{R} \rtimes \mathbb{R}_+ \cong M \rtimes \mathbb{R}_+ \subseteq M_{\mathbb{C}} \rtimes \mathbb{C}^\times \quad \text{with} \quad M = \mathbb{R} \times \{1\}.$$

The invariant cone $C := \mathbb{R}_+ x$ generates the ideal $\mathfrak{m} = \mathbb{R}x$, and $C = C_+$. Then $S = \mathbb{C}_+$ with $S^{\overline{\tau}_h} = i\mathbb{R}_+$. For $s_0 = ir$, $r > 0$, we obtain

$$\varXi = \{(b,a) \in M \times \mathbb{C}^\times : (b,a).s_0 = b + rai \in \mathbb{C}_+\} = \varXi_{+,r}.$$

## 4.3 Nets of real subspaces

In Section 4.2 we constructed examples of crowned Lie groups, introduced in Section 4.1. We are now ready to use this geometric setup to construct nets of real subspaces for antiunitary representations.

Given a domain $\varXi \subseteq G_{\mathbb{C}}$ satisfying (Cr1-3), and an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$, we write

$$\mathcal{H}^\omega(\varXi) \subseteq \mathcal{H}$$

for the subspace of those analytic vectors, whose orbit map extends to $\varXi$. That the non-triviality of this space imposes serious restrictions on $\varXi$ follows in particular from the discussion in the last section of [BN24]. For the group $G = \mathrm{Aff}(\mathbb{R})_e \cong \mathbb{R} \rtimes \mathbb{R}_+$, the domain must be contained in $\mathbb{C} \rtimes \mathbb{C}_r$, where $\mathbb{C}_r$ is the open right half-plane. So one has to understand the boundary behavior of the extended orbit maps on the domain $\varXi$.

For $J = U(\tau_h)$, let

$$\mathcal{H}^J_{\mathrm{temp}} \subseteq \mathcal{H}^J = \mathrm{Fix}(J)$$

be the dense real linear subspace of $\mathcal{H}^J$, consisting of those vectors $v$ for which the orbit map $U^v_h(t) = U(\exp th)v$ extends to the open strip $\mathcal{S}_{\pm\pi/2}$, and the limit

$$\beta^+(v) := \lim_{t \to \pi/2} U^v_h(-it) \tag{125}$$

exists in the subspace $\mathcal{H}^{-\infty}(U_h) \subseteq \mathcal{H}^{-\infty}$ of distribution vectors of the one-parameter group $U_h$. [22] In view of Theorem 1.44, this is equivalent to the existence of $C, N > 0$ such that

$$\|U^v_h(it)\|^2 \leq C\Big(\frac{\pi}{2} - |t|\Big)^{-N} \quad \text{for} \quad |t| < \pi/2. \tag{126}$$

By [FNÓ25a, §2.2, Lemma 2], the limit $\beta^+(v)$ always exists in the larger space $\mathcal{H}^{-\omega}_{U_h,\mathrm{KMS}}$, so that the main point is the additional regularity (temperedness), related to the weak convergence after pairing with elements of $\mathcal{H}^\infty_{U_h}$.

[22] The notation $\mathcal{H}_{\mathrm{temp}}$ refers to the "temperedness" of the boundary values, which in the classical context corresponds to tempered distributions.

The boundary values (125) are actually contained in the space $\mathcal{H}^{-\infty}_{\rm KMS}$ (see Appendix 1.8.3), consisting of those distribution vectors $\alpha$ whose orbit map

$$U_h^{-\infty,\alpha} \colon \mathbb{R} \to \mathcal{H}^{-\infty}, \quad t \mapsto U_h^{-\infty}(t)\alpha = \alpha \circ U(\exp -th)$$

extends analytically to the closed strip $\overline{\mathcal{S}_\pi}$, such that

$$U_h^{-\infty,\alpha}(\pi i) = J\alpha.$$

Using Theorem 1.43, it then follows that smearing with test functions on $\mathbb{R}$ maps $\alpha \in \mathcal{H}^{-\infty}_{\rm KMS}$ into $\mathtt{V} = \mathtt{V}(h, U)$. Therefore any real linear subspace

$$\mathtt{F} \subseteq \mathcal{H}^\omega(\Xi) \cap \mathcal{H}^J_{\rm temp}$$

leads to a real subspace

$$\mathtt{E} := \beta^+(\mathtt{F}) \subseteq \mathcal{H}^{-\infty},$$

and from this space we construct a net of real subspaces on $G$ as follows.

**Definition 4.10** Let $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ be a real linear subspace. Then, for each $\varphi \in C^\infty_c(G, \mathbb{C})$, the operator

$$U^{-\infty}(\varphi) = \int_G \varphi(g) U^{-\infty}(g)\, dg$$

maps $\mathcal{H}^{-\infty}$ into $\mathcal{H}$, because it is an adjoint of the operator $U(\varphi^*) \colon \mathcal{H} \to \mathcal{H}^\infty$. To an open subset $\mathcal{O} \subseteq G$, we associate the closed real subspace

$$\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) := \overline{\operatorname{span}_{\mathbb{R}} U^{-\infty}(C^\infty_c(\mathcal{O}, \mathbb{R}))\mathtt{E}}, \tag{127}$$

where the closure is taken with respect to the topology of $\mathcal{H}$.

*Remark 4.11* It is obvious that the net $\mathsf{H}^G_{\mathtt{E}}$ satisfies (Iso). To see that (Cov) also holds, observe that the left-invariance of the Haar measure on $G$ yields

$$U^{-\infty}(g) U^{-\infty}(\varphi) = U^{-\infty}(\delta_g * \varphi),$$

where $(\delta_g * \varphi)(x) = \varphi(g^{-1}x)$ is the left translate of $\varphi$.

*Remark 4.12* One may also consider subspaces $\mathtt{E} \subseteq \mathcal{H}$, but the key advantage of working with the larger space $\mathcal{H}^{-\infty}$ of distribution vectors is that it contains finite-dimensional subspaces invariant under ad-diagonalizable elements and non-compact subgroups. For finite-dimensional subspaces of $\mathcal{H}$, this is excluded by Moore's Theorem if $\ker U$ is discrete ([Mo80]).

The following proposition is useful to verify the inclusion $\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) \subseteq \mathtt{V}$ for an open subset $\mathcal{O} \subseteq G$.

**Proposition 4.13** *Let $(U,\mathcal{H})$ be an antiunitary representation of $G_{\tau_h}$ and $\mathtt{V} = \mathtt{V}(h,U)$ the corresponding standard subspace. For an open subset $\mathcal{O} \subseteq G$ and a real subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$, the following are equivalent:*

(a) $\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) \subseteq \mathtt{V}$.
(b) *For all $\varphi \in C^\infty_c(\mathcal{O},\mathbb{R})$ we have $U^{-\infty}(\varphi)\mathtt{E} \subseteq \mathtt{V}$.*
(c) *For all $\varphi \in C^\infty_c(\mathcal{O},\mathbb{R})$ we have $U^{-\infty}(\varphi)\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}$.*
(d) *$U^{-\infty}(g)\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}$ for every $g \in \mathcal{O}$.*

To show that $\mathsf{H}^G_{\mathtt{E}}(W^G) \subseteq \mathtt{V}$, we thus need to show that $U^{-\infty}(W^G)\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}$.

***Proof*** ([FNÓ25a, Prop. 9]) By the definition of $\mathsf{H}^G_{\mathtt{E}}(\mathcal{O})$, it is clear that (a) is equivalent to (b). Further, (b) implies (c) because $\mathtt{V} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}$ (Theorem 1.43(b)).

For the implication (c) $\Rightarrow$ (d), let $(\varphi_n)_{n\in\mathbb{N}}$ be a $\delta$-sequence in $C^\infty_c(G,\mathbb{R})$ (cf. Definition 7.23). Then $U(\varphi_n)\xi \to \xi$ in $\mathcal{H}^\infty$ and hence also in $\mathcal{H}^{-\infty}$. It follows in particular that

$$U^{-\infty}(\varphi_n * \delta_g)\eta = U^{-\infty}(\varphi_n)U^{-\infty}(g)\eta \to U^{-\infty}(g)\eta \quad \text{for} \quad \eta \in \mathcal{H}^{-\infty}.$$

Hence the closedness of $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ (Theorem 1.43(a)), shows that (c) implies (d). Here we use that $\varphi_n * \delta_g \in C^\infty_c(\mathcal{O},\mathbb{R})$ for $g \in \mathcal{O}$ if $n$ is sufficiently large.

As the $G$-orbit maps in $\mathcal{H}^{-\infty}$ are continuous and $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ is closed (Theorem 1.43), hence stable under integrals over compact subsets and $U^{-\infty}(C^\infty_c(\mathcal{O},\mathbb{R}))\mathcal{H}^{-\infty} \subset \mathcal{H}^\infty$, we see with

$$\mathcal{H}^{-\infty}_{\mathrm{KMS}} \cap \mathcal{H} = \mathtt{V}$$

(Theorem 1.43) that (d) implies (b). □

**Lemma 4.14** *The following assertions hold:*

(a) *For arbitrary $y \in \mathfrak{g}$, the corresponding Lie derivative operator*

$$L_y \colon C^\infty(\Xi,\mathcal{H}) \to C^\infty(\Xi,\mathcal{H}), \quad (L_y f)(g) := \frac{d}{dt}\Big|_{t=0} f(\exp(ty)g)$$

*satisfies $L_y(\mathcal{O}(\Xi,\mathcal{H})) \subseteq \mathcal{O}(\Xi,\mathcal{H})$.*

(b) *We have $\mathtt{d}U(\mathfrak{g})\mathcal{H}^\omega(\Xi) \subseteq \mathcal{H}^\omega(\Xi)$ and, for $x \in \mathfrak{g}_{\mathbb{C}},\ p \in \Xi,\ v \in \mathcal{H}^\omega(\Xi)$,*

$$U^{\mathtt{d}U(x)v}(p) = \mathtt{d}U(\mathrm{Ad}(\eta_\Xi(p))x)U^v(p) \tag{128}$$

*where $U^v \in \mathcal{O}(\Xi,\mathcal{H})$ is the holomorphic extension of the analytic orbit map $U^v \colon G \to \mathcal{H}$.*

(c) *The closure of $\mathcal{H}^\omega(\Xi)$ in $\mathcal{H}$ is $U(G)$-invariant. If, in particular, $U$ is irreducible and $\mathcal{H}^\omega(\Xi)$ is non-zero, then $\mathcal{H}^\omega(\Xi)$ is dense in $\mathcal{H}$.*

***Proof*** (a) The operator $L_y$ is the Lie derivative with respect to a fundamental vector field defined by the action of $G$ by left-translations on $\Xi$. As this

vector field is holomorphic, it preserves on each open subset the subspace of holomorphic functions.
(b) Let $v \in \mathcal{H}^\omega(\Xi)$ and $x \in \mathfrak{g}$. For arbitrary $g \in G$ we have

$$U(g)\partial U(x)v = \partial U(\mathrm{Ad}(g)x)U(g)v,$$

so that (128) follows for $p = g \in G$. The general case $p \in \Xi$ is then obtained by analytic extension. Since the mapping $G_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$, $g \mapsto \mathrm{Ad}(g)x$, is holomorphic, for any basis $y_1, \ldots, y_m$ of $\mathfrak{g}$, there exist $\chi_1, \ldots, \chi_m \in \mathcal{O}(G_\mathbb{C})$ with

$$\mathrm{Ad}(g)x = \chi_1(g)y_1 + \cdots + \chi_m(g)y_m \quad \text{for all} \quad g \in G_\mathbb{C}.$$

By plugging this in (128), it suffices to prove that, for every $y \in \mathfrak{g}$, the function $w \mapsto \partial U(y)U^v(w)$ is holomorphic on $\Xi$.

We now check this last fact. From the $G$-action on $\Xi$ it follows that, for every $w \in \Xi$, there exists $t_w \in \mathbb{R}_+$, such that for all $t \in (-t_w, t_w)$ we have $\exp(ty)w \in \Xi$, and then $U^v(\exp(ty)w) = U(\exp(ty))U^v(w)$. Taking the derivative at $t = 0$ in this equality, we obtain

$$L_y(U^v)(w) = \frac{d}{dt}\Big|_{t=0} U^v(\exp(ty)w) = \partial U(y)U^v(w)$$

for $w \in \Xi$, where $L_y \colon C^\infty(\Xi, \mathcal{H}) \to C^\infty(\Xi, \mathcal{H})$ is the Lie derivative operator in the direction $y \in \mathfrak{g}$. Since $L_y(\mathcal{O}(\Xi, \mathcal{H})) \subseteq \mathcal{O}(\Xi, \mathcal{H})$ by (a),we are done.
(c) In view of (b), the closure of $\mathcal{H}^\omega(\Xi)$ is $U(G)$-invariant by [HC53, Cor. to Thm. 2, pp. 210–211] or [Wa72, Prop. 4.4.5.6]. Therefore, if the representation $U$ is irreducible and $\mathcal{H}^\omega(\Xi) \neq \{0\}$, then the linear subspace $\mathcal{H}^\omega(\Xi)$ is dense in $\mathcal{H}$. □

As before, $(U, \mathcal{H})$ is an antiunitary representation of $G_{\tau_h}$ and $J = U(\tau_h)$. We will establish in Lemma 4.15 a natural $\mathtt{d}U(\mathfrak{g})$-equivariance property of the mapping $\beta^+ \colon \mathcal{H}^\omega_{U_h}(\mathcal{S}_{\pm\pi/2}) \to \mathcal{H}^{-\omega}_{U_h}$. The domain and the range of $\beta^+$ may not be $\mathtt{d}U(\mathfrak{g}_\mathbb{C})$-invariant, so that such an equivariance property does not make sense on $\mathcal{H}^\omega_{U_h}(\mathcal{S}_{\pm\pi/2})$. However, (Cr2) ensures that $\mathcal{H}^\omega(\Xi) \subseteq \mathcal{H}^\omega_{U_h}(\mathcal{S}_{\pm\pi/2})$, and $\mathcal{H}^\omega(\Xi)$ carries the action of $\mathtt{d}U(\mathfrak{g}_\mathbb{C})$ from Lemma 4.14(b), so that we may study $\mathtt{d}U(\mathfrak{g}_\mathbb{C})$-equivariance for the restriction of $\beta^+$ to $\mathcal{H}^\omega(\Xi)$.

**Lemma 4.15** *The map*

$$\beta^+ \colon \mathcal{H}^\omega(\Xi) \to \mathcal{H}^{-\omega}, \quad \beta^+(v) := \lim_{t \to \pi/2} U^v(\exp(-ith)) = \lim_{t \to \pi/2} e^{-it\partial U(h)}v$$

*satisfies the following equivariance relation with respect to the action of* $\mathfrak{g}_\mathbb{C}$ *on both sides:*

$$\beta^+ \circ \mathtt{d}U(x) = \mathtt{d}U^{-\omega}(\zeta(x)) \circ \beta^+ \quad \textit{for} \quad \zeta := e^{-\frac{\pi i}{2}\,\mathrm{ad}\,h} \in \mathrm{Aut}(\mathfrak{g}_\mathbb{C}), x \in \mathfrak{g}_\mathbb{C}. \tag{129}$$

***Proof*** (cf. [BN25], and [FNÓ25a, §3, Prop. 7(d)] for the semisimple case) Let $v \in \mathcal{H}^\omega(\Xi)$ and $x \in \mathfrak{g}_{\mathbb{C}}$. Then the continuity of the operators $\mathtt{d}U^{-\omega}(z)$, $z \in \mathfrak{g}_{\mathbb{C}}$, implies

$$\mathtt{d}U^{-\omega}(\zeta(x))\beta^+(v) = \lim_{t\to\pi/2} \mathtt{d}U^{-\omega}(\zeta(x))U^v(\exp(-ith)).$$

From (128) in Lemma 4.14, we then obtain

$$\mathtt{d}U(x)U^v(p) = U^{\mathtt{d}U(\mathrm{Ad}(\eta_\Xi(p))^{-1}x)v}(p) \quad \text{for} \quad p \in \Xi. \tag{130}$$

This formula holds obviously for $p \in G$, and for general $p$ it follows by analytic continuation, using that

$$p \mapsto \mathtt{d}U(\mathrm{Ad}(\eta_\Xi(p))^{-1}x)v \in \mathtt{d}U(\mathfrak{g}_{\mathbb{C}})v$$

is a holomorphic function with values in a finite-dimensional space. The relation (130) implies

$$\begin{aligned}\mathtt{d}U(\zeta(x))U^v(\exp(-ith)) &= U^{\mathtt{d}U(e^{it\,\mathrm{ad}\,h}\zeta(x))v}(\exp(-ith)) \\ &= e^{-it\partial U(h)}\mathtt{d}U\big(e^{it\,\mathrm{ad}\,h}\zeta(x)\big)v.\end{aligned}$$

We now observe that $t \mapsto \mathtt{d}U\big(e^{it\,\mathrm{ad}\,h}\zeta(x)\big)v$ is a continuous curve in the finite-dimensional subspace $\mathtt{d}U(\mathfrak{g}_{\mathbb{C}})v$, so that we obtain the limit

$$\lim_{t\to\pi/2} \mathtt{d}U(\zeta(x))U^v(\exp(-ith)) = \beta^+(\mathtt{d}U\big(\zeta^{-1}\zeta(x)\big)v) = \beta^+(\mathtt{d}U(x)v).$$

This proves the lemma. □

The following theorem is a general key tool to construct nets of real subspaces in a rather direct way for an antiunitary representation of $G_{\tau_h}$.

**Theorem 4.16** (Construction Theorem for nets of real subspaces) *Let* $(U, \mathcal{H})$ *be an antiunitary representation of* $G_{\tau_h} = G \rtimes \{\mathbf{1}, \tau_h\}$ *and*

$$\mathtt{F} \subseteq \mathcal{H}^J_{\mathrm{temp}} \cap \mathcal{H}^\omega(\Xi)$$

*be a* $G$*-cyclic subspace of* $\mathcal{H}$*, i.e.,* $U(G)\mathtt{F}$ *is total in* $\mathcal{H}$*. We consider the linear subspace*

$$\mathtt{E} = \beta^+(\mathtt{F}) \subseteq \mathcal{H}^{-\infty}.$$

*Then the net* $\mathsf{H}^G_{\mathtt{E}}$ *on* $G$ *satisfies* (Iso), (Cov) *and* (RS). *It further satisfies* (BW) *in the sense that* $\mathsf{H}^G_{\mathtt{E}}(W^G) = \mathtt{V}$ *holds for* $W^G$ *as in* (121).

***Proof*** We refer to [BN25] for a detailed proof. Here we only give an outline. We have already argued above that (Iso) and (Cov) are satisfied. To verify the Reeh–Schlieder property (RS), one has to show that, for $\emptyset \neq \mathcal{O}$ (w.l.o.g. $\mathcal{O} \subseteq W^G$), we have $\mathsf{H}^G_{\mathtt{E}}(\mathcal{O})^\perp = \{0\}$. This is derived from the fact that, if, for

$\alpha = \beta^+(\xi) \in \mathtt{E}$, the orbit map $U^\xi \colon G \to \mathcal{H}^{-\infty}$ extends to a holomorphic map $\Xi \to \mathcal{H}^{-\infty}$, then it is determined by the values of the orbit map $U^{-\infty,\alpha} \colon G \to \mathcal{H}^{-\infty}$ on every open subset of $G$ ([BN25, Thm. 2.9]).

For the Bisognano–Wichmann property (BW), it suffices to show that

$$U^{-\infty}(W^G)\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}.$$

Then Proposition 4.13 yields $\mathsf{H}^G_{\mathtt{E}}(W^G) \subseteq \mathtt{V}$, and by (RS), $\mathsf{H}^G_{\mathtt{E}}(W^G)$ is cyclic, so that (Cov) and $\exp(\mathbb{R}h)W^G = W^G$, combined with the Equality Lemma 1.9, lead to equality. □

## 4.4 Push-forwards to homogeneous spaces

In the preceding subsection we developed tools to construct from an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ natural nets of real subspaces on open subsets of the group $G$ itself. We now discuss very briefly how such nets can be pushed forward to homogeneous spaces $M = G/H$. This subsection is mainly a collections of tools that can be used to translate between nets on $G$ and $M$.

**Definition 4.17** On a homogeneous space $M = G/H$ with the projection map $q_M \colon G \to M$, we obtain from every net $\mathsf{H}^G$ on open subsets of $G$ a "push-forward net"

$$\mathsf{H}^M(\mathcal{O}) := ((q_M)_*\mathsf{H}^G)(\mathcal{O}) = \mathsf{H}^G(q_M^{-1}(\mathcal{O})). \tag{131}$$

The so-obtained net on $M$ thus corresponds to the restriction of the net $\mathsf{H}^G$, indexed by open subsets of $G$, to those open subsets $\mathcal{O} \subseteq G$ which are $H$-right invariant in the sense that $\mathcal{O} = \mathcal{O}H$; these are the inverse images of open subsets of $M$ under $q_M$.

The following three lemmas are useful tools for understanding the passage from nets on $G$ to $G/H$ better.

**Lemma 4.18** *Let $\mathcal{O} \subseteq G$ be open and $\varphi \in C^\infty_c(\mathcal{O})$. We further assume that $(\mathcal{O}_j)_{j\in J}$ is an open cover of $\mathcal{O}$. Then there exist $j_1, \ldots, j_k \in J$ and $\varphi_\ell \in C^\infty_c(\mathcal{O}_{j_\ell})$ such that $\varphi = \varphi_1 + \cdots + \varphi_k$.*

***Proof*** The family $(\mathcal{O}_j)_{j\in J}$ is an open cover of $\mathrm{supp}(\varphi)$, and there exist $j_1, \ldots, j_k \in J$ such that $\mathrm{supp}(\varphi) \subseteq \mathcal{O}_{j_1} \cup \cdots \cup \mathcal{O}_{j_k}$. Then

$$G \setminus \mathrm{supp}(\varphi), \quad \mathcal{O}_{j_1}, \ldots, \mathcal{O}_{j_k}$$

is an open cover of $G$. Let $\chi_0, \ldots, \chi_k$ be a subordinated partition of unity. Then $\varphi = \sum_{j=1}^k \varphi_j$, where $\varphi_j := \chi_j\varphi$ satisfies $\mathrm{supp}(\varphi_j) \subseteq \mathcal{O}_j$. □

**Lemma 4.19** (Fragmentation Lemma) *For $\emptyset \neq \mathcal{O} \subseteq G$ open, the following assertions hold:*

(a) *If $H \subseteq G$ is a closed subgroup, then*

(i) *every test function $\varphi \in C^\infty_c(\mathcal{O}H, \mathbb{R})$ is a finite sum of test functions of the form*

$$\psi \circ \rho_p : G \to \mathbb{C}, \quad g \mapsto \psi(gp), \quad \psi \in C^\infty_c(\mathcal{O}, \mathbb{R}), p \in H.$$

(ii) *every test function $\varphi \in C^\infty_c(H\mathcal{O}, \mathbb{R})$ is a finite sum of test functions of the form*

$$\psi \circ \lambda_p : G \to \mathbb{C}, \quad g \mapsto \psi(pg), \quad \psi \in C^\infty_c(\mathcal{O}, \mathbb{R}), p \in H.$$

(b) *Every $\varphi \in C^\infty_c(G, \mathbb{R})$ is a finite sum $\sum_{j=1}^n \varphi_j \circ \lambda_{g_j}$ with $\varphi_j \in C^\infty_c(\mathcal{O}, \mathbb{R})$ and $g_j \in G$.*

***Proof*** (a)(i) The family $(\mathcal{O}p)_{p\in H}$ is an open cover of the compact subset $\mathrm{supp}(\varphi)$, so that Lemma 4.18 implies that $\varphi = \sum_{j=1}^n \varphi_j$ with $\mathrm{supp}(\varphi_j) \subseteq \mathcal{O}p_j$. Then $\psi_j := \varphi_j \circ \rho_{p_j} \in C^\infty_c(\mathcal{O}, \mathbb{R})$ and $\varphi = \sum_{j=1}^n \psi_j \circ \rho_{p_j^{-1}}$.
(a)(ii) and (b) are proved along the same lines. For (b), we use the open cover $(g\mathcal{O})_{g\in G}$ of the group $G$. □

**Lemma 4.20** *Let $(U, \mathcal{H})$ be a unitary representation of $G$, let $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ be a real linear subspace, $H \subseteq G$ a closed subgroup and $\emptyset \neq \mathcal{O} \subseteq G$. Then the following assertions hold:*

(a) $\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}H) = \mathsf{H}^G_{\widehat{\mathtt{E}}}(\mathcal{O})$ *for* $\widehat{\mathtt{E}} := \mathrm{span}(U^{-\infty}(H)\mathtt{E})$.

(b) $\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}H) = \mathsf{H}^G_{\mathtt{E}}(\mathcal{O})$ *if* $\mathtt{E}$ *is* $H$*-invariant.*

(c) $\mathsf{H}^G_{\mathtt{E}}(H\mathcal{O})$ *is the closed real span of* $U(H)\mathsf{H}^G_{\mathtt{E}}(\mathcal{O})$.

(d) *The real subspace spanned by* $U(G)\mathsf{H}^G_{\mathtt{E}}(\mathcal{O})$ *is dense in* $\mathsf{H}^G_{\mathtt{E}}(G)$.

***Proof*** (a) For $\varphi = \psi \circ \rho_g$, $\psi \in C^\infty_c(\mathcal{O})$ and $g \in H$, we obtain with (195) in Appendix 7.5.1

$$U^{-\infty}(\varphi)\mathtt{E} = U^{-\infty}(\psi \circ \rho_g)\mathtt{E} = \Delta_G(g)^{-1}U^{-\infty}(\psi)U^{-\infty}(g^{-1})\mathtt{E} \subseteq U^{-\infty}(\psi)\widehat{\mathtt{E}}.$$

Hence Lemma 4.19(a) implies that $\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}H) \subseteq \mathsf{H}^G_{\widehat{\mathtt{E}}}(\mathcal{O})$.
Conversely, for $g \in H$ and $\psi \in C^\infty_c(\mathcal{O}, \mathbb{R})$, we have

$$U^{-\infty}(\psi)U^{-\infty}(g)\mathtt{E} = \Delta_G(g)^{-1}U^{-\infty}(\psi \circ \rho_g)\mathtt{E} \subseteq \mathsf{H}^G_{\mathtt{E}}(\mathcal{O}H),$$

hence also $U^{-\infty}(\psi)\widehat{\mathtt{E}} \subseteq \mathsf{H}^G_{\mathtt{E}}(\mathcal{O}H)$, and this implies that $\mathsf{H}^G_{\widehat{\mathtt{E}}}(\mathcal{O}) \subseteq \mathsf{H}^G_{\mathtt{E}}(\mathcal{O}H)$.
(b) follows from (a).
(c) From Remark 4.11 we know that $U(g)\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) = \mathsf{H}^G_{\mathtt{E}}(g\mathcal{O}) \subseteq \mathsf{H}^G_{\mathtt{E}}(H\mathcal{O})$ for $g \in H$. Now the assertion follows from Lemma 4.19(b).
(d) is an immediate consequence of (c), applied with $H = G$. □

*Remark 4.21* For a net $\mathsf{H}^M_{\mathtt{E}}$ on the homogeneous space $M = G/H$, the preceding lemma implies that, for all open subset $\mathcal{O} \subseteq M$, we have

$$\mathsf{H}^M_{\mathtt{E}}(\mathcal{O}) = \mathsf{H}^G_{\mathtt{E}}(q_M^{-1}(\mathcal{O})) = \mathsf{H}^G_{\mathtt{E}}(q_M^{-1}(\mathcal{O})H) = \mathsf{H}^G_{\widehat{\mathtt{E}}}(q_M^{-1}(\mathcal{O})) = \mathsf{H}^M_{\widehat{\mathtt{E}}}(\mathcal{O}).$$

Therefore the nets $\mathsf{H}^M_{\mathtt{E}}$ on $M$ can always be constructed from $U(H)$-invariant subspaces $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$.

If (BW) holds for some wedge region $W \subseteq M$ containing the base point, if follows in particular from $\mathtt{V} = \mathsf{H}^M_{\mathtt{E}}(W)$ that

$$U^{-\infty}(H)\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}$$

(cf. Corollary 4.24(d) below). This requirement on $\mathtt{E}$ makes it harder to construct nets of the form $\mathsf{H}^M_{\mathtt{E}}$ on homogeneous spaces satisfying the (BW) condition; see in particular Problem 5.28.

**Problem 4.22** Let $(U, \mathcal{H})$ be an irreducible antiunitary representation of $G_{\tau_h}$ and $H \subseteq G$ an integral subgroup. When does $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ contain a non-trivial $U^{-\infty}(H)$-invariant subspace?

We know from [FNÓ25a] that this is always the case if $G$ is semisimple and $G/H$ is a non-compactly causal symmetric space associated to the Euler element $h$ as in Theorem 3.39, but in general the existence of such subspaces is not clear. The case where $G/H$ is a modular compactly causal symmetric space is relevant for an answer to Problem 5.28 below.

*Remark 4.23* (a) If the subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ is invariant under $U^{-\infty}(H)$, then Lemma 4.20(b) implies that $\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) = \mathsf{H}^G_{\mathtt{E}}(\mathcal{O}H)$ for any open subsets $\mathcal{O} \subseteq G$. This means that the nets $\mathsf{H}^G_{\mathtt{E}}$ and $\mathsf{H}^M_{\mathtt{E}}$ contain the same information and that $\mathsf{H}^G_{\mathtt{E}}$ can be recovered from the net $\mathsf{H}^M_{\mathtt{E}}$ on $M$ by

$$\mathsf{H}^G_{\mathtt{E}}(\mathcal{O}) = \mathsf{H}^G_{\mathtt{E}}(\mathcal{O}H) = \mathsf{H}^M_{\mathtt{E}}(q_M(\mathcal{O})).$$

(b) We have already seen in Remark 4.11 that the net $\mathsf{H}^G_{\mathtt{E}}$, and hence also $\mathsf{H}^M_{\mathtt{E}}$, satisfy (Iso) and (Cov). Further, the net $\mathsf{H}^M_{\mathtt{E}}$ inherits (RS) from $\mathsf{H}^G_{\mathtt{E}}$. If (BW) holds for $\mathsf{H}^G_{\mathtt{E}}$ and the wedge region $W^G \subseteq G$, in the sense that $\mathsf{H}^G_{\mathtt{E}}(W^G) = \mathtt{V}$, then it holds for its image in $G/H$ if $\mathtt{E}$ **is** $H$**-invariant**, because this implies with $W^M = q_M(W^G)$ that

$$\mathsf{H}^G_{\mathtt{E}}(W^G) = \mathsf{H}^G_{\mathtt{E}}(W^G H) = \mathsf{H}^G_{\mathtt{E}}(q_M^{-1}(W^M)) = \mathsf{H}^M_{\mathtt{E}}(W^M)$$

(Lemma 4.19).
(c) If $\mathtt{E}$ is not $H$-invariant, then the situation is more complicated. We may enlarge $\mathtt{E}$ to the closed subspace $\widehat{\mathtt{E}}$ of $\mathcal{H}^{-\infty}$ generated by $U^{-\infty}(H)\mathtt{E}$, but then it is not clear if this still has the form $\beta^+(\widehat{\mathtt{F}})$ for some real subspace $\widehat{\mathtt{F}} \subseteq \mathcal{H}^\omega(\Xi) \cap \mathcal{H}^J_{\mathrm{temp}}$.

The following corollary provides a generalization of Proposition 4.13 to homogeneous spaces.

**Corollary 4.24** *Let $(U, \mathcal{H})$ be an antiunitary representation of $G_{\tau_h}$ and $\mathtt{V} = \mathtt{V}(h, U)$ the corresponding standard subspace. For an open subset $\mathcal{O} \subseteq M = G/H$, the projection $q_M : G \to G/H$, and a real subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$, the following are equivalent:*

(a) $\mathsf{H}^M_{\mathtt{E}}(\mathcal{O}) = \mathsf{H}^G_{\mathtt{E}}(q_M^{-1}(\mathcal{O})) \subseteq \mathtt{V}$.
(b) *For all $\varphi \in C^\infty_c(q_M^{-1}(\mathcal{O}), \mathbb{R})$ we have $U^{-\infty}(\varphi)\mathtt{E} \subseteq \mathtt{V}$.*
(c) *For all $\varphi \in C^\infty_c(q_M^{-1}(\mathcal{O}), \mathbb{R})$ we have $U^{-\infty}(\varphi)\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}$.*
(d) *$U^{-\infty}(g)\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}$ for every $g \in q_M^{-1}(\mathcal{O})$.*

***Proof*** Apply the preceding proposition to the open subset $q_M^{-1}(\mathcal{O}) \subseteq G$. □

## 4.5 Representations satisfying a spectral condition

An important source of natural crown domains for Lie groups are complex Olshanski semigroups (Definition 3.17). They arise naturally for antiunitary representations $(U, \mathcal{H})$ of $G_{\tau_h}$ satisfying a spectral condition, asserting that the positive cone $C_U$ is large in a suitable sense. Concretely, suppose that $C \subseteq \mathfrak{g}$ be a pointed closed convex $\mathrm{Ad}(G)$-invariant cone satisfying also

$$-\tau_h(C) = C \quad \text{and} \quad \mathfrak{g} = C - C + \mathbb{R}h. \tag{132}$$

Then the ideal $\mathfrak{m} := C - C \trianglelefteq \mathfrak{g}$ either coincides with $\mathfrak{g}$ or is a hyperplane in $\mathfrak{g}$ for which $\mathfrak{g} \cong \mathfrak{m} \rtimes \mathbb{R}h$. We write $M := \langle \exp \mathfrak{m} \rangle \trianglelefteq G$ for the corresponding integral subgroup. For simplicity, we assume here that the universal complexification $\eta_G : G \to G_{\mathbb{C}}$ is injective, so that we identify $G$ with a closed Lie subgroup of $G$. Accordingly, $\eta_M : M \to M_{\mathbb{C}}$ is injective, and we obtain a complex Olshanski semigroup

$$S = M \exp(iC^\circ) \subseteq M_{\mathbb{C}}$$

(cf. Example 4.8). We pick a point

$$s_0 := \exp(i(x_1 - x_{-1})) \in S^{\overline{\tau}_h} \quad \text{with} \quad x_{\pm 1} \in C^\circ_{\pm}$$

and consider an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$, satisfying the *$C$-spectral condition*

$$C \subseteq C_U, \quad \text{where} \quad C_U = \{x \in \mathfrak{g} : -i \cdot \partial U(x) \geq 0\} \tag{133}$$

is the *positive cone of $U$*.

**Theorem 4.25** *Let $G$ be a connected Lie group for which $\eta_G$ is injective and $G_{\mathbb{C}}$ is simply connected, let $h \in \mathfrak{g}$ be an Euler element for which $\tau_h$ exists on $G$, and let $C \subseteq \mathfrak{g}$ be a pointed closed convex $\mathrm{Ad}(G)$-invariant cone satisfying (132). For $x_{\pm 1} \in C^\circ_{\pm}$, put $s_0 := \exp(i(x_1 - x_{-1}))$. Let $(U, \mathcal{H})$ be* *an antiunitary representation of* $G_{\tau_h} = G \rtimes \{\mathbf{1}, \tau_h\}$ *satisfying* (133)*, and consider*

$$\mathtt{E} = \beta^+(U(s_0)\mathcal{H}^J_{\rm temp}).$$

*Then the net* $\mathsf{H}^G_{\mathtt{E}}$ *on* $G$ *satisfies* (Iso), (Cov), (RS)*, and also* (BW)*, in the sense that* $\mathsf{H}^G_{\mathtt{E}}(W^G) = \mathtt{V}$ *holds for* $W^G = G^h_e.\exp(\Omega_1 + \Omega_{-1}) \subseteq G$*, such that* $\Omega' = \Omega_1 - \Omega_{-1} \subseteq \mathfrak{g}^{-\tau_h^{\mathfrak{g}}}$ *is an open connected* 0*-neighborhood with* $\exp(i\Omega').s_0 \subseteq S$ (cf. (121))*).*

***Proof*** The spectral condition (133) implies that $U$ extends to a strongly continuous contraction representation of $\overline{S} := M\exp(iC)$ by

$$U(m\exp(ix)) = U(m)e^{i\partial U(x)}, \quad m \in M, x \in C,$$

and $U|_S \colon S \to B(\mathcal{H})$ is holomorphic with respect to the operator norm topology (cf. [Ne00, Thm. XI.2.5]). For any $v \in \mathcal{H}^J$, we thus obtain a holomorphic orbit map

$$U^v \colon S \to \mathcal{H}, \quad U^v(s) = U(s)v.$$

It follows in particular, that the map

$$\Xi = \{(g,z) \in M_{\mathbb{C}} \times \mathcal{S}_{\pm\pi/2} \colon (g,z).s_0 = g\alpha_z(s_0) \in S\} \to B(\mathcal{H}),$$
$$(g,z) \mapsto U((g,z).s_0) \quad (134)$$

is holomorphic. We claim that

$$U(s_0)\mathcal{H}^J_{\rm temp} \subseteq \mathcal{H}^\omega(\Xi) \cap \mathcal{H}^J_{\rm temp}, \quad (135)$$

hence in particular that $\mathcal{H}^\omega(\Xi) \cap \mathcal{H}^J_{\rm temp} \neq \{0\}$. So let $w \in \mathcal{H}^J_{\rm temp}$ and $v := U(s_0)w$. Then

$$U^v(g,t) = U(g)U_h(t)U(s_0)w = U(g\alpha_t(s_0))U_h(t)w = U((g,t).s_0)U^w_h(t),$$

where $U_h(t) = U(\exp th)$. The evaluation map $B(\mathcal{H}) \times \mathcal{H} \to \mathcal{H}$ is holomorphic, so that the representation $U \colon S \to B(\mathcal{H})$, and thus the orbit map $U^w \colon S \to \mathcal{H}$, are holomorphic. For $w \in \mathcal{H}^J_{\rm temp} \subseteq \mathcal{H}^\omega_{U_h}(\mathcal{S}_{\pm\pi/2})$, the prescription

$$U^v(g,z) := U((g,z).s_0)U^w_h(z)$$

defines a holomorphic map $\Xi \to \mathcal{H}$, extending the orbit map $U^v$ on $G$. This proves $v \in \mathcal{H}^\omega(\Xi)$. To see that also $v \in \mathcal{H}^J_{\rm temp}$, note that

$$U^v(e,t) = U(\exp th)U(s_0)w = U(\alpha_t(s_0))U^w_h(t) \quad \text{for} \quad t \in \mathbb{R}$$

implies by analytic continuation

$$e^{it\partial U(h)}v = U^v(e,it) = U(\alpha_{it}(s_0))U^w_h(it) \quad \text{for} \quad |t| < \pi/2.$$

As $U(S) = U(M)e^{i\partial U(C)}$ consists of contractions, we obtain

$$\|e^{it\partial U(h)}v\| \leq \|U^w_h(it)\|.$$

It thus follows with (126) that $w \in \mathcal{H}^J_{\rm temp}$ implies that $v \in \mathcal{H}^J_{\rm temp}$.

We conclude that the dense subspace $U(s_0)\mathcal{H}^J_{\rm temp}$ is contained $\mathcal{H}^\omega(\Xi) \cap \mathcal{H}^J_{\rm temp}$, so that this subspace is dense as well, and now Theorem 4.16 applies to all antiunitary representations $(U, \mathcal{H})$ of $G_{\tau_h}$ with $C \subseteq C_U$. □

## 4.6 Semisimple Lie groups

The following theorem builds on the analytic extension results from [KSt04], and their generalization to non-linear groups by T. Simon in [Si24], which also contains the result on the temperedness, resp., the growth condition (126). These results were used in [FNÓ25a] to construct nets of real subspaces on non-compactly causal symmetric spaces.

**Theorem 4.26** (The crown domain of a semisimple group) *Suppose that $\mathfrak{g}$ is semisimple, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition, and that $h \in \mathfrak{p}$ an Euler element. We consider three cases for a connected Lie group $G$ with Lie algebra on which $\tau_h$ exists:*

(a) *If $G \subseteq G_{\mathbb{C}}$, $K = \exp_G \mathfrak{k}$ and $K_{\mathbb{C}} = K\exp(i\mathfrak{k}) \subseteq G_{\mathbb{C}}$, then we consider the domain*

$$\Xi_{G_{\mathbb{C}}} = G\exp(i\Omega_{\mathfrak{p}})K_{\mathbb{C}} \subseteq G_{\mathbb{C}}, \qquad \Omega_{\mathfrak{p}} = \{x \in \mathfrak{p} \colon \mathrm{Spec}(\mathrm{ad}\, x) \subseteq (-\pi/2, \pi/2)\},$$

*and*

$$W^{\rm cr} = G^h_e \exp(i\Omega_{\mathfrak{p}}^{-\tau_h^{\mathfrak{g}}})K_{\mathbb{C}}^{\overline{\tau}_h}. \tag{136}$$

(b) *If $G$ is simply connected, $K = \exp_G(\mathfrak{k})$, and $K_{\mathbb{C}}$ is the universal complexification of $K$, then we consider the simply connected covering manifold $\Xi := \widetilde{\Xi}_{G_{\mathbb{C}}}$ of the complex manifold $\Xi_{G_{\mathbb{C}}}$ from* (a) *and*

$$W^{\rm cr} = G^h_e.\exp(i\Omega_{\mathfrak{p}}^{-\tau_h^{\mathfrak{g}}}).K_{\mathbb{C}}^{-\overline{\tau}_h}, \tag{137}$$

*with respect to the $G$-action from the left and the $K_{\mathbb{C}}$-action from the right on $\Xi$.*

(c) *If $G \cong \widetilde{G}/\Gamma$ is a connected Lie group with Lie algebra $\mathfrak{g}$ and $\Gamma$ is a $\tau_h$-invariant discrete central subgroup, so that $\tau_h$ can also be implemented on $G$, then we put*

$$\Xi := \widetilde{\Xi}_{G_{\mathbb{C}}}/\Gamma$$

*for the simply connected crown domain $\widetilde{\Xi}_{G_{\mathbb{C}}}$ from* (b). *Then $G$ acts on $\Xi$ from the left, and, for $K := \exp_G \mathfrak{k}$ and its universal complexification*

*$K_{\mathbb{C}}$, the group $K_{\mathbb{C}}$ acts naturally on $\Xi$ from the right, so that we obtain a domain*

$$W^{\rm cr} = G_e^h.\exp(i\Omega_{\mathfrak p}^{-\tau_h^{\mathfrak g}}).K_{\mathbb{C}}^{\overline{\tau}_h}. \tag{138}$$

*Then, in all cases* (a), (b), (c), *the conditions* (Cr1-3) *are satisfied.*

***Proof*** (a) As $\Xi_{G_{\mathbb{C}}}$ is the inverse image of the open crown domain

$$\Xi_{G_{\mathbb{C}}/K_{\mathbb{C}}} = G\exp(i\Omega_{\mathfrak p})K_{\mathbb{C}} \cong G \times_K i\Omega_{\mathfrak p}$$

in $G_{\mathbb{C}}/K_{\mathbb{C}}$, it is an open subset of $G_{\mathbb{C}}$ which is a holomorphic $K_{\mathbb{C}}$-principal bundle over the contractible space $\Xi_{G_{\mathbb{C}}/K_{\mathbb{C}}}$. Condition (Cr1) follows from the invariance of $G$, $\exp(i\Omega_{\mathfrak p})$ and $K_{\mathbb{C}}$ under the antiholomorphic involution $\overline{\tau}_h$. To verify (Cr2), we first observe that $W^{\rm cr} \subseteq \Xi_{G_{\mathbb{C}}}^{\overline{\tau}_h}$ follows from the fact that all 3 factors in (136) consist of fixed points of $\overline{\tau}_h$. The set of $\overline{\tau}_h$-fixed points in $\Xi_{G_{\mathbb{C}}}$ is a fiber bundle over

$$\Xi_{G_{\mathbb{C}}/K_{\mathbb{C}}}^{\overline{\tau}_h} \cong G_e^h \times_{K_e^h} i\Omega_{\mathfrak p}^{-\tau_h^{\mathfrak g}} \hookrightarrow G_{\mathbb{C}}^{\overline{\tau}_h}/K_{\mathbb{C}}^{\overline{\tau}_h}$$

([NÓ23b, Thm. 6.1]), so that the $K_{\mathbb{C}}$-principal bundle structure of $\Xi_{G_{\mathbb{C}}}$ implies that $\Xi_{G_{\mathbb{C}}}^{\overline{\tau}_h}$ is a $K_{\mathbb{C}}^{\overline{\tau}_h}$-principal bundle. We conclude that $W^{\rm cr} = \Xi_{G_{\mathbb{C}}}^{\overline{\tau}_h}$. The inclusion

$$\exp(\mathcal{S}_{\pm\pi/2}h)W^{\rm cr} \subseteq \Xi_{G_{\mathbb{C}}} \quad \text{is equivalent to} \quad \exp(\mathcal{S}_{\pm\pi/2}h)\Xi_{G_{\mathbb{C}}/K_{\mathbb{C}}}^{\overline{\tau}_h} \subseteq \Xi_{G_{\mathbb{C}}/K_{\mathbb{C}}},$$

which is shown in [MNÓ24, §8].

(b) We now assume that $G$ is simply connected, so that $\eta_G \colon G \to G_{\mathbb{C}}$ has discrete kernel and $G_{\mathbb{C}}$ is simply connected. The discussion in the proof of [FNÓ25a, §3, Prop. 5] shows that the simply connected covering $\Xi$ of $\Xi_{G_{\mathbb{C}}}$ is a complex manifold which is a $K_{\mathbb{C}}$-principal bundle over the contractible space $\Xi_{G_{\mathbb{C}}/K_{\mathbb{C}}}$. This implies that $\eta_G \colon G \hookrightarrow \Xi_{G_{\mathbb{C}}}$ lifts to an embedding $G \hookrightarrow \Xi$ and $\pi_1(\eta_G(G)) \cong \ker(\eta_G) \cong \pi_1(\Xi_{G_{\mathbb{C}}})$ acts as a group of deck transformations on $\Xi$. We thus obtain a free action of $G$ on $\Xi$ from the left, and a free holomorphic action of the simply connected universal complexification $K_{\mathbb{C}}$ of the integral subgroup $K = \exp_G \mathfrak{k} \subseteq G$ from the right. The exponential map $i\Omega_{\mathfrak p} \to \Xi_{G_{\mathbb{C}}}$ also lifts to a map $\exp \colon i\Omega_{\mathfrak p} \to \widetilde{\Xi}_{G_{\mathbb{C}}}$. In this sense, we obtain a factorization

$$\Xi = G\exp(i\Omega_{\mathfrak p})K_{\mathbb{C}}.$$

The involution $\tau_h$ of $G$ extends to an antiholomorphic involution $\overline{\tau}_h$ on $\Xi$ (by the Lifting Theorem for Coverings), and we obtain a connected open subset

$$W^{\rm cr} := G_e^h \exp(i\Omega_{\mathfrak p}^{-\tau_h^{\mathfrak g}})K_{\mathbb{C}}^{\overline{\tau}_h} \subseteq \Xi^{\overline{\tau}_h}.$$

Here $K_{\mathbb{C}}^{\overline{\tau}_h}$ is connected because $K_{\mathbb{C}}$ is simply connected ([Lo69, Thm. IV.3.4]). We thus obtain crowned Lie groups in both situations.

(c) follows easily from (b) by factorization of the discrete central subgroup $\pi_1(G) \cong \Gamma \subseteq \widetilde{G}$. $\square$

We take a closer look at de Sitter space $\mathrm{dS}^d$, considered as an irreducible ncc symmetric space of the Lorentz group $\mathrm{SO}_{1,d}(\mathbb{R})_e$.

**Example 4.27** (cf. Exercise 3.56) For de Sitter space $M = \mathrm{dS}^d \subseteq \mathrm{V} := \mathbb{R}^{1,d}$ and the Lorentzian forms $x^2 = x_0^2 - \mathbf{x}^2$ on $\mathbb{R}^{1,d}$, a natural complexification is the complex sphere

$$M_{\mathbb{C}} := \{z = (z_0, \mathbf{z}) \in \mathbb{C}^{1+d} \colon z_0^2 - \mathbf{z}^2 = -1\}.$$

It contains $M = M_{\mathbb{C}} \cap \mathbb{R}^{1,d}$ and also the Riemannian symmetric spaces

$$\mathbb{H}_{\pm} := \{(iy_0, i\mathbf{y}) \colon y_0^2 - \mathbf{y}^2 = 1, \pm y_0 > 0\} \cong \mathrm{SO}_{1,d}(\mathbb{R})_e / \mathrm{SO}_d(\mathbb{R}).$$

Here $G = \mathrm{SO}_{1,d}(\mathbb{R})_e \subseteq G_{\mathbb{C}} = \mathrm{SO}_{1,d}(\mathbb{C}) \cong \mathrm{SO}_{1+d}(\mathbb{C})$ and $K = \mathrm{SO}_d(\mathbb{R}) \subseteq K_{\mathbb{C}} = \mathrm{SO}_d(\mathbb{C})$. The crown domains of the hyperbolic spaces $\mathbb{H}_{\pm} \cong G/K$ are the intersections with the tube domains $\mathrm{V} \pm i\mathrm{V}_+$:

$$\Xi_{\pm} := M_{\mathbb{C}} \cap (\mathrm{V} \pm i\mathrm{V}_+).$$

For both domains,

$$\mathrm{dS}^d = \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d} \colon x_0^2 - \mathbf{x}^2 = -1\} \subseteq \partial_{M_{\mathbb{C}}} \Xi_{\pm}.$$

For the Euler element $h$, given by the Lorentz boost

$$h.(x_0, x_1, \ldots, x_{d-1}) = (x_1, x_0, 0, \ldots, 0),$$

$\zeta := \exp(-\frac{\pi i}{2} h)$ acts by

$$\zeta.x = (-ix_1, -ix_0, x_2, \ldots, x_d),$$

so that $\zeta.i\mathbf{e}_0 = \mathbf{e}_1 \in \mathrm{dS}^d$.

We also note that, for $\mathrm{V} \subseteq \partial(\mathrm{V}+i\mathrm{V}_+)$ and $C := \overline{\mathrm{V}_+}$, the set of KMS-points in V is

$$\mathrm{V}_{\mathrm{KMS}} = C_+^\circ + \mathrm{V}_0 + C_-^\circ = W_{\mathrm{V}}^+(h), \qquad \text{where} \qquad C_{\pm} = \mathbb{R}_{\geq 0}(\mathbf{e}_1 \pm \mathbf{e}_0)$$

(cf. Examples 3.53(a)). Accordingly,

$$\mathrm{dS}^d_{\mathrm{KMS}} = \mathrm{V}_{\mathrm{KMS}} \cap \mathrm{dS}^d = W^+_{\mathrm{dS}^d}(h).$$

**Example 4.28** (cf. Exercise 4.34) For $G = \mathrm{SL}_n(\mathbb{R})$, $n = p+q$, and the Euler element

$$h := h_q := \frac{1}{n}\begin{pmatrix} q\mathbf{1}_p & 0 \\ 0 & -p\mathbf{1}_q \end{pmatrix} \in \mathfrak{sl}_n(\mathbb{R})$$

from (47), the corresponding non-compactly causal involution $\tau = \tau_h\theta$ is

$$\tau(x) = -I_{p,q}x^\top I_{p,q}$$

(cf. Section 3.5.3). Therefore $G^\tau = \mathrm{SO}_{p,q}(\mathbb{R})$ and, for the action of $G$ on $\mathrm{Sym}_n(\mathbb{R})$, we have

$$M := G.I_{p,q} = \{gI_{p,q}g^\top : g \in \mathrm{SL}_n(\mathbb{R})\}.$$

This space carries a causal structure for which $M \hookrightarrow (\mathrm{Sym}_n(\mathbb{R}), \mathrm{Sym}_n(\mathbb{R})_+)$ becomes an embedding of causal manifolds.

Here $M_r := G.I_n = G.\mathbf{1} \cong G/K$ is the corresponding Riemannian symmetric space. For

$$\zeta := \exp\Big(-\frac{\pi i}{2}h_q\Big)$$

we have

$$\zeta.I_n = \exp(-\pi i h_q) = e^{-\pi i q/n}\mathbf{1}_p \oplus e^{\pi i p/n}\mathbf{1}_q = e^{-\pi i q/n}I_{p,q},$$

so that $G.(\zeta.I_n) \cong G.I_{p,q} \cong M$.

Now that we know what the natural crown domains for semisimple Lie groups are, we can address the construction of nets of real subspaces. Theorem 4.31 is surprisingly general because it applies to **all irreducible** antiunitary representations of $G_{\tau_h}$. A key ingredient of its proof is Simon's Growth Theorem. This is [Si24, Thm. 3.2.6] (and [Si25, Thm. 3.1, Rem. 3.3.15]), where, in addition, we use [FNÓ25a, Thm. 3] for the existence of the limit in the smaller subspace

$$\mathcal{H}^{-\infty}(\partial U(x)) \subseteq \mathcal{H}^{-\infty}.$$

This result generalizes the extension results by Krötz and Stanton [KSt04] by removing the condition on the group $G$ that its universal complexification is injective.

**Theorem 4.29** (Simon's Growth Theorem) *Let $G$ be a connected semisimple Lie group with Cartan decomposition $G = K\exp\mathfrak{p}$ and let $(\pi, \mathcal{H})$ be an* **irreducible** *unitary representation of $G$. Then there exist for every $K$-finite vector $v \in \mathcal{H}$ constants $C, n > 0$ such that, for every $x \in \mathfrak{p}$ with spectral radius $r_{\mathrm{Spec}}(\mathrm{ad}\, x) < \pi/2$, we have*

$$\|e^{i\partial U(x)}v\| \le C\Big(\frac{\pi}{2} - r_{\mathrm{Spec}}(\mathrm{ad}\, x)\Big)^{-n}.$$

*In particular, $\lim_{t\to\frac{\pi}{2}-} e^{it\partial U(h)}v$ exists in $\mathcal{H}^{-\infty}(U_h)$ for $h \in \mathfrak{p}$ with*

$$r_{\mathrm{Spec}}(\mathrm{ad}\, h) = 1,$$

*where we put $U_h(t) = U(\exp th)$.*

The last statement uses the equivalence of (b) and (c) in Theorem 1.44.

*Remark 4.30* The conclusion of the preceding theorem does not hold for reducible unitary representations without suitable restrictions. It prevails for finite sum or irreducible representations, but in general not for infinite sums or direct integrals.

Consider, for instance, the Lie subalgebra $\mathfrak{g} = \mathfrak{so}_{1,2}(\mathbb{R}) \subseteq \mathfrak{g}^\sharp := \mathfrak{su}_{1,2}(\mathbb{C})$ and an Euler element $h \in \mathfrak{g}$ (a Lorentz boost). Then $h$ is **not** an Euler element in $\mathfrak{g}^\sharp$ because the eigenvalues of $\mathrm{ad}_{\mathfrak{g}^\sharp}(h)$ are $\{0, \pm 1, \pm 2\}$.

Now consider a unitary representation $(U, \mathcal{H})$ of $G$ that extends to a unitary representation of the larger group $G^\sharp$. Then $K^\sharp = G^\sharp \cap \mathrm{U}_3(\mathbb{C})$ is a maximal compact subgroup. We consider a $K^\sharp$-finite vector $v \in \mathcal{H}$. Then Simon's Growth Theorem provides estimates for $e^{it\partial U(h)}v$ for $t \to \pm\frac{\pi}{4}$, and in general the limit does not exist in $\mathcal{H}$, so that $e^{it\partial U(h)}v$ is not defined for $\frac{\pi}{4} \leq t < \frac{\pi}{2}$.

**Theorem 4.31** *Let $(U, \mathcal{H})$ be an irreducible antiunitary representation of $G_{\tau_h} = G \rtimes \{\mathbf{1}, \tau_h\}$, where $G$ is a connected semisimple Lie group. Let $\mathcal{F} \subseteq \mathcal{H}$ be a finite-dimensional subspace invariant under $U(K)$ and $J = U(\tau_h)$. Then*

$$\mathtt{F} := \mathcal{F}^J \subseteq \mathcal{H}^\omega(\Xi) \cap \mathcal{H}^J_{\mathrm{temp}} \quad \text{and} \quad \mathtt{E} = \beta^+(\mathtt{F}) \subseteq \mathcal{H}^{-\infty},$$

*for $\beta^+$ from (125). The push-forward net $\mathsf{H}^M_{\mathtt{E}}$ from (131) on the non-compactly causal symmetric space $M = G/H$ for $H = K^{\tau_h, h}\exp(\mathfrak{h}_\mathfrak{p})$ from* Definition 3.41 *satisfies* (Iso), (Cov), (RS) *and* (BW), *where $W = W^+_M(h)_{eH}$ is the connected component of the positivity domain of $h$ on $M$, containing the base point $eH$.*

Note that $\tau_h(K) = K$ implies that $J$ leaves the dense subspace $\mathcal{H}^{[K]}$ of $K$-finite vectors invariant. Therefore $J$-invariant finite-dimensional $K$-invariant subspaces exist in abundance.

***Proof*** (Sketch) The Krötz–Stanton Extension Theorem and Simon's generalization [Si24] for non-linear groups imply that the space $\mathcal{H}^{[K]}$ of $K$-finite vectors is contained in $\mathcal{H}^\omega(\Xi)$. By [Si24, Thm. 3.2.6] or [Si25, Thm. 3.1, Rem. 3.3.15], the space $\mathcal{H}^{[K]} \cap \mathcal{H}^J$ of $J$-fixed $K$-finite vectors, which is dense in $\mathcal{H}^J$, is contained in $\mathcal{H}^J_{\mathrm{temp}}$, so that

$$\mathtt{F} \subseteq \mathcal{H}^\omega(\Xi) \cap \mathcal{H}^J_{\mathrm{temp}}$$

(cf. Theorem 1.44). Irreducibility of $U$ further implies that $U(G)\mathtt{F}$ is total in $\mathcal{H}$. Therefore the assumptions of the Construction Theorem 4.16 are satisfied for the finite-dimensional subspace $\mathtt{E} = \beta^+(\mathtt{F}) \subseteq \mathcal{H}^{-\infty}$, and natural equivariance properties, such as Lemma 4.15 then imply that it is $H$**-invariant**. Therefore the net $\mathsf{H}^M_{\mathtt{E}} = (q_M)_*\mathsf{H}^G_{\mathtt{E}}$ on $M = G/H$, defined as in (131), also satisfies (RS) and (BW) (Remark 4.23). □

### 4.7 The Poincaré group

In this subsection we apply the general machinery to positive energy representatiuon of the Poincaré group, where the crown domain in complexified Minkowski space is obtained from a tube domain (the complex Olshanski semigroup corresponding to the future light cone) (Section 4.7).

We consider the Poincaré group

$$G := \mathbb{R}^{1,d} \rtimes \mathrm{SO}_{1,d}(\mathbb{R})_e \subseteq G_{\mathbb{C}} := \mathbb{C}^{1+d} \rtimes \mathrm{SO}_{1+d}(\mathbb{C})$$

and the Euler element $h \in \mathfrak{so}_{1,d}(\mathbb{R}) \subseteq \mathfrak{g}$, generating a Lorentz boost:

$$h(x_0, x_1, \ldots, x_d) := (x_1, x_0, 0, \cdots, 0).$$

The corresponding involution

$$e^{\pi i h} = \mathrm{diag}(-1, -1, 1, \ldots, 1) \in \mathrm{SO}_{1,d}(\mathbb{R})$$

acts by conjugation on $G_{\mathbb{C}}$ (denoted $\tau_h$) and we also obtain an antiholomorphic involution on $G_{\mathbb{C}}$ by $\overline{\tau}(g) := \tau_h(\overline{g})$.

We consider the action of $G_{\mathbb{C},\overline{\tau}_h} = G_{\mathbb{C}} \rtimes \{\mathbf{1}, \overline{\tau}_h\}$ on $M = \mathbb{C}^{1+d}$ by real affine maps, and

$$\tau_M(z_0, \cdots, z_d) = (-\overline{z}_0, -\overline{z}_1, \overline{z}_2, \cdots, \overline{z}_d).$$

Write

$$\mathrm{V}_+ := \{x = (x_0, \mathbf{x}) \in \mathbb{R}^{1,d} \colon x_0 > \sqrt{\mathbf{x}^2}\}$$

for the open future light one in Minkowski space $\mathrm{V} := \mathbb{R}^{1,d}$. Then

$$\Xi_M := \mathbb{R}^{1,d} + i\mathrm{V}_+ = \{z \in \mathbb{C}^{1+d} \colon \mathrm{Im}\, z \in \mathrm{V}_+\}$$

is an open tube domain in $M$, invariant under $G_{\tau_h}$, and the subset of $\tau_M$-fixed points is

$$\Xi_M^{\tau_M} = \{(iy_0, iy_1, y_2, \ldots, y_d) \colon y_j \in \mathbb{R}, y_0 > |y_1|\} \subseteq i\mathbb{R}^2 \oplus \mathbb{R}^{d-1}.$$

For $z := (iy_0, iy_1, y_2, \ldots, y_d) \in \Xi_M^{\tau_M}$, we have

$$\mathrm{Im}(e^{ith} z) = (\cos(t) y_0, \cos(t) y_1, 0, \ldots, 0),$$

so that

$$e^{ith} \Xi_M^{\tau_M} \subseteq \Xi_M \quad \text{for} \quad |t| < \pi/2.$$

We thus put $W^{M,\mathrm{cr}} := \Xi_M^{\tau_M}$.

For $m_0 := i\mathbf{e}_0 \in \Xi_M^{\tau_M}$ we now obtain a crown domain

$$\Xi := \{g \in G_{\mathbb{C}} \colon g.m_0 \in \Xi_M\},$$

and we put

$$W^{\mathrm{cr}} := \{g \in G_{\mathbb{C}}^{\overline{\tau}_h} \colon g.m_0 \in \Xi_M\} = \{g \in G_{\mathbb{C}}^{\overline{\tau}_h} \colon g.m_0 \in W^{M,c}\} = \Xi^{\overline{\tau}_h}.$$

By Lemma 4.6, this data defines a crowned Lie group $(G, h, \Xi)$.

The unitary representations of $G$ that are most relevant in Physics can be realized on a Hilbert space $\mathcal{H}$ of holomorphic functions $f \colon \Xi_M \to \mathcal{K}$, where $\mathcal{K}$ is a finite-dimensional Hilbert space, and $\mathcal{H}$ is specified by a reproducing kernel of the form

$$K(z, w) = \widetilde{\mu}(z - \overline{w}) = \int_{\mathrm{V}_+^\star} e^{\alpha(z-\overline{w})}\, d\mu(\alpha), \tag{139}$$

where $\mu$ is a tempered $\mathrm{Herm}(\mathcal{K})_+$-valued measure on the dual cone

$$\mathrm{V}_+^\star = \{\lambda \in \mathrm{V}^* \colon \lambda(\mathrm{V}_+) \subseteq [0, \infty[\}.$$

The Fourier transform of $\mu$ is considered as a holomorphic function $\widetilde{\mu} \colon \Xi_M \to B(\mathcal{K})$, whose boundary values define an element in $\mathcal{S}'(\mathbb{R}^{1+d}, B(\mathcal{K}))$. More concretely, there exists a finite-dimensional representation $\rho \colon G \to \mathrm{GL}(\mathcal{K})$ such that

$$(U(x, g)f)(z) = \rho(g) f(g^{-1}.(z - x)), \quad (x, g) \in G, z \in \Xi_M.$$

We refer to [NÓØ21] for a detailed discussion of the analytic aspects of such Hilbert spaces and the standard subspaces associated to $h$. We extend $U$ to an antiunitary representation of $G_{\tau_h}$ by

$$(Jf)(z) := J_{\mathcal{K}} f(\tau_M(z)),$$

where $J_{\mathcal{K}}$ is a conjugation on $\mathcal{K}$. Then $\mathcal{H}^J$ consists of those functions satisfying $f(\Xi_M^{\tau_M}) \subseteq \mathcal{K}^{J_{\mathcal{K}}}$ ([NÓØ21, Lemma 2.5]).

Let

$$K_z \colon \mathcal{H} \to \mathcal{K}, \quad K_z(f) := f(z)$$

denote the evaluation operator in $z \in \Xi_M$ and $K_z^* \colon \mathcal{K} \to \mathcal{H}$ its adjoint. Then the functions

$$K_w^* \xi, \quad w \in \Xi_M^{\tau_M}, \xi \in \mathcal{K}^{J_{\mathcal{K}}} \tag{140}$$

are contained in $\mathcal{H}^J$ and span a dense subspace thereof ([NÓØ21, Lemma 3.11]). A straightforward calculation shows that

$$U(g) K_w^* \xi = K_{g.w}^* \rho(g^{-1})^* \xi.$$

As the representation $\rho$ extends to a holomorphic representation of $G_{\mathbb{C}}$, it follows that $K_w^* \xi \in \mathcal{H}^\omega(\Xi)$ if $w \in W^{M,c} = \Xi_M^{\tau_M}$, and thus all functions (140) are contained in $\mathcal{H}^\omega(\Xi)^J$. To see that they are actually contained in $\mathcal{H}^J_{\mathrm{temp}}$, we need to estimate the norms $\|e^{it\partial U(h)}K_w^*\xi\|$ for $|t| \to \pi/2$ (cf. (126)). As the finite-dimensional one-parameter group $\rho(\exp(ith))$ is bounded for $|t| \leq \pi/2$, we have to verify that

$$\|K(e^{ith}w, e^{ith}w)\| \leq C\Big(\frac{\pi}{2} - |t|\Big)^{-N} \quad \text{for some} \quad C, N > 0.$$

The operator $K(e^{ith}w, e^{ith}w)$ is the Fourier transform of $\mu$, evaluated in

$$e^{ith}w - e^{-ith}\overline{w}, \quad w = (iy_0, iy_1, y_2, \ldots, y_d),$$

so that

$$e^{ith}w - e^{-ith}\overline{w} = 2\cos(th)(iy_0\mathbf{e}_0 + iy_1\mathbf{e}_1) \in \mathbb{R}i\mathbf{e}_0 + \mathbb{R}i\mathbf{e}_1. \tag{141}$$

In view of [FNÓ25a, Prop. 4, §2.3], that we also used in the proof of Theorem 1.44, the temperedness of the measure $\mu$ yields an estimate

$$\|\widetilde{\mu}(x+iy)\| \leq C\|y\|^{-N} \quad \text{for} \quad x+iy \in \mathbb{R}^{1,d} + i\mathrm{V}_+,$$

and we conclude from (141) that the functions $K_w^*\xi$ are contained in $\mathcal{H}^J_{\rm temp}$. Therefore all our assumptions are satisfied for the finite-dimensional space

$$\mathtt{F} := \{K_{i\mathbf{e}_0}^*\xi \colon \xi \in \mathcal{K}^{J_\mathcal{K}}\}.$$

We refer to [NÓØ21] for detailed descriptions of the corresponding standard subspaces $\mathtt{V} \subseteq \mathcal{H}$.

## 4.8 Exercises for Section 4

**Exercise 4.32** (Crown domains of convex cones) Let $C \subseteq E$ be a generating closed convex cone in the finite-dimensional real linear space $E$ and $h \in \mathrm{End}(E)$ be diagonalizable with eigenvalues $\{-1, 0, 1\}$, such that

$$e^{\mathbb{R}h}C = C \quad \text{and} \quad -\tau_h(C) = C \quad \text{for} \quad \tau_h = e^{\pi i h}.$$

Show that

(a) If $C_\pm := \pm C \cap E_{\pm 1}(h)$, then $\pm C_\pm = p_{\pm 1}(C)$, where $p_{\pm 1} \colon E \to E_{\pm 1}(h)$ is the projection along the other eigenspaces of $h$.
(b) $\overline{\tau}_h(z) := \overline{\tau_h(z)}$ defines an antilinear involution on $E_\mathbb{C}$, preserving the tube domain

$$\Xi := E + iC^\circ, \quad \text{and} \quad \Xi^{\overline{\tau}_h} = iC^\circ.$$

(c) The set $E_{\rm KMS}$ of those elements $v$ for which the orbit map $\alpha^v(z) := e^{zh}v, \mathbb{C} \to E_\mathbb{C}$ satisfies

$$\alpha^v(\mathcal{S}_\pi) \subseteq \Xi \quad \text{and} \quad \alpha^v(\pi i) = \tau_h v$$

coincides with

$$E_{\rm KMS} = C_+^\circ \oplus E_0(h) \oplus C_-^\circ.$$

**Exercise 4.33** (The crown of real hyperbolic space) Let $\mathrm{V} := \mathbb{R}^{1,d}$ be $(d+1)$-dimensional Minkowski space and

$$C := \{(x_0, \mathbf{x}) \colon x_0 \geq 0, x_0^2 - \mathbf{x}^2 \geq 0\}$$

the closed positive light cone. We consider the action of the group $G := \mathrm{SO}_{1,d}(\mathbb{R})_e$ on V and its complexification $\mathrm{V}_{\mathbb{C}} = \mathrm{V} + i\mathrm{V}$. Show that:

(a) $\mathbb{H}^d := \{(ix_0, i\mathbf{x}) \colon x_0^2 - \mathbf{x}^2 = 1\}$ is the orbit $G.i\mathbf{e}_0 \cong G/K$ for $K \cong \mathrm{SO}_d(\mathbb{R})$.
(b) $\mathrm{dS}^d := \{(x_0, \mathbf{x}) \colon x_0^2 - \mathbf{x}^2 = -1\} = G.\mathbf{e}_1$ and both lie in the orbit

$$M_{\mathbb{C}} := G_{\mathbb{C}}.\mathbf{e}_1 = G_{\mathbb{C}}.i\mathbf{e}_0 = \{(z_0, \mathbf{z}) \in \mathbb{C}^{1+d} \colon z_0^2 - \mathbf{z}^2 = -1\}$$

(the complex sphere) of the complex orthogonal group $G_{\mathbb{C}} = \mathrm{SO}_{1,d}(\mathbb{C})$.
(c) Let $h(x_0, \mathbf{x}) = (x_1, x_0, 0, \ldots, 0)$ denote the Lorentz boost and

$$\overline{\tau}_h(z) = (-\overline{z}_0, -\overline{z}_1, \overline{z}_2, \cdots, \overline{z}_d)$$

the corresponding antilinear involution. Then

$$\mathrm{V}_{\mathrm{KMS}} := \{x \in \mathrm{V} \colon \alpha^x(\mathcal{S}_\pi) \subseteq \mathrm{V}+iC^\circ, \alpha^x(\pi i) = \tau_h(x)\} = W_R = \{(x_0, \mathbf{x}) \colon x_1 > |x_0|\}$$

is the Rindler wedge.
(d) The open domain

$$\Xi := (\mathrm{V} + iC^\circ) \cap M_{\mathbb{C}}$$

has the following properties:

(i) It is a $G$-invariant open neighborhood of $\mathbb{H}^d$.
(ii) $\mathrm{dS}^d \subseteq \partial \Xi$.
(iii) $\Xi^{\overline{\tau}_h} = \{(ix_0, ix_1, x_2, \ldots, x_d) \in M_{\mathbb{C}} \colon x_j \in \mathbb{R}, x_0 > |x_1|\}$ and

$$e^{-\frac{\pi i}{4} h} \Xi^{\overline{\tau}_h} = W_R \cap \mathrm{dS}^d.$$

(iv) $\mathrm{dS}^d_{\mathrm{KMS}} = \{x \in \mathrm{dS}^d \colon \alpha^x(\mathcal{S}_\pi) \subseteq \Xi, \alpha^x(\pi i) = \tau_h(x)\} = W_R \cap \mathrm{dS}^d$.

**Exercise 4.34** (Crowns of the ncc spaces $\mathrm{GL}_{p+q}(\mathbb{R})/\mathrm{O}_{p,q}(\mathbb{R})$ from Example 4.28) For $n = p+q, 0 < p < n$, we consider in $\mathrm{V} = \mathrm{Sym}_n(\mathbb{R})$ the causal symmetric space

$$M = \mathrm{GL}_n(\mathbb{R}).I_{p,q} = \{gI_{p,q}g^\top \colon g \in \mathrm{GL}_n(\mathbb{R})\} \subseteq E := \mathrm{Sym}_n(\mathbb{R}), \quad I_{p,q} = \mathbf{1}_p \oplus -\mathbf{1}_q$$

and the Euler element

$$h_p := \mathbf{1}_p \oplus 0 = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0) \in \mathfrak{gl}_n(\mathbb{R}).$$

Show that:

(a) $M$ is the set of all symmetric matrices of signature $(p,q)$. In particular, $M$ is open in V.
(b) $M_{\mathbb{C}} := \mathrm{Sym}_n(\mathbb{C}) \cap \mathrm{GL}_n(\mathbb{C})$ is a complex homogeneous space of $\mathrm{GL}_n(\mathbb{C})$.
(c) For $\Xi := M_{\mathbb{C}} \cap (\mathrm{V} + i\mathrm{V}_+)$ ($\mathrm{V}_+ = \{x \colon x \gg 0\}$ is the open cone of positive definite matrices), we have

$$M_{\mathrm{KMS}} = M \cap \left\{ \begin{pmatrix} a & b \\ b^\top & d \end{pmatrix} \in \mathrm{Sym}_{p+q}(\mathbb{R}) \colon a \gg 0, d \ll 0 \right\}.$$

# 5 Minimal and maximal nets for unitary representations

At this point, we have seen large families of causal homogeneous spaces $M = G/H$, for which $\mathfrak{g}$ contains an Euler element $h$. In Section 4 we have seen concrete tools that can be used to construct nets satisfying (Iso), (Cov), (RS) and (BW), by using the geometric structure of a crowned Lie group. For these constructions to work, we used rather fine information on the unitary representations $(U, \mathcal{H})$ of $G_{\tau_h}$, that can typically only be verified for irreducible representations and finite direct sums of such representations. In this section we adopt a global perspective. In particular, we want to construct nets satisfying (BW) for representations that can also be direct integrals of irreducible representations. The key observation is that the wedge region $W \subseteq M$ and the standard subspace $\mathtt{V} = \mathtt{V}(h, U) \subseteq \mathcal{H}$ have to be compatible in the sense that

$$g.W \subseteq W \qquad \Rightarrow \qquad U(g)\mathtt{V} \subseteq \mathtt{V}.$$

Our setting is as follows: $G$ is a connected Lie group, $h \in \mathfrak{g}$ an Euler element, and we assume that the involution $\tau_h^{\mathfrak{g}} = e^{\pi i \operatorname{ad} h}$ on $\mathfrak{g}$ integrates to an involution $\tau_h$ on $G$, so that we can form the semidirect product $G_{\tau_h} = G \rtimes \{\mathbf{1}, \tau_h\}$.

We also fix a homogeneous space $M = G/H$, in which we consider an open subset $W \neq \emptyset$, invariant under the one-parameter group $\exp(\mathbb{R}h)$. We call the translates $(gW)_{g\in G}$ of $W$ *wedge regions* in $M$. At the outset, we do not assume any specific properties of $W$ or $M$, but Lemma 5.2 below will indicate which properties good choices of $W$ should have.

## 5.1 Minimal and maximal nets

We consider an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ and the canonical standard subspace $\mathtt{V} = \mathtt{V}(h, U) \subseteq \mathcal{H}$, specified by $\Delta_{\mathtt{V}} = e^{2\pi i \cdot \partial U(h)}$ and $J_{\mathtt{V}} = U(\tau_h)$ (cf. The Euler Element Theorem 2.3). We associate to the open $\exp(\mathbb{R}h)$-invariant subset $W \subseteq M = G/H$ and the antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ the two nets $\mathsf{H}_M^{\min}$ and $\mathsf{H}_M^{\max}$, defined on open subsets of $M$ by

$$\mathsf{H}_M^{\max}(\mathcal{O}) := \bigcap_{g\in G, \mathcal{O}\subseteq gW} U(g)\mathtt{V} \quad \text{and} \quad \mathsf{H}_M^{\min}(\mathcal{O}) := \overline{\sum_{g\in G, gW\subseteq \mathcal{O}} U(g)\mathtt{V}}. \tag{142}$$

We call $\mathsf{H}_M^{\max}$ the *maximal net* and $\mathsf{H}_M^{\min}$ the *minimal net* associated to $U, M, W$ (cf. [MN24]). This is justified by Lemma 5.8 below. By construction, these nets are isotone and covariant, and we shall see in Lemma 5.2 below that they assign $\mathtt{V}$ to $W \subseteq M$ if and only if we have the semigroup inclusions

$$S_W = \{g \in G \colon g.W \subseteq W\} \subseteq S_{\mathtt{V}} = \{g \in G \colon U(g)\mathtt{V} \subseteq \mathtt{V}\}. \qquad (143)$$

Any other properties of these nets require a more detailed analysis.

*Remark 5.1* (a) If there exists no $g \in G$ with $\mathcal{O} \subseteq gW$, i.e., $\mathcal{O}$ is not contained in any wedge region, then $\mathsf{H}_M^{\max}(\mathcal{O}) = \mathcal{H}$ (the "empty intersection"). Otherwise $\mathsf{H}_M^{\max}(\mathcal{O})$ is always separating because it is contained in a standard subspace $U(g)\mathtt{V}$.

We likewise get $\mathsf{H}_M^{\min}(\mathcal{O}) := \{0\}$ (the "empty sum") if there exists no $g \in G$ with $gW \subseteq \mathcal{O}$, i.e., $\mathcal{O}$ contains no wedge region. Otherwise $\mathsf{H}_M^{\min}(\mathcal{O})$ is always cyclic because it contains a standard subspace $U(g)\mathtt{V}$.
(b) If $\emptyset \neq W \neq M$, then we have in particular

$$\mathsf{H}_M^{\min}(\emptyset) = \{0\} \subseteq \mathsf{H}_M^{\max}(\emptyset) = \bigcap_{g \in G} U(g)\mathtt{V}$$

and

$$\mathsf{H}_M^{\min}(M) = \overline{\sum_{g \in G} U(g)\mathtt{V}} \subseteq \mathsf{H}_M^{\max}(M) = \mathcal{H}.$$

As for locality issues, we note that

$$\mathtt{V}' = \mathtt{V}(-h, U) = J\mathtt{V}$$

(Lemma 1.7) need not be contained in $U(G)\mathtt{V}$, and even if this is the case, then locality properties of $\mathsf{H}_M^{\max}$ are not immediate. We refer to [NÓ26] for a discussion of locality properties of nets on non-compactly causal symmetric spaces.

The following lemma is elementary. It only uses the Equality Lemma 1.9 to verify the equality of standard subspaces.

**Lemma 5.2** *The following assertions hold:*

(a) *The nets* $\mathsf{H}_M^{\max}$ *and* $\mathsf{H}_M^{\min}$ *on* $M$ *satisfy* (Iso) *and* (Cov).
(b) *The set of all open subsets* $\mathcal{O} \subseteq M$ *for which* $\mathsf{H}_M^{\max}(\mathcal{O})$ *is cyclic is* $G$*-invariant.*
(c) *The following are equivalent:*

(i) $S_W \subseteq S_{\mathtt{V}}$.
(ii) $\mathsf{H}_M^{\max}(W) = \mathtt{V}$.
(iii) $\mathsf{H}_M^{\max}(W)$ *is standard.*
(iv) $\mathsf{H}_M^{\max}(W)$ *is cyclic.*
(v) $\mathsf{H}_M^{\min}(W) = \mathtt{V}$.
(vi) $\mathsf{H}_M^{\min}(W)$ *is standard.*
(vii) $\mathsf{H}_M^{\min}(W)$ *is separating.*

***Proof*** (a) Isotony is clear and covariance of the maximal net follows from

$$\mathsf{H}_M^{\max}(g_0\mathcal{O}) = \bigcap_{g_0\mathcal{O}\subseteq gW} U(g)\mathtt{V} = U(g_0) \bigcap_{g_0\mathcal{O}\subseteq gW} U(g_0^{-1}g)\mathtt{V} = U(g_0)\mathsf{H}_M^{\max}(\mathcal{O}).$$

The argument for the minimal net is similar.
(b) follows from covariance.
(c) (i) $\Leftrightarrow$ (ii): Clearly, $\mathsf{H}_M^{\max}(W) \subseteq \mathtt{V}$, and equality holds if and only if $W \subseteq gW$ implies $U(g)\mathtt{V} \supseteq \mathtt{V}$, which is equivalent to $S_W^{-1} \subseteq S_{\mathtt{V}}^{-1}$, and this is equivalent to (i).

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial.

(iv) $\Rightarrow$ (ii): By covariance and $\exp(\mathbb{R}h).W = W$, the subspace $\mathsf{H}_M^{\max}(W) \subseteq \mathtt{V}$ is invariant under the modular group $U(\exp \mathbb{R}h)$ of $\mathtt{V}$. If $\mathsf{H}_M^{\max}(W)$ is cyclic, then the Equality Lemma 1.9 implies $\mathsf{H}_M^{\max}(W) = \mathtt{V}$.

(i) $\Leftrightarrow$ (v) is obvious.

(v) $\Rightarrow$ (vi) $\Rightarrow$ (vii) are trivial.

(vii) $\Rightarrow$ (v): By covariance and $\exp(\mathbb{R}h).W = W$, the subspace $\mathsf{H}_M^{\min}(W) \supseteq \mathtt{V}$ is invariant under the modular group $U(\exp \mathbb{R}h)$ of $\mathtt{V}$. If $\mathsf{H}_M^{\min}(W)$ is separating, then the Equality Lemma 1.9 implies $\mathsf{H}_M^{\min}(W) = \mathtt{V}$. □

For later applications, we record the following observation.

**Lemma 5.3** *Suppose that $(U, \mathcal{H}) = \otimes_{j=1}^n (U_j, \mathcal{H}_j)$ is a tensor product of antiunitary representations of $G_{\tau_h}$. Then the standard subspace $\mathtt{V} = \mathtt{V}(h, U) = \mathrm{Fix}(Je^{\pi i \cdot \partial U(h)})$ is a tensor product*

$$\mathtt{V} = \mathtt{V}_1 \otimes \cdots \otimes \mathtt{V}_n,$$

*and, for every non-empty subset $A \subseteq G$, the subset $\mathtt{V}_A := \bigcap_{g\in A} U(g)\mathtt{V}$ satisfies*

$$\mathtt{V}_A \supseteq \mathtt{V}_{1,A} \otimes \cdots \otimes \mathtt{V}_{n,A}. \tag{144}$$

***Proof*** We have $\xi \in \mathtt{V}_A$ if and only if $U(A)^{-1}\xi \subseteq \mathtt{V}$. This shows that any $\xi = \xi_1 \otimes \cdots \otimes \xi_n$ with $\xi_j \in \mathtt{V}_{j,A}$ is contained in $\mathtt{V}_A$, which is (144). □

The following lemma is a consequence of the naturality of the minimal and the maximal net.

**Lemma 5.4** *For $A := \{g \in G \colon g^{-1}\mathcal{O} \subseteq W\}$, we have*

$$\mathsf{H}_M^{\max}(\mathcal{O}) = \mathtt{V}_A := \bigcap_{g\in A} U(g)\mathtt{V}, \tag{145}$$

*and the cyclicity of this subspace is inherited by subrepresentations, direct sums, direct integrals and finite tensor products.*

***Proof*** For a direct sum representation $U = U_1 \oplus U_2$, we have $\mathtt{V} = \mathtt{V}_1 \oplus \mathtt{V}_2$, which leads to

$$\mathtt{V}_A = \mathtt{V}_{1,A} \oplus \mathtt{V}_{2,A} \tag{146}$$

because $U(g)^{-1}(v_1, v_2) \in \mathtt{V}$ is equivalent to $U_j(g)^{-1}v_j \in \mathtt{V}_j$ for $j = 1, 2$. We thus obtain

$$\mathsf{H}^{\max}_M(\mathcal{O}) = \mathsf{H}^{\max}_{M,1}(\mathcal{O}) \oplus \mathsf{H}^{\max}_{M,2}(\mathcal{O}).$$

This proves that cyclicity of $\mathsf{H}^{\max}_M(\mathcal{O})$ is inherited by subrepresentations and direct sums. For finite tensor products, the assertion follows from Lemma 5.3. If $U = \int_X^{\oplus} U_x \, d\mu(x)$ is a direct integral, then (145) and Lemma 7.28(a) in Appendix 7.6 imply that

$$\mathsf{H}^{\max}_M(\mathcal{O}) = \int_X^{\oplus} \mathsf{H}^{\max}_{M,x}(\mathcal{O}) \, d\mu(x) \tag{147}$$

for direct integrals. So Lemma 7.26 shows that $\mathsf{H}^{\max}_M(\mathcal{O})$ is cyclic if every $\mathsf{H}^{\max}_{M,x}(\mathcal{O})$ is cyclic in $\mathcal{H}_x$. □

*Remark 5.5* (Inner and outer $W$-saturation of subsets) If we write

$$\mathcal{O}^\wedge := \Big( \bigcap_{gW \supseteq \mathcal{O}} gW \Big)^\circ \supseteq \mathcal{O} \quad \text{and} \quad \mathcal{O}^\vee := \bigcup_{gW \subseteq \mathcal{O}} gW \subseteq \mathcal{O},$$

then $\mathcal{O}^\wedge$ and $\mathcal{O}^\vee$ are open subsets satisfying $(\mathcal{O}^\wedge)^\wedge = \mathcal{O}^\wedge$, $(\mathcal{O}^\vee)^\vee = \mathcal{O}^\vee$, and

$$\mathsf{H}^{\max}_M(\mathcal{O}^\wedge) = \mathsf{H}^{\max}_M(\mathcal{O}) \quad \text{and} \quad \mathsf{H}^{\min}_M(\mathcal{O}^\vee) = \mathsf{H}^{\min}_M(\mathcal{O}). \tag{148}$$

So all values of the maximal net are taken on the subset of open subsets $\mathcal{O} \subseteq M$ satisfying $\mathcal{O} = \mathcal{O}^\wedge$ (interiors of intersections of wedge regions) and the minimal net takes all values on those open subsets satisfying $\mathcal{O} = \mathcal{O}^\vee$ (unions of wedge regions)

*Remark 5.6* (The case where $S_W$ is a group) If the semigroup $S_W$ is a group, i.e., $S_W = G_W$ and $\ker(U)$ is discrete, then the inclusion $S_W \subseteq S_{\mathtt{V}}$ is equivalent to

$$G_W \subseteq G_{\mathtt{V}} = G^{h,J} = \{g \in G^h \colon JU(g)J = U(g)\} \tag{149}$$

(cf. Exercise 1.57). In the context of causal homogeneous spaces, the definition of $W$ as a connected component of $W_M^+(h)$ (Definition 3.1) implies that $\exp(\mathbb{R}h) \subseteq G_e^h \subseteq G_W$, and we have in many concrete examples that $G_W \subseteq G^h$, and always $\mathbf{L}(G_W) = \mathfrak{g}^h$ (Proposition 3.13). However, $U(G_W)$ need not commute with $J$, so that (149) may fail. Examples arise already for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$; see Remark 5.7 below.

*Remark 5.7* (cf. Remark 3.45) If $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ and $G$ is a connected Lie group with Lie algebra $\mathfrak{g}$, then $G_{\mathrm{ad}} \cong \mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{SO}_{1,2}(\mathbb{R})_e$, and $H_{\mathrm{ad}} = \exp(\mathbb{R} \operatorname{ad} h)$, so that $G_{\mathrm{ad}}/H_{\mathrm{ad}} \cong \mathrm{dS}^2$ (Example 3.11).

If $M = G/H$ is a covering of $\mathrm{dS}^2$, where $H = \exp(\mathbb{R}h)$ and $Z(G)$ is non-trivial, then $\tau_h$ acts on $Z(\widetilde{G}) \cong \mathbb{Z}$ by inversion, so that it also exists on $G$. Moreover, [MN21, Thm. 4.24] shows that all irreducible unitary representations of $G$ extend to antiunitary representations of $G_{\tau_h}$.

In $M$ the connected components of $W_M^+(h)$ can be labeled by the elements of $Z(G)$ because this subgroup acts non-trivially on $M$, leaving the positivity region $W_M^+(h)$ invariant. In any irreducible representation $(U, \mathcal{H})$ we have $U(Z(G)) \subseteq \mathbb{T}$ by Schur's Lemma, but this subgroup preserves the standard subspace $\mathtt{V}$ if and only if it is contained in $\{\pm 1\}$. To be more concrete, recall that the element $z_{\mathfrak{k}} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ from (173) satisfis $\exp(2\pi\mathbb{Z}z_{\mathfrak{k}}) = Z(G)$, and that, for every $\lambda < 0$, there exists a unitary highest weight representation $U_\lambda$ with $U_\lambda(\exp 2\pi z_{\mathfrak{k}}) = e^{2\pi i\lambda}$.

The following lemma justifies the terminology "minimal" and "maximal".

**Lemma 5.8** *Let* $(U, \mathcal{H})$ *be an antiunitary representation of* $G_{\tau_h}$ *and* $\mathsf{H}$ *a net of real subspaces on open subsets of* $M$*, satisfying* (Iso), (Cov), *and for which the Bisognano–Wichmann property holds in the sense that* $\mathsf{H}(W) = \mathtt{V}(h, U)$*. Then*

$$\mathsf{H}_M^{\min}(\mathcal{O}) \subseteq \mathsf{H}(\mathcal{O}) \subseteq \mathsf{H}_M^{\max}(\mathcal{O}) \qquad \text{for } \mathcal{O} \subseteq M \text{ open}, \tag{150}$$

*and equality holds for all domains of the form* $\mathcal{O} = g.W$, $g \in G$*, i.e., the wedge regions in* $M$*.*

***Proof*** The three properties (Iso), (Cov) and $\mathsf{H}(W) = \mathtt{V}$ of the net $\mathsf{H}$ entail $S_W \subseteq S_{\mathtt{V}}$ because $g.W \subseteq W$ implies

$$U(g)\mathtt{V} = U(g)\mathsf{H}(W) \overset{\text{(Cov)}}{=} \mathsf{H}(g.W) \overset{\text{(Iso)}}{\subseteq} \mathsf{H}(W) \overset{\text{(BW)}}{\subseteq} \mathtt{V}.$$

From Lemma 5.2(c) we thus obtain $\mathsf{H}_M^{\max}(W) = \mathsf{H}_M^{\min}(W) = \mathtt{V}$. Hence

$$\mathsf{H}(gW) = U(g)\mathtt{V} = \mathsf{H}_M^{\max}(gW) = \mathsf{H}_M^{\min}(gW)$$

by covariance for any $g \in G$ (Lemma 5.2(a)). By (Iso), $\mathcal{O} \subseteq gW$ implies $\mathsf{H}(\mathcal{O}) \subseteq \mathsf{H}(gW) = U(g)\mathtt{V}$, so that $\mathsf{H}(\mathcal{O}) \subseteq \mathsf{H}_M^{\max}(\mathcal{O})$. Likewise, $gW \subseteq \mathcal{O}$ implies $U(g)\mathtt{V} = \mathsf{H}(gW) \subseteq \mathsf{H}(\mathcal{O})$, and thus $\mathsf{H}_M^{\min}(\mathcal{O}) \subseteq \mathsf{H}(\mathcal{O})$. □

*Remark 5.9* The construction of the minimal and the maximal net can also be carried out on $G$ itself with respect to $W^G = q_M^{-1}(W)$. It then makes sense to compare the minimal/maximal net on $M$ with the push-forwards of the minimal/maximal net on $G$.

For $\mathcal{O} \subseteq M$, the relation $q_M^{-1}(\mathcal{O}) \subseteq gW^G$ is equivalent to $\mathcal{O} \subseteq gW$, so that

$$(q_M)_*\mathsf{H}_G^{\max} = \mathsf{H}_M^{\max}.$$

Likewise, $q_M^{-1}(\mathcal{O}) \supseteq gW^G$ is equivalent to $\mathcal{O} \supseteq gW$, which shows that

$$(q_M)_*\mathsf{H}_G^{\min} = \mathsf{H}_M^{\min}.$$

If, however, $W^G \subseteq G$ is not the full inverse image of $W \subseteq M$, then these relations may fail.

## 5.2 The endomorphism semigroup of a standard subspace

Motivated by the important relation $S_W \subseteq S_{\mathtt{V}}$ from Lemma 5.2, we take in this subsection a closer look at the semigroup $S_{\mathtt{V}}$ for an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ with discrete kernel ([Ne21, Ne22]). Here we consider the standard subspace $\mathtt{V} := \mathtt{V}(h, U) \subseteq \mathcal{H}$ from (2) and Definition 2.20. To describe $S_{\mathtt{V}}$, we need the *positive cone*

$$C_U := \{x \in \mathfrak{g}\colon\ -i \cdot \partial U(x) \geq 0\}, \qquad \partial U(x) = \frac{d}{dt}\Big|_{t=0} U(\exp tx) \qquad (151)$$

of a unitary representation $U$. It is a closed, convex, $\mathrm{Ad}(G)$-invariant cone in $\mathfrak{g}$ ([Ne00, Prop. X.1.5]).

The key point of the identity

$$S(h, C_U) = \{g \in G\colon\ \mathrm{Ad}(g)h \in h - C_U\} = \exp(C_+)G^h \exp(C_-)$$

for

$$C_\pm := \pm C_U \cap \mathfrak{g}_{\pm 1}(h)$$

in Theorem 3.18 is that it provides two different perspectives on the same subsemigroup of $G$, and this is instrumental for the descriptions of the semigroups $S_{\mathtt{V}}$. To see this connection, we recall that the Monotonicity Theorem [Ne22, Thm. 3.3] asserts that

$$S_{\mathtt{V}} \subseteq S(h, C_U). \qquad (152)$$

Its proof is based on the fact that, for two standard subspaces $\mathtt{V}_1 \subseteq \mathtt{V}_2$, we have $\log \Delta_{\mathtt{V}_2} \leq \log \Delta_{\mathtt{V}_1}$ in the sense of quadratic forms. Since these selfadjoint operators are typically not semibounded, the order relation requires some explanation, provided in an appendix to [Ne22]. Put differently, the Monotonicity Theorem asserts that the well-defined $G$-equivariant map

$$\mathcal{O}_{\mathtt{V}} = U(G)\mathtt{V} \cong G/G_{\mathtt{V}} \to \mathcal{O}_h \cong G/G^h, \quad U(g)\mathtt{V} \mapsto \mathrm{Ad}(g)h$$

is monotone with respect to the $C_U$-order on $\mathfrak{g}$ and the inclusion order on $\mathcal{O}_{\mathtt{V}} \subseteq \mathrm{Stand}(\mathcal{H})$ (cf. Section 3.3), hence the name.

**Theorem 5.10** ([Ne22, Thm. 3.4]) *If $(U, \mathcal{H})$ is an antiunitary representation of $G_{\tau_h}$ with discrete kernel, then*

$$S_{\mathtt{V}} = \exp(C_+)G_{\mathtt{V}} \exp(C_-) = G_{\mathtt{V}} \exp(C_+ + C_-) \quad \textit{for} \quad C_\pm := \pm C_U \cap \mathfrak{g}_{\pm 1}(h).$$

Here the second equality follows from Theorem 3.18. The Borchers–Wiesbrock Theorem 5.42 in Appendix 5.8.1 immediately shows that $\exp(C_+) \subseteq S_{\mathtt{V}}$. Applying it again with $-h$ and $\mathtt{V}' = \mathtt{V}(-h, U)$, we also get $\exp(C_-) \subseteq S_{\mathtt{V}}$, which leads with (152) to

$$\exp(C_+)G_{\mathtt{V}}\exp(C_-) \subseteq S_{\mathtt{V}} \subseteq S(h, C_U).$$

Therefore the main point is to show that

$$S(h, C_U) \subseteq \exp(C_+)G^h \exp(C_-)$$

and to identify the connected components of $G^h$ fixing $\mathtt{V}$.

**Example 5.11** (Poincaré group) In Quantum Field Theory on Minkowski space, the natural symmetry group is the proper Poincaré group $P(d) \cong \mathbb{R}^{1,d-1} \rtimes \mathrm{O}_{1,d-1}(\mathbb{R})^{\uparrow}$ acting by causal isometries on $d$-dimensional Minkowski space $M := \mathbb{R}^{1,d-1}$. Its Lie algebra is $\mathfrak{g} := \mathfrak{p}(d) \cong \mathbb{R}^{1,d-1} \rtimes \mathfrak{so}_{1,d-1}(\mathbb{R})$ and the closed forward light cone

$$C_{\mathfrak{g}} := \{(x_0, \mathbf{x}) \in \mathbb{R}^{1,d-1} \colon x_0 \geq \sqrt{\mathbf{x}^2}\} \tag{153}$$

is a pointed invariant cone in $\mathfrak{g} = \mathfrak{p}(d)$.

The generator $h \in \mathfrak{so}_{1,d-1}(\mathbb{R})$ of the Lorentz boost on the $(x_0, x_1)$-plane

$$h(x_0, x_1, \ldots, x_{d-1}) = (x_1, x_0, 0, \ldots, 0)$$

is an Euler element and $\tau_h = e^{\pi i \,\mathrm{ad}\, h}$ defines an involution on $\mathfrak{g}$, acting on the ideal $\mathbb{R}^{1,d-1}$ (Minkowski space) by

$$\tau_M(x_0, x_1, \ldots, x_{d-1}) = (-x_0, -x_1, x_2, \ldots, x_{d-1}).$$

We apply the results in this section to the identity component

$$G := P(d)_e \cong \mathbb{R}^{1,d-1} \rtimes \mathrm{SO}_{1,d-1}(\mathbb{R})_e$$

which has trivial center $Z(G) = \{e\}$. A unitary representation $(U, \mathcal{H})$ of $G$ is called a *positive energy representation* if $C_{\mathfrak{g}} \subseteq C_U$. If $\ker(U)$ is discrete, then $C_U$ is pointed, and $C_{\mathfrak{g}} = C_U$ follows from the fact that the only non-zero pointed invariant cone in the Lie algebra $\mathfrak{g} = \mathfrak{p}(d)$ for $d > 2$ are $\pm C_{\mathfrak{g}}$, and, for $d = 2$, there are four pointed invariant cones which are quarter planes in $\mathbb{R}^{1,1}$.

The centralizer of the Lorentz boost is

$$\mathfrak{g}_0 = (\{(0,0)\} \times \mathbb{R}^{d-2}) \rtimes (\mathfrak{so}_{1,1}(\mathbb{R}) \oplus \mathfrak{so}_{d-2}(\mathbb{R})) \cong (\mathbb{R}^{d-2} \rtimes \mathfrak{so}_{d-2}(\mathbb{R})) \oplus \mathbb{R}h,$$

and,

$$C_{\pm} = \pm C_{\mathfrak{g}} \cap \mathfrak{g}_{\pm 1} = \mathbb{R}_{\geq 0}(\mathbf{e}_1 \pm \mathbf{e}_0). \tag{154}$$

The subsemigroup

$$S(h, C_{\mathfrak{g}}) = \{g \in G \colon h - \mathrm{Ad}(g)h \in C_{\mathfrak{g}}\}$$

is easy to determine. The relation $\mathrm{Ad}(g)h - h \in \mathbb{R}^d$ implies that $g = (v, \ell)$ with $\mathrm{Ad}(\ell)h = h$, and then $\mathrm{Ad}(g)h = \mathrm{Ad}(v, \mathbf{1})h = -hv \in -C_{\mathfrak{g}}$ is equivalent to $hv \in C_{\mathfrak{g}}$, which specifies the closure $\overline{W_R}$ of the standard right wedge

$$W_R = \{x \in \mathbb{R}^{1,d-1} \colon x_1 > |x_0|\}.$$

The two cones $C_\pm$ generate a proper Lie subalgebra of $\mathfrak{g}$. We therefore obtain with Lemma 3.15

$$S(h, C_{\mathfrak{g}}) = \overline{W_R} \rtimes \big(\mathrm{SO}_{1,1}(\mathbb{R})^\uparrow \times \mathrm{SO}_{d-2}(\mathbb{R})\big) \overset{3.15}{=} \{g \in G \colon gW_R \subseteq W_R\} = S_{W_R}, \tag{155}$$

where $\mathrm{SO}_{1,1}(\mathbb{R})^\uparrow = \exp(\mathbb{R}h)$. We claim that, for any antiunitary positive energy representation of $G_{\tau_h}$ with discrete kernel, the semigroup $S_{\mathtt{V}}$ corresponding to the standard subspace $\mathtt{V} = \mathtt{V}(h, U)$ is given by

$$S_{\mathtt{V}} = S(h, C_{\mathfrak{g}}) = S_{W_R}. \tag{156}$$

To verify this claim, we first observe that (152) implies $S_{\mathtt{V}} \subseteq S(h, C_{\mathfrak{g}})$. We further have

$$S(h, C_{\mathfrak{g}}) = S_{W_R} = \exp(C_+) G_{W_R} \exp(C_-),$$

and the group $G_{W_R}$ is connected, hence contained in $G^{h,\tau_h} \subseteq G_{\mathtt{V}}$. Now our claim follows from Theorem 5.10.

- For $d \geq 4$, the simply connected covering group $q_G \colon \widetilde{G} \to G$ is a 2-fold covering. Then we obtain for $\widetilde{G}$ the same picture because the involution $\tau_h$ acts trivially on the covering group $\widetilde{G}^h$ of $G^h$ by (155), and this implies that $U(\widetilde{G}^h)$ fixes $\mathtt{V}$.
- For $d = 2$, the group $G \cong \mathbb{R}^2 \rtimes \mathbb{R}$ is simply connected.
- But for $d = 3$ the picture is quite different. Then $\mathrm{SO}_{1,2}(\mathbb{R})_e \cong \mathrm{PSL}_2(\mathbb{R})$ and $\pi_1(G) \cong \mathbb{Z}$. In this case $\tau_h$ acts by inversion on the center $Z(\widetilde{G})$. So

  $$\widetilde{G}^h \cong \mathbb{R}\mathbf{e}_2 \times (\exp(\mathbb{R}h) Z(\widetilde{G})) \cong \mathbb{R} \times \mathbb{R} \times \mathbb{Z}$$

  and

  $$\widetilde{G}_{\mathtt{V}} = \{g \in \widetilde{G}^h \colon g\tau_h(g)^{-1} \in \ker U\} = \mathbb{R} \times \mathbb{R} \times \{n \in \mathbb{Z} \colon 2n \in \ker U\}.$$

  Therefore $\widetilde{G}_{W_R} = \widetilde{G}^h$ is equivalent to $U(Z(\widetilde{G})) \subseteq \{\pm\mathbf{1}\}$. In this case the nets $\mathsf{H}^{\min}$ and $\mathsf{H}^{\max}$ on $M = \mathbb{R}^{1,d-1}$ satisfy (BW) for $W = W_R$.

**Example 5.12** (Conformal groups $\mathrm{SO}_{2,d}(\mathbb{R})$) The Lie algebra of the conformal group $G := \mathrm{SO}_{2,d}(\mathbb{R})_e$ of Minkowski space is $\mathfrak{g} = \mathfrak{so}_{2,d}(\mathbb{R})$, which contains the Poincaré–Lie algebra as those elements corresponding to affine vector fields on $\mathrm{V} := \mathbb{R}^{1,d-1}$. For $d \geq 3$ it is a simple hermitian Lie algebra. All its Euler elements $h$ are mutually conjugate (Proposition 2.13). One arises from the element $h = \mathrm{id}_{\mathrm{V}}$ corresponding to the Euler vector field on V. Then $\mathfrak{g}_j(h)$, $j = -1, 0, 1$, are spaces of vector fields on V which are linear (for $j = 0$), constant (for $j = 1$) and quadratic (for $j = -1$).

Another important example is the element $h_{01} \in \mathfrak{so}_{1,d-1}(\mathbb{R}) \subseteq \mathfrak{so}_{2,d-1}(\mathbb{R})$ corresponding to a Lorentz boost in the Poincaré–Lie algebra (see Example 5.11).

We consider the minimal invariant cone $C_{\mathfrak{g}} \subseteq \mathfrak{g}$ which intersects V in the positive light cone $C_+ \subseteq$ V. Then we obtain a complete description of the endomorphism semigroup of the standard subspace $\mathtt{V}$

$$S_{\mathtt{V}} = \exp(C_+)G_{\mathtt{V}}\exp(C_-)$$

(Theorem 5.10), and these semigroups have interior points because the cones $C_\pm$ generate the subspaces $\mathfrak{g}_{\pm 1}$ (see [MNÓ26b] for more details).

**Example 5.13** Another interesting example which is neither semisimple nor an affine group is given by the Lie algebra

$$\mathfrak{g} = \mathfrak{hcsp}(V,\omega) := \mathfrak{heis}(V,\omega) \rtimes \mathfrak{csp}(V,\omega)$$

from Example 2.17. Now we turn to the corresponding group and one of its irreducible unitary representations. Choosing a symplectic basis in $V$, we obtain an isomorphism

$$V \cong V_{-1} \oplus V_1 \cong \mathbb{R}^n \oplus \mathbb{R}^n$$

with the canonical symplectic form specified by $\omega((q,0),(0,p)) = \langle q,p\rangle$ and $\tau_V(q,p) = (-q,p)$. Let $\mathrm{Mp}_{2n}(\mathbb{R})$ denote the *metaplectic group*, which is the unique non-trivial double cover of $\mathrm{Sp}_{2n}(\mathbb{R})$. We consider the group

$$G := \mathrm{Heis}(\mathbb{R}^{2n}) \rtimes_\alpha (\mathbb{R}^\times_+ \times \mathrm{Mp}_{2n}(\mathbb{R})),$$

where $\mathbb{R}^\times$ acts on $\mathrm{Heis}(\mathbb{R}^{2n}) = \mathbb{R}\times\mathbb{R}^{2n}$ by $\alpha_r(z,v) = (r^2z, rv)$. Its Lie algebra is $\mathfrak{g} = \mathfrak{hcsp}(V,\omega)$. Then the Hilbert space

$$\mathcal{H} := L^2\Big(\mathbb{R}^\times_+, \frac{d\lambda}{\lambda}; L^2(\mathbb{R}^n)\Big) \cong L^2\Big(\mathbb{R}^\times_+ \times \mathbb{R}^n, \frac{d\lambda}{\lambda} \otimes dx\Big),$$

carries an irreducible unitary representation of $G$, where $L^2(\mathbb{R}^n) \cong L^2(V_{-1})$ carries the oscillator representation $U_0$ of $\mathrm{Heis}(\mathbb{R}^{2n}) \rtimes \mathrm{Mp}_{2n}(\mathbb{R})$ (cf. [Ne00, §IX.4]). The Heisenberg group $\mathrm{Heis}(\mathbb{R}^{2n})$ is represented on $\mathcal{H}$ by

$$\begin{aligned}
(U(z,0,0)f)(\lambda,x) &= e^{i\lambda^2 z} f(\lambda,x),\\
(U(0,q,0)f)(\lambda,x) &= e^{i\lambda\langle q,x\rangle} f(\lambda,x),\\
(U(0,0,p)f)(\lambda,x) &= f(\lambda, x-\lambda p).
\end{aligned}$$

The group $\mathrm{Mp}_{2n}(\mathbb{R})$ acts by the metaplectic representation on $L^2(\mathbb{R}^n)$ via

$$(U(g)f)(\lambda,\cdot) := U_0(g)f(\lambda,\cdot),$$

independently of $\lambda$. For $h_0 = \mathrm{id}_V$, the one-parameter group $\mathbb{R}_+^\times \cong \exp(\mathbb{R}h_0)$ acts by

$$(U'(r)f)(\lambda, x) := f(r\lambda, x) \quad \text{for} \quad r > 0.$$

The element

$$h := \tfrac{1}{2}(\tau_V + \mathrm{id}_V) \in \mathfrak{csp}(V, \omega)$$

is Euler in $\mathfrak{g}$ (see (61)).

We also have a conjugation $J$ on $\mathcal{H}$ defined by

$$(Jf)(\lambda, x) := \overline{f(\lambda, -x)} \quad \text{satisfying} \quad JU(g)J = U(\tau_h(g)),$$

where $\tau_h$ induces on $\mathfrak{g}$ the involution $e^{\pi i \,\mathrm{ad}\, h} = (-\tau_V)^{\sim}$ (cf. Example 2.17).

The positive cone $C_U \subseteq \mathfrak{g}$ is the same as the one of the metaplectic representation. It intersects $\mathfrak{sp}(V, \omega)$ in the cone of non-negative homogeneous quadratic polynomials on $V$. This implies that $(C_U)_- = C_-$. To determine $(C_U)_+ = C_U \cap \mathfrak{g}_1$, we observe that $\mathfrak{g}_1$ acts on $L^2(\mathbb{R}^n) \cong L^2(V_-)$ by multiplication operators. This shows that we also have $(C_U)_+ = C_+$, so that we get with Theorem 5.10 the semigroup $S_{\mathtt{V}}$ for the standard subspace $\mathtt{V} = \mathtt{V}(h, U)$:

$$S_{\mathtt{V}} = \exp(C_+) G_{\mathtt{V}} \exp(C_-).$$

Here $G_{\mathtt{V}} = G^h$ is a double cover of $\mathrm{Aff}(\mathbb{R}^n)_e$, its inverse image in $\mathrm{Mp}_{2n}(\mathbb{R})$.

## 5.3 Regularity of unitary representations

The regularity concept for an antiunitary representation $(U, \mathcal{H})$ is closely related to the assumptions of the Euler Element Theorem 2.3, but here we are already given an Euler element and an antiunitary representation of $G_{\tau_h}$. It is a challenging open problem (Conjecture 5.15) to show that **all** antiunitary representations of $G_{\tau_h}$ are $h$-regular, without any additional structural assumption on $G$. We refer to [MN24] for a detailed discussion, and to [BN25] for the case of the 4-dimensional split oscillator group. In this section we describe the connection between regularity and the existence of nets of real subspaces satisfying (Iso), (Cov), (RS) and (BW) on a very general level.

**Definition 5.14** We call an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ *regular with respect to* $h$, or $h$*-regular*, if there exists an $e$-neighborhood $N \subseteq G$ such that,

$$\mathtt{V}_N = \bigcap_{g \in N} U(g)\mathtt{V}(h, U) \quad \text{for} \quad \mathtt{V}(h, U) = \mathrm{Fix}(U(\tau_h) e^{\pi i \partial U(h)}),$$

is cyclic. Replacing $N$ by its interior, we may always assume that $N$ is open.

*Conjecture 5.15* (Regularity Conjecture) If $h \in \mathfrak{g}$ is an Euler element, then any antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ is $h$-regular.

This conjecture holds for connected reductive groups by Corollary 5.23 below and for several specific classes of groups and representations (see [MN24] and [BN25] for details).

**Lemma 5.16** *For an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$, the following assertions hold:*

(a) *If $U = U_1 \oplus U_2$ is a direct sum, then $U$ is $h$-regular if and only if $U_1$ and $U_2$ are $h$-regular.*
(b) *If $U$ is $h$-regular, then every subrepresentation is $h$-regular.*
(c) *Let $U = \int_X^{\oplus} U_m \, d\mu(m)$ be an antiunitary direct integral representation of $G_{\tau_h}$, then $U$ is regular if and only if there exists an $e$-neighborhood $N \subseteq G$ such that, for $\mu$-almost every $m \in X$, the subspace $\mathtt{V}_{m,N}$ is cyclic.*

***Proof*** (a) If $U \cong U_1 \oplus U_2$, then (146) implies that $\mathtt{V}_N = \mathtt{V}_{1,N} \oplus \mathtt{V}_{2,N}$ for every $e$-neighborhood $N \subseteq G$. In particular, $\mathtt{V}_N$ is cyclic if and only if $\mathtt{V}_{1,N}$ and $\mathtt{V}_{2,N}$ are.
(b) follows immediately from (a).
(c) Applying Lemma 7.28(b) in Appendix 7.6 to $A := N$, we obtain (c). □

Note that the Regularity Characterization Theorem 5.19 below does not require any assumption concerning the irreducibility of the representation. Proposition 5.17 (cf. [MN24, Prop. 2.26]) is a convenient tool to reduce to irreducible representations.

**Proposition 5.17** *Consider a group of the form $G_{\tau_h} = G \rtimes \{\mathbf{1}, \tau_h\}$ and a subset $A \subseteq G$. Then the following are equivalent:*

(a) *For all antiunitary representations $(U, \mathcal{H})$ of $G_{\tau_h}$, the subspace $\mathtt{V}_A := \bigcap_{g \in A} U(g)\mathtt{V}$ is cyclic.*
(b) *For all irreducible antiunitary representations $(U, \mathcal{H})$ of $G_{\tau_h}$, the subspace $\mathtt{V}_A$ is cyclic.*
(c) *For all irreducible unitary representations $(U, \mathcal{H})$ of $G$, the subspace $\widetilde{\mathtt{V}}_A$ is cyclic in $\widetilde{\mathcal{H}}$, where $\widetilde{\mathtt{V}} := \mathtt{V}(h, \widetilde{U})$ and $(\widetilde{U}, \widetilde{\mathcal{H}})$ is the canonical antiunitary extension of $U$ from* Lemma 7.19 *in* Appendix 7.4*.*
(d) (Characterization in terms of unitary representations) *For all unitary representations $(U, \mathcal{H})$ of $G$, the subspace $\widetilde{\mathtt{V}}_A$ is cyclic in $\widetilde{\mathcal{H}}$.*

**Proposition 5.18** (Localizability implies regularity) *Let $\emptyset \neq \mathcal{O} \subseteq W \subseteq M$ be open subsets such that $N := \{g \in G \colon g^{-1}\mathcal{O} \subseteq W\}$ is an $e$-neighborhood. If $(U, \mathcal{H})$ is an antiunitary representation for which $\mathsf{H}_M^{\max}(W) = \mathtt{V}$ and $\mathsf{H}_M^{\max}(\mathcal{O})$ is cyclic, then $U$ is regular.*

***Proof*** By assumption $\mathsf{H}^{\max}_M(\mathcal{O})$ is cyclic, and we obtain with (Iso) and (Cov) (Lemma 5.2) the relation

$$\mathsf{H}^{\max}_M(\mathcal{O}) \subseteq \bigcap_{g\in N} \mathsf{H}^{\max}_M(gW) = \bigcap_{g\in N} U(g)\mathsf{H}^{\max}_M(W) = \bigcap_{g\in N} U(g)\mathtt{V} = \mathtt{V}_N.$$

It follows that $\mathtt{V}_N$ is cyclic. □

### 5.3.1 Regularity versus existence of nets

The following surprising theorem show that regularity already implies the existence of a net of real subspaces on open subsets of $G$, satisfying (Iso), (Cov), (RS) and (BW) for a suitably chosen subset $W \subseteq G$.

**Theorem 5.19** (Regularity Characterization Theorem) *Let $(U, \mathcal{H})$ be an antiunitary representation of $G_{\tau_h}$ and $\mathtt{V} = \mathtt{V}(h, U) \subseteq \mathcal{H}$ the corresponding standard subspace. Then there exists a net $(\mathsf{H}(\mathcal{O}))_{\mathcal{O}\subseteq G}$ on open subsets of $G$ satisfying* (Iso), (Cov), (RS), *and* (BW) *for some open connected subset $W \subseteq G$ if and only if $U$ is $h$-regular, i.e., $\mathtt{V}_N$ is cyclic for some $e$-neighborhood $N \subseteq G$.*

***Proof*** "⇒": If a net $\mathsf{H}$ with the asserted properties exists, then $\mathtt{V} = \mathsf{H}(W)$, and for any relatively compact open subset $\mathcal{O} \subseteq W$, there exists an identity neighborhood $N \subseteq G$ with $N\mathcal{O} \subseteq W$. Then, for $g^{-1} \in N$, we have

$$U(g)^{-1}\mathsf{H}(\mathcal{O}) = \mathsf{H}(g^{-1}.\mathcal{O}) \subseteq \mathsf{H}(W) = \mathtt{V}, \quad \text{hence} \quad \mathsf{H}(\mathcal{O}) \subseteq \mathtt{V}_N,$$

as in the proof of Proposition 5.18. Now (RS) implies that $U$ is $h$-regular.
"⇐": Assume that $\mathtt{V}_N$ is cyclic for an $e$-neighborhood $N$. Pick an open connected identity neighborhood $N_1 \subseteq N$ with $N_1 N_1^{-1} \subseteq N$. Then

$$W := \exp(\mathbb{R}h)N_1$$

is an open connected subset of $G$. We consider the net $\mathsf{H} := \mathsf{H}^{\max}_G$, defined by

$$\mathsf{H}(\mathcal{O}) = \bigcap_{g\in G, \mathcal{O}\subseteq gW} U(g)\mathtt{V}.$$

This net satisfies (Iso) and (Cov) by Lemma 5.2.

We now verify the Reeh–Schlieder property (RS). So let $\emptyset \neq \mathcal{O} \subseteq G$ be an open subset. By (Iso) and (Cov), it suffices to show that $\mathsf{H}(\mathcal{O})$ is cyclic if $\mathcal{O} \subseteq N_1$. Then $\mathcal{O} \subseteq gW = g\exp(\mathbb{R}h)N_1$ implies

$$g \in \mathcal{O}N_1^{-1}\exp(\mathbb{R}h) \subseteq N_1 N_1^{-1}\exp(\mathbb{R}h) \subseteq N\exp(\mathbb{R}h),$$

so that

$$\mathsf{H}(\mathcal{O}) \supseteq \bigcap_{g\in N\exp(\mathbb{R}h)} U(g)\mathtt{V} = \bigcap_{g\in N} U(g)\mathtt{V} = \mathtt{V}_N$$

implies that $\mathsf{H}(\mathcal{O})$ is cyclic. This proves (RS). It follows in particular that $\mathsf{H}(W)$ is cyclic, so that Lemma 5.2(c) implies $\mathsf{H}(W) = \mathtt{V}$. Therefore (BW) is also satisfied. □

*Remark 5.20* (Regularity versus orbit maps in $\mathtt{V}$)
(a) Note that $v \in \mathsf{H}^{\max}(\mathcal{O})$ is equivalent to

$$g^{-1}\mathcal{O} \subseteq W \quad \Rightarrow \quad U(g)^{-1}v \in \mathtt{V}.$$

If $\mathcal{O} \subseteq W$ is relatively compact, this condition holds for $g$ in an $e$-neighborhood. Therefore $\mathsf{H}^{\max}(\mathcal{O})$ consists of vectors $v \in \mathcal{H}$ whose orbit map $U^v \colon G \to \mathcal{H}$ maps an identity neighborhood into $\mathtt{V}$ (cf. Proposition 4.13(d)). Put differently, the subset $(U^v)^{-1}(\mathtt{V}) \subseteq G$ has interior points.
(b) Suppose that $v \in \mathtt{V} \cap \mathcal{H}^\omega$ is an analytic vector and $U(N)v \subseteq \mathtt{V}$ holds for an identity neighborhood $N \subseteq G$. Then the connectedness of $G$ and uniqueness of analytic continuation imply $U(G)v \subseteq \mathtt{V}$, i.e., $v \in \mathtt{V}_G = \bigcap_{g\in G} U_g\mathtt{V}$.

If, in addition, $v$ is $G$-cyclic, then $\mathtt{V}_G$ is a cyclic real subspace, so that its invariance under the modular group of $\mathtt{V}$ implies with the Equality Lemma 1.9 that $\mathtt{V} = \mathtt{V}_G$, i.e., that $\mathtt{V}$ is $G$-invariant. If $U$ has discrete kernel, this implies that $h \in \mathfrak{z}(\mathfrak{g})$. Hence $\tau_h$ is trivial and therefore $J_\mathtt{V}$ commutes with $G$. Therefore $\mathcal{H}^{J_\mathtt{V}} = \mathtt{V}$ is a real orthogonal representation of $G$, and $U$ is its complexification, considered as a representation of $G_{\tau_h}$ on $\mathcal{H} \cong (\mathcal{H}^{J_\mathtt{V}})_\mathbb{C}$. This is the context where $\partial U(h)$ and $J_\mathtt{V}$ commute with $G_{\tau_h}$.
(c) Another perspective on (b) is that the cyclic subrepresentation $U_v$ generated by any $v \in \mathcal{H}^\omega \cap \mathtt{V}_N$ is such that $\partial U_v(h)$ and $J_\mathtt{V}$ commute with $G$. So $v$ is fixed by the normal subgroup $B$ with Lie algebra

$$\mathfrak{b} := \mathfrak{g}_1 + [\mathfrak{g}_1, \mathfrak{g}_{-1}] + \mathfrak{g}_{-1} \subseteq \ker(\mathtt{d}U_v).$$

### 5.3.2 Regularity for reductive Lie groups

In this subsection we show that all antiunitary representations of connected reductive Lie groups are regular with respect to any Euler element $h \in \mathfrak{g}$ (Corollary 5.23). This is based on the existence of nets of real subspaces on the naturally associated ncc symmetric space $M = G/H$ from Definition 3.41, satisfying (Iso), (Cov), (RS) and (BW) (Theorem 5.22). An interesting byproduct of this result is that, for a ncc symmetric space, the semigroup $S_W$ of a wedge region $W \subseteq M$ is actually a group (Theorem 5.24).

So let $\mathfrak{g}$ be reductive, $h \in \mathfrak{g}$ an Euler elements, and let $G$ be a corresponding connected Lie group on which $\tau_h$ exists. As in (105) in Definition 3.41, we specify the corresponding non-compactly causal symmetric space $M = G/H$ (cf. Section 3.5.3) and the corresponding wedge region $W = W_M^+(h)_{eH}$.

**Definition 5.21** We say that the (anti-)unitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ is $(h, W)$-*localizable* in those open subsets $\mathcal{O} \subseteq M$ for which $\mathsf{H}_M^{\max}(\mathcal{O})$ is cyclic.

**Theorem 5.22** (Localization for reductive groups) *If $\mathfrak{g}$ is reductive and $(U, \mathcal{H})$ is an antiunitary representation of $G_{\tau_h}$, then the net $\mathsf{H}_M^{\max}$ on the non-compactly causal symmetric space $M = G/H$ from* (105) *in Definition 3.41, and $W = W_M^+(h)_{eH}$ satisfy* (Iso), (Cov), (RS) *and* (BW).

***Proof*** (cf. [MN24, Thm. 4.23]) As the standard subspace $\mathtt{V}$ is invariant under $G_1 \subseteq G^{h,\tau_h}$, and $G_1$ acts trivially on $M$, the real subspaces $\mathsf{H}_M^{\max}(\mathcal{O})$ only depend on $U|_{G_2}$. We may therefore assume that $G = G_2$, i.e., that $G$ is semisimple and that $\mathfrak{g}_0(h)$ contains no non-zero ideal.

In view of Lemma 5.2(c), $\mathtt{V} = \mathsf{H}^{\max}(W)$ follows from the cyclicity of all subspaces $\mathsf{H}^{\max}(\mathcal{O})$, $\mathcal{O} \neq \emptyset$. So it suffices to verify the latter. By Proposition 5.17 and Lemma 5.4, we may further assume that $(U, \mathcal{H})$ is irreducible. Then Theorem 4.31 provides a net $\mathsf{H}_{\mathtt{E}}^M$ satisfying (Iso), (Cov), (RS) and (BW), and this net satisfies $\mathsf{H}_{\mathtt{E}}^M(\mathcal{O}) \subseteq \mathsf{H}_M^{\max}(\mathcal{O})$ for each $\mathcal{O} \subseteq M$ (Lemma 5.8). Thus $\mathsf{H}_M^{\max}(\mathcal{O})$ is cyclic. □

**Corollary 5.23** (Regularity for reductive groups) *Let $G$ be a connected reductive Lie group. Then there exists an $e$-neighborhood $N \subseteq G$ such that for* **every** *separable antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ and $\mathtt{V} = \mathtt{V}(h, U)$, the real subspace*

$$\mathtt{V}_N = \bigcap_{g \in N} U(g)\mathtt{V}$$

*is cyclic. In particular, $(U, \mathcal{H})$ is $h$-regular.*

***Proof*** Let $\mathcal{O} \subseteq W \subseteq M = G/H$ (with $W \subseteq M$ as in Theorem 5.22) be an open subset whose closure $\overline{\mathcal{O}}$ is relatively compact. In Theorem 5.22 we have seen that $\mathsf{H}_M^{\max}(\mathcal{O})$ is cyclic. Further

$$N := \{g \in G \colon g\mathcal{O} \subseteq W\} \supseteq \{g \in G \colon g\overline{\mathcal{O}} \subseteq W\}$$

is an $e$-neighborhood because $\overline{\mathcal{O}} \subseteq W$ is compact. Therefore the $h$-regularity of $(U, \mathcal{H})$ follows from Proposition 5.18. □

**Theorem 5.24** (Triviality of the semigroups of wedge regions in ncc symmetric spaces) *If $G$ is a connected reductive Lie group and $M = G/H$ a corresponding non-compactly causal symmetric space as in* (105)*, with causal Euler element $h$, and the maximal causal structure, then the following assertions hold:*

(a) $S_W = G_W = \{g \in G^h \colon g.W = W\}$.
(b) $S_W = G^h$ *if $\mathfrak{g}$ is simple and $Z(G) = \{e\}$.*

***Proof*** (a) First we apply the Localization Theorem 5.22 to a unitary representation with discrete kernel and $C_U = \{0\}$; for instance the regular representation on $L^2(G)$. Then $\mathsf{H}^{\max}(W) = \mathtt{V}$ by the Localization Theorem, and since $S_{\mathtt{V}} = G_{\mathtt{V}} \subseteq G^h$ by Theorem 5.10, we obtain $S_W \subseteq S_{\mathtt{V}} \subseteq G^h$ from Lemma 5.2(c).

As $W \subseteq W_M^+(h)$ is a connected component and $G^h$ preserves $W_M^+(h)$, it follows that $S_W \subseteq G^h$ is the stabilizer subgroup $(G^h)_W$ of $W$.
(b) follows from $G_W = G^h$ in this case ([MNÓ24, Prop. 7.3]). □

The argument in the preceding proof is somewhat unnatural because it uses rather deep information on unitary representations to derive the geometric fact that $S_W$ is a group. It would be nice to have a direct geometric argument for this fact.

### 5.4 Left invariant nets on causal Lie groups

A particularly simple situation arises for $M = G$, with $G$ acting by left multiplication.

Let us start with an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ with discrete kernel, so that

$$S_{\mathtt{V}} = G_{\mathtt{V}} \exp(C_+ + C_-) \quad \text{for} \quad C_\pm = \pm C_U \cap \mathfrak{g}_{\pm 1}(h)$$

follows from Theorem 5.10. If this semigroup has interior points, it immediately leads to left invariant nets on $G$. We have the following theorem:

**Theorem 5.25** *Suppose that $(U, \mathcal{H})$ is an antiunitary representation of $G_{\tau_h}$ with discrete kernel and that $C_\pm^\circ \neq \emptyset$.*

(a) *For $\mathtt{E} := \mathtt{V}$, the net $\mathsf{H}_{\mathtt{E}}^G$ satisfies* (Iso), (Cov) *and* (BW) *with respect to $W = S_{\mathtt{V},e}^\circ$.*
(b) *If, in addition, $C_U^\circ \neq \emptyset$, then also* (RS) *is satisfied. In particular $\mathsf{H}_G^{\max}$ has this property.*

***Proof*** That $S_{\mathtt{V}}$ has interior points follows from $C_\pm^\circ \neq \emptyset$. We consider the open connected subsemigroup $W := S_{\mathtt{V},e}^\circ = G_e^h \exp(C_+^\circ + C_-^\circ)$. Then

$$S_W = \{g \in G \colon g.W \subseteq W\} = \{g \in G \colon g.S_{\mathtt{V},e} \subseteq S_{\mathtt{V},e}\} = S_{\mathtt{V},e}$$

follows from the fact that $W$ is dense in the identity component $S_{\mathtt{V},e}$ of $S_{\mathtt{V}}$. Therefore the equality $S_W = S_{\mathtt{V},e}$ implies that the corresponding maximal net $\mathsf{H}_G^{\max}$ satisfies the Bisognano–Wichmann condition in the form

$$\mathsf{H}_G^{\max}(W) = \mathsf{H}_G^{\max}(S_{\mathtt{V},e}^\circ) = \mathtt{V}.$$

In addition, we obtain for the real subspace $\mathtt{E} := \mathtt{V} \subseteq \mathcal{H}_{\mathrm{KMS}}^{-\infty}$ a left invariant isotone net $\mathsf{H}_{\mathtt{E}}^G$ on open subsets of $G$. As $W \subseteq S_{\mathtt{V},e}$ is dense, this net satisfies

$$\mathsf{H}_{\mathtt{E}}^G(W) = \overline{U(S_{\mathtt{V},e}^\circ)\mathtt{V}} = \mathtt{V},$$

which is the (BW) condition. If, in addition, $C_U^\circ \neq \emptyset$, then Theorem 5.46 in Appendix 5.8.2 further implies that $\mathsf{H}_{\mathsf{E}}^G$ has the Reeh–Schlieder property (RS). The inclusion

$$\mathsf{H}_{\mathsf{E}}^G(\mathcal{O}) \subseteq \mathsf{H}_G^{\max}(\mathcal{O}),$$

implies that the net $\mathsf{H}_G^{\max}$ also has the Reeh–Schlieder property. □

## 5.5 Causal symmetric spaces

Regularity of antiunitary representations has interesting consequences for causal symmetric spaces, which are by far not completely explored. He we state some for non-compactly and compactly causal symmetric spaces.

The following theorem, concerning non-compactly causal symmetric spaces, is a consequence of the Localization Theorem 5.22 for reductive groups.

**Theorem 5.26** (Maximal nets on ncc spaces) *If $M = G/H$ is a semisimple non-compactly causal symmetric space and $(U, \mathcal{H})$ an antiunitary representation of $G$, then the net $\mathsf{H}_M^{\max}$ satisfies* (Iso), (Cov), (RS) *and* (BW) *for $W = W_M^+(h)_{eH}$.*

***Proof*** In this case, $S_W = G_W = G_e^h H^h \subseteq G^h$ is a group by Theorem 5.24. Since $\tau = \tau_h\theta$ coincides on $K$ with $\tau_h$, we further have $H^h \subseteq K^{h,\tau} \subseteq K^{\tau_h}$, so that $S_W \subseteq G^{h,\tau_h} \subseteq G_{\mathsf{V}}$. Therefore Lemma 5.2 shows that $\mathsf{H}_M^{\max}$ satisfies (BW), and the condition (RS) follows from Theorem 5.22. □

For compactly causal spaces, we presently only have the following weaker result:

**Theorem 5.27** (Maximal nets on cc spaces) *Let $M = G/H$ be an irreducible modular compactly causal symmetric space, $C_{\mathfrak{g}} \subseteq \mathfrak{g}$ an invariant closed convex cone with $C = C_{\mathfrak{g}} \cap \mathfrak{q}$, and $(U, \mathcal{H})$ an antiunitary representation of $G$ with discrete kernel. Then the net $\mathsf{H}_M^{\max}$ satisfies* (Iso), (Cov) *and* (BW) *if and only if*

(a) *the positive spectrum condition $C_+ = C_{\mathfrak{g}} \cap \mathfrak{g}_1 \subseteq C_U$ is satisfied, and*
(b) *$U(G_W)$ commutes with $J$, i.e., $\tau_h(g)g^{-1} \in \ker U$ for $g \in H^h$.*

***Proof*** From (104) in Section 3.5 it follows that the semigroup $S_W$ can be written as

$$S_W = G_e^h H^h \exp(C_+ + C_-) \quad \text{for} \quad C_\pm = \pm C_{\mathfrak{g}} \cap \mathfrak{g}_{\pm 1}. \tag{157}$$

Therefore $S_W \subseteq S_{\mathsf{V}}$ is equivalent to the two conditions

$$C_\pm \subseteq \pm C_U \quad \text{and} \quad U(g)J = JU(g) \quad \text{for} \quad g \in H^h.$$

The first condition implies that $C_U \neq \{0\}$. As $\ker U$ is discrete and $\mathfrak{g}$ is simple or a sum of two simple ideals ([MNÓ23, Rem. 4.24]), $C_U$ is a pointed generating invariant cone with $C_{U,\pm} = C_\pm$. This shows that the condition $S_W \subseteq S_{\mathtt{V}}$ (which is equivalent to (BW) for $\mathsf{H}_M^{\max}$ by Lemma 5.2) implies (a) and (b).

To see the converse, suppose that (a) and (b) are satisfied. Then $C_+ \subseteq C_U$ implies that $C_{U,\pm} = C_\pm = (C_{\mathfrak{g}})_\pm$ ([NÓ23a, Prop. 2.7(d)]). Then $S_W \subseteq S_{\mathtt{V}}$ follows from (157). This proves the theorem because the (BW) property of $\mathsf{H}_M^{\max}$ is equivalent to $S_W \subseteq S_{\mathtt{V}}$ by Lemma 5.2. □

**Problem 5.28** Show that, in the context of Theorem 5.27, (a) and (b) also imply the Reeh–Schlieder property (RS) of $\mathsf{H}_M^{\max}$.

*Remark 5.29* Here are some remarks that may be useful to solve this problem.

- Using Proposition 5.17 and that $\mathsf{H}_M^{\max}(\mathcal{O}) = \mathtt{V}_A$ for $A := \{g \in G \colon \mathcal{O} \subseteq g.W\}$, it suffices to solve the problem for irreducible representations.
- From (157) we know that

$$S_W = G_W \exp(C_+ + C_-) \quad \text{with} \quad G_W = G_e^h H^h \quad \text{and} \quad G_{W,e} = G_e^h,$$

so that $S_{W,e} = G_e^h \exp(C_+ + C_-)$, $S_{W,e}^\circ = G_e^h \exp(C_+^\circ + C_-^\circ)$, and

$$W = S_{W,e}^\circ.eH \cong G_e^h \times_{H_e^h} \exp(C_+^{\circ,-\tau} + C_-^{\circ,-\tau}).$$

For $W^G = q_M^{-1}(W)$ we then have

$$W^G = S_{W,e}^\circ H \supseteq S_{W,e}^\circ,$$

which is strictly larger than $S_{W,e}^\circ$.
From Remark 5.9 we know that $\mathsf{H}_M^{\max} = (q_M)_* \mathsf{H}_G^{\max}$, with respect to the wedge regions $W \subseteq M$ and $W^G \subseteq G$. It follows in particular that

$$\mathsf{H}_G^{\max}(S_{W,e}^\circ) \subseteq \mathsf{H}_G^{\max}(W^G) = \mathtt{V}.$$

Next we argue that we actually have equality.
- For $g \in G$, the relation $S_{W,e}^\circ \subseteq gW^G$ is equivalent to

$$q_M(S_{W,e}^\circ) = q_M(W^G) = W \subseteq g.W,$$

i.e., to $g \in S_W^{-1}$. Now $S_W \subseteq S_{\mathtt{V}}$ implies that any such $g$ satisfies $U(g)\mathtt{V} \supseteq \mathtt{V}$, so that

$$\mathsf{H}_G^{\max}(S_{W,e}^\circ) = \mathtt{V} = \mathsf{H}_G^{\max}(W^G). \tag{158}$$

- The net $\mathsf{H}_{G,S_{W,e}^\circ}^{\max}$ on $G$, constructed from the wedge region $S_{W,e}^\circ$, which is smaller than $W^G$, also satisfies

$$\mathsf{H}_{G,S_{W,e}^\circ}^{\max}(S_{W,e}^\circ) = \mathtt{V}$$

because $S_{W,e} = G_e^h \exp(C_+ + C_-) \subseteq S_{\mathtt{V}}$. We thus obtain the two maximal nets $\mathsf{H}^{\max}_{G,S^\circ_{W,e}}$ and $\mathsf{H}^{\max}_{G,W^G}$ on $G$, corresponding to the wedge regions $S^\circ_{W,e}$ and $W^G$. They satisfy the relation

$$\mathsf{H}^{\max}_{G,S^\circ_{W,e}}(\mathcal{O}) \supseteq \mathsf{H}^{\max}_{G,W^G}(\mathcal{O}). \tag{159}$$

Note that this also follows from Lemma 5.8 and the maximality of the net $\mathsf{H}^{\max}_{G,S^\circ_{W,e}}$ among the nets mapping $S^\circ_{W,e}$ to $\mathtt{V}$ (see (158)).

- If, in addition, $\mathsf{H}^{\max}_{G,S^\circ_{W,e}}(W^G) \supseteq \mathsf{H}^{\max}_{G,S^\circ_{W,e}}(S^\circ_{W,e}) = \mathtt{V}$ is separating, then the Equality Lemma 1.9 and the $\exp(\mathbb{R}h)$-invariance of $W^G$ imply that both are equal. Now the maximality of $\mathsf{H}^{\max}_{G,W^G}$ and Lemma 5.8 imply that the nets $\mathsf{H}^{\max}_{G,S^\circ_{W,e}}$ and $\mathsf{H}^{\max}_{G,W^G}$ coincide.

The situation looks much better if there are nets of the form $\mathsf{H}^M_{\mathtt{E}}$, where $\mathtt{E}$ is invariant under $H$:

**Theorem 5.30** *Suppose that, in addition to the assumptions of* Theorem 5.27*, there exists a real linear subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}$ which is $U^{-\infty}(H)$-invariant and satisfies $\mathsf{H}^M_{\mathtt{E}}(W) = \mathtt{V}$. Then $\mathsf{H}^{\max}_M$ also satisfies* (RS)*.*

***Proof*** We have already seen in the proof of Theorem 5.27 that $C_U^\circ \neq \emptyset$, so that Theorem 5.46 in Appendix 5.8.2 implies that the net $\mathsf{H}^G_{\mathtt{E}}$ on $G$ satisfies (RS), and so does the net $\mathsf{H}^M_{\mathtt{E}} = (q_M)_* \mathsf{H}^G_{\mathtt{E}}$ (Remark 4.23(b)). By assumption, $\mathsf{H}^M_{\mathtt{E}}$ satisfties (BW), hence is contained in $\mathsf{H}^{\max}_M$, and therefore the latter net also satisfies (RS). □

*Remark 5.31* The assumption of $\mathtt{E}$ to be $U^{-\infty}(H)$-invariant may seem to be quite strong, but it is close to being necessary. In fact, what we need to argue as in the proof of the preceding theorem, is that $\mathsf{H}^G_{\mathtt{E}}(W^G) = \mathsf{H}^M_{\mathtt{E}}(W) = \mathtt{V}$, which is equivalent to $U^{-\infty}(W^G)\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}$ by Proposition 4.13. As $eH \in \overline{W}$, we have $H \subseteq \overline{W_G}$, so that the weak-$*$ closedness of $\mathcal{H}^{-\infty}_{\mathrm{KMS}}$ (Theorem 1.43) implies that $U^{-\infty}(H)\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}$.

## 5.6 Causal flag manifolds

After we have seen how the methods developed in this chapter apply to nets on Lie groups (Section 5.4) and on causal symmetric spaces (Section 5.5), we now turn to causal flag manifolds (Section 3.4). The results in this section can be found in [MN26].

We briefly recall the key features of irreducible causal flag manifolds $M = G/Q_h$. We identitfy the tangent space $T_{eQ_h}(M)$ in the base point with $\mathfrak{g}_1(h)$, and recall that the causal structure on $M$ is determined by

$$C_{eQ_h} = C_+ = C_{\mathfrak{g}} \cap \mathfrak{g}_1(h),$$

where $C_{\mathfrak{g}}$ is an invariant convex cone in $\mathfrak{g}$.

The fundamental group $\pi_1(M)$ is isomorphic to $\mathbb{Z}$ ([Wig98, Thm. 1.1] and [MN26, §3.1]), so that there exists for every $k \in \mathbb{N} \cup \{\infty\}$ a $k$-fold covering $M_k$, where $M_\infty$ is simply connected.

- $M_\infty$ is a simple spacetime manifold in the sense of Mack/de Riese [MdR07]. It carries a global causal order (no closed causal curves).
- $M_k, k < \infty$, has closed causal curves, hence no global causal order. To see how such curves arise, we start with the Euler element $h \in \mathfrak{g}$ and pick $e \in \mathfrak{g}_1(h)$ (a unit of the euclidean Jordan algebra), so that $h, e$ and $\theta(e)$ span a Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ (see (93)). Then the element $z_{\mathfrak{k}} := \frac{1}{2}(e + \theta(e)) \in C_{\mathfrak{g}}^\circ$ corresponds to the generator of $\mathfrak{so}_2(\mathbb{R}) \subseteq \mathfrak{sl}_2(\mathbb{R})$, and generates a periodic flow on every $M_k, k < \infty$, whose flow lines are causal curves.
- The open embedding $\iota_M \colon \mathrm{V} = \mathfrak{g}_1 \hookrightarrow M$ of the euclidean Jordan algebra V (see the appendix to Section 3.4 for more on euclidean Jordan algebras) lifts to an open embedding $\iota_{M_k} \colon \mathrm{V} \hookrightarrow M_k$.
- In $M_k$ the canonical wedge region is
$$W_{M_k} := \iota_{M_k}(\mathrm{V}_+) \subseteq M_k.$$
It is a connected component of the positivity domain $W^+_{M_k}(h)$ of the Euler vector field $X^h_{M_k}$ on $M_k$. In $M_k$ the positivity domain $W^+_{M_k}(h)$ has $k$ connected components which are permuted by a transitive action of the group $\mathbb{Z}_k$ of deck transformations of the covering $M_k \to M$.

**Examples 5.32** (a) For Minkowski space $\mathrm{V} = \mathbb{R}^{1,d-1}$, the conformal completion
$$M \cong (\mathbb{S}^1 \times \mathbb{S}^{d-1})/\{\pm \mathbf{1}\} \subseteq \mathbb{P}(\mathbb{R}^{2,d})$$
is the isotropic quadric in $\mathbb{P}(\mathbb{R}^{2,d})$ (see (96) in Example 3.33) on which $G = \mathrm{SO}_{2,d}(\mathbb{R})_e$ acts transitively. In this case,
$$M_\infty \cong \mathbb{R} \times \mathbb{S}^{d-1}$$
is the so-called *Einstein universe*
(b) For the euclidean Jordan algebra $\mathrm{V} = \mathrm{Herm}_r(\mathbb{C})$, we have $M \cong \mathrm{U}_r(\mathbb{C})$, on which $G = \mathrm{SU}_{r,r}(\mathbb{C})$ acts by fractional linear transformations
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .z = (az+b)(cz+d)^{-1}.$$
Here
$$\widetilde{\mathrm{U}}_r(\mathbb{C}) \cong \mathbb{R} \times \mathrm{SU}_r(\mathbb{C}).$$
(c) For the euclidean Jordan algebra $\mathrm{V} = \mathrm{Sym}_r(\mathbb{R})$, the conformal compactification is the space $M$ of Lagrangian subspaces in the symplectic vec-

tor space $(\mathbb{R}^{2r}, \omega)$, on which $G = \mathrm{Sp}_{2r}(\mathbb{R})$ acts naturally. Here $M_\infty \cong \mathbb{R} \times (\mathrm{SU}_r(\mathbb{C})/\mathrm{SO}_r(\mathbb{R}))$.

To formulate existence criteria for nets on the spaces $M_k$, we observe that the simply connected covering group $\widetilde{G}$ acts transitively on every $M_k$. The centralizer $\widetilde{G}^h$ of $h$ in this group satisfies

$$\pi_0(\widetilde{G}^h) \cong \pi_1(M) \cong \mathbb{Z}$$

([MN26, §3.1]). We pick $g_h \in \widetilde{G}^h$ so that its connected component generates $\pi_0(\widetilde{G}^h)$ and

$$\tau_h(g_h) = g_h^{-1}. \tag{160}$$

This element can be chosen to be central in an $\widetilde{\mathrm{SL}}_2(\mathbb{R})$-subgroup with $g_h^2 \in Z(\widetilde{G})$ (see (4.6) in the proof of [MNÓ26a, Thm. 4.1] and [MN26, §3.1]).

**Theorem 5.33** (Existence of nets) *For an antiunitary representation $(U, \mathcal{H})$ of $\widetilde{G}_{\tau_h}$ with discrete kernel, a net $\mathsf{H}$ on open subsets of $M_k$ satisfying* (Iso), (Cov), (RS) *and* (BW) *exists if and only if*

- *$U$ satisfies the positive energy condition $C_+ \subseteq C_U$, and,*
- *if $k < \infty$, then $g_h^{2k} \in \ker U$.*

***Proof*** (Sketch; see [MN26]) In view of Lemmas 5.2 and 5.8, the existence of a net $\mathsf{H}$ satisfying (Iso), (Cov) and (BW) is equivalent to $S_{W_{M_k}} \subseteq S_{\mathsf{V}}$. These semigroups are of the form

$$S_{W_{M_k}} = \widetilde{G}_{W_{M_k}} \exp(C_+ + C_-) \quad \text{and} \quad S_{\mathsf{V}} = \widetilde{G}_{\mathsf{V}} \exp(C_+^U + C_-^U)$$

for $C_\pm = \pm C_{\mathfrak{g}} \cap \mathfrak{g}_{\pm 1}(h)$ and $C_\pm^U = \pm C_U \cap \mathfrak{g}_{\pm 1}(h)$ (cf. Theorem 5.10). This reduces the inclusion $S_{W_{M_k}} \subseteq S_{\mathsf{V}}$ to the inclusion $\widetilde{G}_{W_{M_k}} \subseteq \widetilde{G}_{\mathsf{V}}$ and the positive energy condition $C_+ \subseteq C_U$. As $\widetilde{G}_{W_{M_k}} \subseteq \widetilde{G}_{W_M} = \widetilde{G}^h$ (Proposition 3.26) commutes with the Euler element $h$, the inclusion $\widetilde{G}_{W_{M_k}} \subseteq \widetilde{G}_{\mathsf{V}}$ is equivalent to

$$\{g\tau_h(g)^{-1} \colon g \in \widetilde{G}_{W_{M_k}}\} = g_h^{2k\mathbb{Z}} \subseteq \ker U.$$

So it only remains to verify the Reeh–Schlieder condition. We refer to [MN26, Thm. 6.16] (which builds on [NÓ21]) for details. □

The following theorem extends results by Brunetti, Guido and Longo [BGL93] for the Jordan algebra $\mathrm{V} = \mathbb{R}^{1,d-1}$ (Minkowski space) and the group $G = \mathrm{SO}_{2,d}(\mathbb{R})_e$ to general causal flag manifolds.

**Theorem 5.34** (Existence and uniqueness of additive nets) *For every antiunitary representation $(U, \mathcal{H})$ of $\widetilde{G}$ satisfying the positive energy condition $C_+ \subseteq C_U$, there exists a unique additive net $\mathsf{H}$ on open subsets of $M_\infty$, satisfying* (Iso), (Cov), (RS) *and* (BW). *On $M_k$ such nets exist for $U$ if and only if, in addition, $g_h^{2k} \subseteq \ker U$.*

***Proof*** For details we refer to [MN26, §6.4]. Here we only sketch the argument.
**Uniqueness:** On every $M_k$, the wedge regions form a basis for the topology. Every additive covariant net $\mathsf{H}$ satisfying (BW) thus satisfies

$$\mathsf{H}(\mathcal{O}) = \mathsf{H}\Big(\bigcup_{g.W_{M_k}\subseteq\mathcal{O}} g.W_{M_k}\Big) = \overline{\sum_{g.W_{M_k}\subseteq\mathcal{O}} U(g)\mathsf{H}(W_{M_k})} = \overline{\sum_{g.W_{M_k}\subseteq\mathcal{O}} U(g)\mathtt{V}},$$

so that $\mathsf{H}$ is determined by the representation $U$ via $\mathsf{H}(W_{M_k}) = \mathtt{V} = \mathtt{V}(h,U)$.
**Existence:** The argument for existence builds on nets for irreducible representations constructed in [NÓ21] (see also [MN26]), and direct integral techniques. The main point is to find a finite-dimensional subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\rm KMS}$ invariant under the parabolic subgroup $Q_h$ and that $\mathsf{H}^{M_k}_{\mathtt{E}}(W_{M_k}) = \mathtt{V}$. □

### 5.6.1 Locality

Locality conditions concern open $G$-invariant subsets $\mathcal{D}_{\rm loc} \subseteq M \times M$. Here are some relevant facts:

- $M \times M$ contains a unique open $G$-orbit $\mathcal{D}^*$.
- $M_\infty \times M_\infty$ contains infinitely many open $\widetilde{G}$-orbits $(\mathcal{D}^*_n)_{n\in\mathbb{Z}}$, permuted transitively by the group $\pi_1(M) \cong \mathbb{Z}$, acting by deck transformations.
- $M_k \times M_k$ contains $k$ open $\widetilde{G}$-orbits $\mathcal{D}^*_n$, $n \in \mathbb{Z}/k\mathbb{Z}$, permuted by deck transformations in $\mathrm{Deck}(M_k) \cong \pi_1(M)/\pi_1(M_k) \cong \mathbb{Z}/k\mathbb{Z}$.

Let $g_h \in \widetilde{G}^h$ be as above and pick $z_{\mathfrak{k}} \in \mathfrak{z}(\mathfrak{k})$ such that $\theta := \exp(\pi \,\mathrm{ad}\, z_{\mathfrak{k}}) \in \mathrm{Aut}(\mathfrak{g})$ is a Cartan involution.

**Theorem 5.35** (Locality properties of the nets) *For the unique additive net associated to the positive energy representation $(U,\mathcal{H})$ of $\widetilde{G}_{\tau_h}$ on $M_k$, $k \in \mathbb{N}\cup\{\infty\}$ and $n \in \mathbb{Z}_k$, the following are equivalent:*

(a) *The $n$-locality condition: $\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{D}^*_n \Rightarrow \mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)'$*
(b) $\mathsf{H}(g_h^n.W_{M_\infty}) \subseteq \mathsf{H}(W'_{M_\infty})'$ *for the dual wedge $W'_{M_\infty} := \theta.W_{M_\infty}$.*
(c) $\exp(2\pi z_{\mathfrak{k}})g_h^{2n} \in \ker U$. [23]

### 5.6.2 The massless spin 0 representation on Minkowski space

We consider Minkowski space $\mathrm{V} = \mathbb{R}^{1,d-1}$ and its conformal compactification $M$. On $\widetilde{G} = \widetilde{\mathrm{SO}}_{2,d}(\mathbb{R})_e$ the "minimal" positive energy representation $(U,\mathcal{H})$ is the extension of the Poincaré representation corresponding to *massless spin* $0$*-particles.* In the momentum picture, the Hilbert space is $\mathcal{H} = L^2(\partial \mathrm{V}^\star_+)$,

[23] Note that the discrete normal subgroup $\ker U \trianglelefteq G$ is central because $G$ is connected.

endowed with the natural action of the Poincaré group, where translations act by multiplications.

It depends on the dimension $d$, to which quotient group of $\widetilde{G}$ the representation $U$ descends, and on which covering of $M$ the net can be implemented (Theorem 5.34). We have the following properties (cf. [BGL93, BGL02], and also [MNÓ26b], [MN26], [LM23]):

- $d \in 2+4\mathbb{Z}$: $U$ is defined on the adjoint group $\mathrm{SO}_{2,d}(\mathbb{R})_e/\{\pm\mathbf{1}\}$ and the net lives on $M$.
- $d \in 4\mathbb{Z}$: $U$ is defined on $\mathrm{SO}_{2,d}(\mathbb{R})_e$ with $U(-1) = -\mathbf{1}$ and the net lives on $M$.
- $d$ odd: $U$ is defined on a 2-fold covering of $\mathrm{SO}_{2,d}(\mathbb{R})_e$ and the net lives on on $M_2 \cong \mathbb{S}^1 \times \mathbb{S}^{d-1}$.

*Remark 5.36* The $n$-locality condition on $M_2$ (for $n = 0, 1$) is $(n+1)d \in 2\mathbb{Z}$ (cf. Theorem 5.35(a) and also [MNÓ26b, §4.1, App. A.3] for more details).

- For $d$ even, the net therefore is 0- and 1-local, which corresponds to spacelike and timelike locality on Minkowski space.
- For $d$ odd, it is only 1-local, which corresponds to spacelike locality on Minkowski space.

These locality conditions relate to support properties of the fundamental solutions of the Klein–Gordon equation (Huygens' Principle). We refer to [MN26] and [MNÓ26b] for details.

## 5.7 Localizability for the Poincaré group

The following result is well-known in AQFT ([BGL02, Thm. 4.7]; see also [Mu01, Mu03]). Below we derive it naturally in the context of our theory for general Lie groups (cf. [MN24, Thm. 4.25]). It connects regularity, resp., localizability with the positive energy condition.

**Definition 5.37** An open subset $\mathcal{O}$ in Minkowski space $\mathbb{R}^{1,d}$ is called *spacelike* if $x_0^2 < \mathbf{x}^2$ holds for all $(x_0, \mathbf{x}) \in \mathcal{O}$ (cf. Section 1.6). A spacelike open subset is called a *spacelike (convex) cone* if, in addition, it is a (convex) cone.

**Theorem 5.38** (Localization for the Poincaré group) *Let $(U, \mathcal{H})$ be an (anti-)unitary representation of the proper Poincaré group $P(d)_+ = \mathbb{R}^{1,d} \rtimes \mathrm{SO}_{1,d}(\mathbb{R})$ (identified with $P(d)_{\tau_h}$) and consider the standard boost $h$ and the corresponding Rindler wedge $W_R \subseteq \mathbb{R}^{1,d}$. Then $(U, \mathcal{H})$ is $(h, W_R)$-localizable in the set of all spacelike open cones if and only if it is a positive energy representation, i.e.,*

$$C_U \supseteq \overline{V_+} := \{(x_0, \mathbf{x}) \colon x_0 \geq \sqrt{\mathbf{x}^2}\}. \tag{161}$$

*These representations are $h$-regular.*

Note that $\mathrm{Ad}(P(d)^{\uparrow}_{+})$ acts transitively on the set $\mathcal{E}(\mathfrak{p})$ of Euler elements (Example 2.10), so that the choice of a specific Euler element $h$ is inessential.

***Proof*** First we show that the positive energy condition is necessary for localizability in spacelike cones. In fact, the localizability condition implies in particular that $\mathsf{H}(W_R)$ is cyclic, so that Lemma 5.2 implies $S_{W_R} \subseteq S_{\mathtt{V}}$. We recall from Lemma 3.15 that

$$S_{W_R} = \{g \in P(d)^{\uparrow}_{+} : gW_R \subseteq W_R\} = \overline{W_R} \rtimes \mathrm{SO}_{1,d}(\mathbb{R})^{\uparrow}_{W_R},$$

where

$$\mathrm{SO}_{1,d}(\mathbb{R})^{\uparrow}_{W_R} = \mathrm{SO}_{1,1}(\mathbb{R})^{\uparrow} \times \mathrm{SO}_{d-2}(\mathbb{R})$$

is connected, hence coincides with $\mathrm{SO}_{1,d}(\mathbb{R})^h_e$. It follows that

$$S_{W_R} = G^h_e \exp(\mathbb{R}_{\geq 0}(\mathbf{e}_1 + \mathbf{e}_0)) \exp(\mathbb{R}_{\geq 0}(\mathbf{e}_1 - \mathbf{e}_0)).$$

Theorem 5.10, combined with $S_{W_R} \subseteq S_{\mathtt{V}}$ now implies that

$$\mathbf{e}_1 + \mathbf{e}_0 \in \mathbf{L}(S_{\mathtt{V}}) \cap \mathfrak{g}_1(h) \subseteq C_U,$$

and thus $\overline{V_+} \subseteq C_U$ by Lorentz invariance of $C_U$. Therefore $(U, \mathcal{H})$ is a positive energy representation.

Assume, conversely, that $(U, \mathcal{H})$ is a positive energy representation. For the standard boost, we have $h \in \mathfrak{l} \cong \mathfrak{so}_{1,d}(\mathbb{R})$, and the restriction $(U|_{\mathrm{SO}_{1,d}(\mathbb{R})}, \mathcal{H})$ is $(h, W)$-localizable in the family of all non-empty open subsets of $\mathrm{dS}^d$, where $W = W_R \cap \mathrm{dS}^d$ is the canonical wedge region (Theorem 5.22). In view of Theorem 5.10, the positive energy condition $\overline{\mathrm{V}_+} \subseteq C_U$ and

$$\mathbb{R}_{\geq 0}(\mathbf{e}_1 \pm \mathbf{e}_0) \subseteq C_{U,\pm} := \pm C_U \cap \mathfrak{g}_{\pm 1}(h),$$

imply that $S_{W_R}$ is contained in

$$\exp(C_{U,+}) G^h_e \exp(C_{U,-}) \subseteq \exp(C_{U,+}) G_{\mathtt{V}} \exp(C_{U,-}) = S_{\mathtt{V}}$$

(Theorem 5.10). By Lemma 5.2(c), the net $\mathsf{H}^{\max}$ satisfies $\mathsf{H}^{\max}(W_R) = \mathtt{V}$.

Now suppose that $\mathcal{C} \subseteq W_R$ is a spacelike cone, so that $\mathcal{C} = \mathbb{R}_+(\mathcal{C} \cap \mathrm{dS}^d)$, where $\mathcal{C} \cap \mathrm{dS}^d$ is an open subset of the wedge region $W = W_R \cap \mathrm{dS}^d$ in de Sitter space. For $g^{-1} = (v, \ell) \in P(d)^{\uparrow}_{+}$, the condition $\mathcal{C} \subseteq g.W_R$ is equivalent to

$$g^{-1}.\mathcal{C} = v + \ell.\mathcal{C} \subseteq W_R,$$

which in turn means that $v \in \overline{W_R}$ and $\ell.\mathcal{C} \subseteq W_R$. Below we shall write $U_L := U|_{\mathrm{SO}_{1,d}(\mathbb{R})}$ for the restriction of $U$ to the Lorentz group. Then

$$U(g)\mathtt{V} = U(\ell)^{-1} U(v)^{-1} \mathtt{V} \supseteq U(\ell)^{-1} \mathtt{V} \tag{162}$$

follows from $\overline{W_R} \subseteq S_{\mathtt{V}}$, and therefore

$$\mathsf{H}^{\max}(\mathcal{C}) = \bigcap_{\mathcal{C}\subseteq g.W_R} U(g)\mathtt{V} \overset{(162)}{\supseteq} \bigcap_{\mathcal{C}\subseteq \ell^{-1}.W_R} U(\ell)^{-1}\mathtt{V}$$
$$= \bigcap_{\mathcal{C}\cap \mathrm{dS}^d\subseteq \ell^{-1}.(W_R\cap \mathrm{dS}^d)} U_L(\ell)^{-1}\mathtt{V} = \mathsf{H}^{\max}_{U_L}(\mathcal{C}\cap \mathrm{dS}^d).$$

We conclude that, on spacelike cones with vertex in 0, the net $\mathsf{H}^{\max}$ can be expressed in terms of the net $\mathsf{H}^{\max}_{U|_{\mathcal{L}}}$ on de Sitter space. As the latter net has the Reeh–Schlieder property by Theorem 5.22, and all spacelike cones can be translated to one with vertex 0, localization in spacelike cones follows.

Finally we show that $(U,\mathcal{H})$ is regular. For $v\in W_R$ and a pointed closed spacelike cone $C$ with $v+C\subseteq W_R$, there exists an $e$-neighborhood $N\subseteq G$ with $v+C\subseteq g.W$ for all $g\in N$. This implies that $\mathsf{H}^{\max}(v+C^\circ)\subseteq \mathtt{V}_N$, so that $(U,\mathcal{H})$ is regular. □

**Definition 5.39** (a) (Causal complements) The quadratic form $xy = x_0y_0 - \mathbf{xy}$ on Minkowski space $\mathrm{V}=\mathbb{R}^{1,d}$ allows us to define the *causal complement (or the spacelike complement)* of an open subset $\mathcal{O}\subset \mathrm{V}$ by

$$\mathcal{O}' = \{x\in \mathrm{V} : (\forall y\in\mathcal{O})\,(y-x)^2<0\}^\circ. \tag{163}$$

This is the interior of the set of all points that cannot be reached from $\mathcal{O}$ with a timelike or lightlike curve. We set $\mathcal{O}'' := (\mathcal{O}')'$ is called the (open) causal closure of $\mathcal{O}$.
(b) (Double cone) A *double cone* is, up to Poincaré covariance, the *causal closure*

$$\mathbb{B}_r'' = (r\mathbf{e}_0 - \mathrm{V}_+)\cap(-r\mathbf{e}_0+\mathrm{V}_+)$$

of an open ball of the time zero hyper-plane $\mathbb{B}_r = \{(0,\mathbf{x})\in\mathbb{R}^{1,d} : \mathbf{x}^2<r^2\}$.

*Remark 5.40* Infinite helicity representations $(U,\mathcal{H})$ of $P(d)_+$ in $\mathbb{R}^{1,d}$ are **not** localizable in double cones (Definition 5.39). Let $\mathtt{V}=\mathtt{V}(h,U)$ for $h$ as in Example 1.28. In [LMR16, Thm. 6.1] it is shown that, if $\mathcal{O}\subseteq\mathbb{R}^{1,d}$ is a double cone, then

$$\mathsf{H}^{\max}(\mathcal{O}) = \bigcap_{\mathcal{O}\subseteq g.W_R} U(g)\mathtt{V} = \{0\}. \tag{164}$$

The argument for (164) can be sketched as follows. Infinite spin representations are massless representations, i.e., the support of the spectral measure of the spacetime translation group is

$$\partial\mathrm{V}_+ = \{(x_0,\mathbf{x})\in\mathbb{R}^{1,d} : x_0^2-\mathbf{x}^2=0, x_0\geq 0\}.$$

Covariant nets of standard subspaces on double cones in massless representations are also dilation covariant in the sense that the representation of $P(d)_+$ extends to the Poincaré and dilation group $\mathbb{R}^{1,d}\rtimes(\mathbb{R}^+\times\mathcal{L})$, and that the net is also covariant under this larger group, cf. [LMR16, Prop. 5.4]. When $d=3$, this follows from the fact that, due to Huygens' Principle, one can associate a standard subspace to the forward light cone by

$$\mathsf{H}(\mathrm{V}_+) = \overline{\sum_{\mathcal{O}\subset \mathrm{V}_+} \mathsf{H}(\mathcal{O})}$$

(sum over all double cones in $\mathrm{V}_+$), and the modular group of $\mathsf{H}(\mathrm{V}_+)$ is geometrically implemented by the dilation group. As massless infinite helicity representations are not dilation covariant, it follows that they do not permit localization in double cones. Properties of the free wave equation permit to extend this argument to any space dimension $d \geq 2$, including even space dimensions, for which Huygens' Principle fails ([LMR16, Sect. 8.2]). However, in Theorem 5.38, we recover in our general setting the result contained in [BGL02, Thm. 4.7] that all positive energy representations of $P(d)_+$ are localizable in spacelike cones.

## 5.8 Appendices to Section 5

We conclude this chapter with two appendices providing interesting background on the general results concerning Lie groups. The first is the Borchers–Wiesbrock Theorem, characterizing so-called standard pairs in terms of antiunitary representations of the affine group of the line.

The second is a general argument that derives the Reeh–Schlieder property of nets of real subspaces for positive energy representations from the general fact that holomorphic functions on a complex domain are uniquely determined by the restriction of their boundary values to an open subset of the Shilov boundary.

### 5.8.1 Standard pairs

**Definition 5.41** *Positive standard pairs* $(U, \mathtt{V})$ consist of a standard subspace $\mathtt{V}$ of a complex Hilbert space $\mathcal{H}$ and a unitary one-parameter group $(U_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$ such that $U_t\mathtt{V} \subseteq \mathtt{V}$ for $t \geq 0$ and $U_t = e^{itH}$ with $H \geq 0$.

**Theorem 5.42** (Borchers–Wiesbrock Theorem) *Any standard pair defines an antiunitary positive energy representation of* $\mathrm{Aff}(\mathbb{R}) \cong \mathbb{R} \rtimes \mathbb{R}^\times$ *by*

$$U(b, e^s) := U_b \Delta_{\mathtt{V}}^{-is/2\pi} \quad \textit{and} \quad U(0, -1) := J_{\mathtt{V}}. \tag{165}$$

*Conversely, all these representations define positive standard pairs.*

***Proof*** This is the Borchers–Wiesbrock Theorem ([NÓ17, Thm. 3.18], see also [Bo92], [Wi93a]). □

We refer to [NOe22] for classification results for pairs $(h, x)$ in Lie algebras, where $h$ is an Euler element and $x$ is contained in a pointed generating invariant cone satisfying $[h, x] = x$.

**Proposition 5.43** *Consider a Lie group $G_\sigma = G \rtimes \{\mathbf{1}, \sigma\}$, where $\sigma \in \mathrm{Aut}(G)$ is an involution. Let $(U, \mathcal{H})$ be an antiunitary representation of $G_\sigma$. Suppose that $(\mathtt{V}, U^j)$, $j = 1, 2$, are positive standard pairs for which there exists a graded homomorphism $\gamma\colon \mathbb{R}^\times \to G$ and $x_1, x_2 \in \mathfrak{g}$ such that*

$$J_{\mathtt{V}} = U(\gamma(-1)), \quad \Delta_{\mathtt{V}}^{-it/2\pi} = U(\gamma(e^t)),$$

*and*

$$U^j(t) = U(\exp t x_j), \quad t \in \mathbb{R}, j = 1, 2.$$

*Then the unitary one-parameter groups $U^1$ and $U^2$ commute.*

***Proof*** The positive cone $C_U \subseteq \mathfrak{g}$ of the representation $U$ is a closed convex $\mathrm{Ad}(G)$-invariant cone. As we may w.l.o.g. assume that $U$ is injective, the cone $C_U$ is pointed.

Writing $\Delta_{\mathtt{V}}^{-it/2\pi} = U(\exp th)$ and $U_t^j = U(\exp t x_j)$ with $h, x_1, x_2 \in \mathfrak{g}$, the structure of the Lie algebra of $\mathrm{Aff}(\mathbb{R})$ implies by (165) that $[h, x_j] = x_j$ for $j = 1, 2$ and $x_1, x_2 \in C_U$. If

$$\mathfrak{g}_\lambda(h) = \ker(\mathrm{ad}\, h - \lambda \mathbf{1})$$

is the $\lambda$-eigenspace of $\mathrm{ad}\, h$ in $\mathfrak{g}$, then $[\mathfrak{g}_\lambda(h), \mathfrak{g}_\mu(h)] \subseteq \mathfrak{g}_{\lambda+\mu}(h)$, so that $\mathfrak{g}_+ := \sum_{\lambda>0} \mathfrak{g}_\lambda(h)$ is a nilpotent Lie subalgebra. Therefore $\mathfrak{n} := (C_U \cap \mathfrak{g}_+) - (C_U \cap \mathfrak{g}_+)$ is a nilpotent Lie algebra generated by the pointed invariant cone $C_U \cap \mathfrak{g}_+$. By [Ne00, Ex. VII.3.21], $\mathfrak{n}$ is abelian. Finally $x_j \in C_U \cap \mathfrak{g}_1(h) \subseteq \mathfrak{n}$ implies that $[x_1, x_2] = 0$. □

One may expect that one-parameter groups $U^1$ and $U^2$, for which $(\mathtt{V}, U^j)$ form a standard pair, commute. By Proposition 5.43 this is true if they both come from an antiunitary representation of a finite-dimensional Lie group. The following example shows that this is not true in general, not even if the two one-parameter groups are conjugate under the stabilizer group $\mathrm{U}(\mathcal{H})_{\mathtt{V}}$.

**Example 5.44** On $L^2(\mathbb{R})$ we consider the selfadjoint operators

$$(Qf)(x) = xf(x) \quad \text{and} \quad (Pf)(x) = if'(x),$$

satisfying the canonical commutation relations $[P, Q] = i\mathbf{1}$. For both operators, the Schwartz space $\mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$ is a core. Actually it is the space of smooth vectors for the representation of the 3-dimensional Heisenberg group generated by the corresponding unitary one-parameter groups

$$(e^{itQ}f)(x) = e^{itx}f(x) \quad \text{and} \quad (e^{itP}f)(x) = f(x-t).$$

Since $e^{ix^3}$ is a smooth function for which all derivatives grow at most polynomially, it defines a continuous multiplication operator on $\mathcal{S}(\mathbb{R})$ ([Tr67, Thm. 25.5]). Therefore the unitary operator $T := e^{iQ^3}$ maps $\mathcal{S}(\mathbb{R})$ continuously onto itself, and thus

$$\widetilde{P} := TPT^{-1} = e^{iQ^3} P e^{-iQ^3}$$

is a selfadjoint operator for which $\mathcal{S}(\mathbb{R})$ is a core. For $f \in \mathcal{S}(\mathbb{R})$, we obtain

$$(\widetilde{P} f)(x) = i e^{ix^3} \frac{d}{dx} e^{-ix^3} f(x) = i(-i3x^2 f(x) + f'(x)),$$

so that $\widetilde{P} = P + 3Q^2$ on $\mathcal{S}(\mathbb{R})$.

The two selfadjoint operators $Q$ and $e^P$ are the infinitesimal generators of the irreducible antiunitary representation of $\mathrm{Aff}(\mathbb{R}) = \mathbb{R} \rtimes \mathbb{R}^\times$, given by

$$U(b, e^t) = e^{ibe^P} e^{itQ} \quad \text{and} \quad (U(0,-1)f)(x) = \overline{f(-x)}.$$

Accordingly, the pair $(\Delta, J)$ with

$$\Delta = e^{-2\pi Q} \quad \text{and} \quad J = U(0,-1)$$

specifies a standard subspace $\mathtt{V}$ which combines with $U^1_t := e^{ite^P}$ to an irreducible standard pair $(\mathtt{V}, U^1)$. The unitary operator $T$ commutes with $\Delta$ and with $J$ because $JQJ = -Q$, so that $T(\mathtt{V}) = \mathtt{V}$. Therefore the unitary one-parameter group $U^2_t := e^{iQ^3} U^1_t e^{-iQ^3} = e^{ite^{\widetilde{P}}}$ also defines a standard pair $(\mathtt{V}, U^2)$. These two one-parameter groups do not commute because otherwise the selfadjoint operators $P$ and $P + 3Q^2$ would commute in the strong sense, hence in particular on their core $\mathcal{S}(\mathbb{R})$; contradiction.

### 5.8.2 The Reeh–Schlieder property

The following lemma from [NÓ21, Lemma 2.13] is a key tool in the derivation of the Reeh–Schlieder property for representations for which the positive cone $C_U$ has interior points.

**Lemma 5.45** *Let $(U, \mathcal{H})$ be a unitary representation of the connected Lie group $G$ for which the positive cone $C_U$ has interior points. If $\xi, \eta \in \mathcal{H}$ are such that the matrix coefficient*

$$U^{\xi,\eta} \colon G \to \mathbb{C}, \quad g \mapsto \langle \xi, U(g)\eta \rangle$$

*vanishes on an open subset of $G$, then $U^{\xi,\eta} = 0$ on $G$.*

***Proof*** Passing to the quotient Lie group $G/\ker(U)$, we may w.l.o.g. assume that $U$ is injective. Then the closed convex cone $C_U \subseteq \mathfrak{g}$ is pointed, and by assumption it is also generating, so that it defines a closed complex Olshanski semigroup $S_{C_U} = G\exp(iC_U)$ (Definition 3.17), and the matrix coefficient $U^{\xi,\eta}$ extends to a continuous function on $S_{C_U}$ which is holomorphic on the interior $S_{C_U}^\circ = G\exp(iC_U^\circ)$, which is a complex manifold ([Ne00, Thm. XI.2.5, Prop. XI.3.7]).

Suppose that $U^{\xi,\eta}$ vanishes on the non-empty open subset $\mathcal{O}$. Replacing $\eta$ by $U(g_0)\eta$ for some $g_0 \in \mathcal{O}$, we may assume that $e \in \mathcal{O}$. The exponential function $\exp\colon \mathfrak{g} + iC_U \to S_{C_U}$ is continuous, and holomorphic on $\mathfrak{g} + iC_U^\circ$. Therefore the function $U^{\xi,\eta} \circ \exp\colon \mathfrak{g} \to \mathbb{C}$ extends to a continuous function on the closed wedge $\mathfrak{g} + iC_U$, holomorphic on the interior and vanishing on $\exp^{-1}(\mathcal{O}) \neq \emptyset$. Now [Ne00, Lemma A.III.6] implies that $U^{\xi,\eta} \circ \exp = 0$. Therefore the regularity of the exponential function $\mathfrak{g} + iC_U^\circ \to S_{C_U}^\circ$ near 0 implies that $U^{\xi,\eta} = 0$ on $S_{C_U}$, and hence its restriction to $G$ vanishes. □

**Theorem 5.46** (Positive energy implies Reeh–Schlieder) *Let $(U, \mathcal{H})$ be an antiunitary representation of $G_{\tau_h}$ for which $C_U^\circ \neq \emptyset$. We consider a net $\mathsf{H}$ of real subspaces on open subsets of the homogeneous space $M = G/H$. Then $\mathsf{H}$ has the Reeh–Schlieder property, i.e., $\mathsf{H}(\mathcal{O})$ is cyclic whenever $\mathcal{O} \neq \emptyset$, if one of the following conditions is satisfied:*

(a) $\mathsf{H}$ *satisfies* (Iso), (Cov), (Add) *and* $\mathsf{H}(M)$ *is cyclic.*
(b) *There exists a real subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ with $\mathsf{H} = \mathsf{H}_{\mathtt{E}}^M$ and an open subset $W \subseteq M$ with $\mathsf{H}_{\mathtt{E}}^M(W) = \mathtt{V} = \mathtt{V}(h, U)$.*

***Proof*** (a) Let $\mathcal{O} \subseteq M$ be non-empty. We choose $\emptyset \neq \mathcal{O}_1 \subseteq \mathcal{O}$ relatively compact and an $e$-neighborhood $V \subseteq G$ with $V.\mathcal{O}_1 \subseteq \mathcal{O}$. For $\xi \in \mathsf{H}(\mathcal{O})^\perp$, $\eta \in \mathsf{H}(\mathcal{O}_1)$ and $g \in V$, we then have

$$U^{\xi,\eta}(g) = \langle \xi, U(g)\eta \rangle \in \langle \xi, \mathsf{H}(\mathcal{O}) \rangle = \{0\}.$$

As the cone $C_U$ has interior points, Lemma 5.45 now implies that $U^{\xi,\eta} = 0$. We conclude that $\xi \perp U(G)\mathsf{H}(\mathcal{O}_1)$. Since the net $\mathsf{H}$ is additive, $U(G)\mathsf{H}(\mathcal{O}_1)$ generates $\mathsf{H}(M)$, so that our assumption that $\mathsf{H}(M)$ is cyclic entails that $\xi = 0$.
(b) The nets $\mathsf{H}_{\mathtt{E}}^M$ satisfies (Iso) and (Cov) by Remark 4.11 and they are additive by Proposition 6.2 below. Further $\mathsf{H}_{\mathtt{E}}^M(M) \supseteq \mathsf{H}_{\mathtt{E}}^M(W) = \mathtt{V}$ is cyclic. Now (b) follows from (a). □

*Remark 5.47* If $\mathsf{H}(M)$ is not cyclic, then the preceding theorem applies to the representation on the closed complex subspace spanned by $\mathsf{H}(M)$, which is $U(G)$-invariant.

To shed some more light on the assumption that $\mathsf{H}(M)$ is cyclic, the following lemma is useful:

**Lemma 5.48** *Let $(U, \mathcal{H})$ be an antiunitary representation of $G_{\tau_h}$. For a real subspace $\mathtt{E} \subseteq \mathcal{H}_{\mathrm{KMS}}^{-\infty}$ and $M = G/H$, the following are equivalent:*

(a) $\mathsf{H}^M_{\mathtt{E}}(M)$ *is cyclic in* $\mathsf{H}$.
(b) $U^{-\infty}(C^\infty_c(G,\mathbb{R}))\mathtt{E}$ *is total in* $\mathcal{H}$.
(c) $U^{-\infty}(G)\mathtt{E}$ *spans a weak-$*$ dense subspace of* $\mathcal{H}^{-\infty}$.
(d) $U^{-\infty}(G_{\tau_h})\mathtt{E}$ *spans a weak-$*$ dense subspace of* $\mathcal{H}^{-\infty}$.

***Proof*** (a) $\Leftrightarrow$ (b) follows from the fact that $\mathsf{H}^M_{\mathtt{E}}(M)$ is the closed real subspace of $\mathcal{H}$ generated by $U^{-\infty}(C^\infty_c(G,\mathbb{R}))\mathtt{E}$.
(b) $\Rightarrow$ (c) follows from the fact that the inclusion $\mathcal{H} \hookrightarrow \mathcal{H}^{-\infty}$ is continuous with dense range and that the weak-$*$ closed subspace generated by $U^{-\infty}(C^\infty_c(G,\mathbb{R}))\mathtt{E}$ coincides with the one generated by $U^{-\infty}(G)\mathtt{E}$.
(c) $\Rightarrow$ (d) is trivial.
(d) $\Rightarrow$ (c): We have to show that $J\mathtt{E} = U^{-\infty}(\tau_h)\mathtt{E}$ is contained in the weak-$*$ closed subspace generated by $U^{-\infty}(G)\mathtt{E}$. So let $\xi \in \mathcal{H}^\infty$ be orthogonal to $U^{-\infty}(G)\mathtt{E}$. Then $\xi \bot U^{-\infty}(\exp \mathbb{R}h)\mathtt{E}$, and since the $U_h$-orbit map $U^\alpha_h$ of $\alpha \in \mathtt{E}$ extends holomorphically to $\mathcal{S}_\pi$ with $U^\alpha_h(\pi i) = J\alpha$, it follows that $\xi \bot J\mathtt{E}$. Now the assertion follows from the duality between $\mathcal{H}^\infty$ and $\mathcal{H}^{-\infty}$.
(c) $\Rightarrow$ (b): Let $\xi \in \mathcal{H}$ be orthogonal to $U(\varphi)\mathtt{E}$ for each $\varphi \in C^\infty_c(G,\mathbb{R})$. Then we also have $\xi \bot U(\varphi)U^{-\infty}(G)\mathtt{E}$, and now $U(\varphi^*)\xi \bot U^{-\infty}(G)\mathtt{E}$ implies with (c) that $U(\varphi^*)\xi = 0$. Using for $\varphi$ an approximate identity in $C^\infty_c(G,\mathbb{R})$, we conclude that $\xi = 0$. This proves (b). □

# 6 Perspectives

In this final section we briefly discuss several issues that are under current investigation and for which the existing results are much less complete.

The first section concerns additivity of nets of real subspaces (Section 6.1), a property that is difficult to verify in general, but always satisfied for nets of the form $\mathsf{H}^M_{\mathtt{E}}$. The second section contains some remarks on locality of nets, an important property which still has to be explored for nets on causal symmetric spaces (Section 6.2). Closely related to locality for nets of real subspaces is how this property reflects locality conditions for nets of operator algebras. In this context there are at least two natural ways, one is by the commuting property (Loc) in the introduction, but this condition can also be modified to fit graded Hilbert spaces and anticommutators, which leads to antiunitary representations of Lie supergroups (Section 6.3). Section 6.4 concerns the geometric structure on $M$ that we need to study nets of real subspaces. In Section 3.1 we argued with the Thermal Time Hypothesis that it is a natural assumption that $M$ carries a $G$-invariant causal structure. This is a rather big conceptual step, and it would be nice to derive this causal structure more directly from nets of operator algebras or real subspaces. Finally, one would also like to "classify" nets of real subspaces in terms of the geometry of $M$ and the unitary representation $(U,\mathcal{H})$ of $G_{\tau_h}$, but this seems presently far out of reach (Section 6.5).

## 6.1 Additivity

In this subsection, we take a closer look at the additivity condition (Add) for nets of real subspaces. We show in particular that the nets $\mathsf{H}_{\mathtt{E}}^M$ are always additive. For causal flag manifolds, this implies already that nets satisfying (Iso), (Cov), (BW) and (Add) are uniquely determined by the representation $(U, \mathcal{H})$ of $G_{\tau_h}$ (cf. Theorem 5.34).

**Definition 6.1** We call a net $\mathsf{H}$ on open subsets of $M$ *additive* if $\mathcal{O} = \bigcup_{j\in J} \mathcal{O}_j$ implies $\mathsf{H}(\mathcal{O}) = \overline{\sum_{j\in J} \mathsf{H}(\mathcal{O}_j)}$. We call it *countably additive*, it this relation holds for countable index sets.

The following proposition shows that a large class of nets of real subspaces is additive.

**Proposition 6.2** *For a real subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$, the net $\mathsf{H}_{\mathtt{E}}^M$ is additive.*

***Proof*** Let $\mathcal{O} \subseteq M$ be open and $(\mathcal{O}_j)_{j\in J}$ an open covering of $\mathcal{O}$. We write

$$q_M \colon G \to M = G/H, \qquad g \mapsto gH$$

for the quotient map and consider some $\varphi \in C_c^\infty(q_M^{-1}(\mathcal{O}))$.

The open subsets $q_M^{-1}(\mathcal{O}_j)$ form an open cover of $q_M^{-1}(\mathcal{O})$. Therefore Lemma 4.18 implies the existence of $j_1, \ldots, j_k$ and test functions $\varphi_\ell$, supported in $q_M^{-1}(\mathcal{O}_{j_\ell})$, such that $\varphi = \varphi_1 + \cdots + \varphi_k$. Now

$$U^{-\infty}(\varphi)\mathtt{E} \subseteq \sum_{\ell=1}^k U^{-\infty}(\varphi_\ell)\mathtt{E} \subseteq \sum_{\ell=1}^k \mathsf{H}_{\mathtt{E}}^M(\mathcal{O}_{j_\ell}) \subseteq \sum_{j\in J} \mathsf{H}_{\mathtt{E}}^M(\mathcal{O}_j)$$

implies that $\mathsf{H}_{\mathtt{E}}^M(\mathcal{O}) \subseteq \overline{\sum_{j\in J} \mathsf{H}_{\mathtt{E}}^M(\mathcal{O}_j)}$, and additivity thus follows from isotony. □

**Tools to verify additivity.** The following lemma is a general observation for which $M$ does not have to be connected.

**Lemma 6.3** *If $M$ has a countable basis for its topology, then every countably additive net on open subsets of $M$ is additive.*

***Proof*** Let $(\mathcal{O}_j)_{j\in J}$ be a family of open subsets of $M$. Further, let $\mathfrak{B}$ be a countable basis for the topology of $M$. Then each $\mathcal{O}_j$ is the union of the countable set $\mathfrak{B}_j$ of basis elements contained in $\mathcal{O}_j$, and therefore

$$\mathcal{O} = \bigcup\{\mathcal{B} \colon \mathcal{B} \in \mathfrak{B}_{\mathcal{O}}\}, \quad \mathfrak{B}_{\mathcal{O}} := \bigcup_{j\in J} \mathfrak{B}_j,$$

where $\mathfrak{B}_{\mathcal{O}}$ is countable. Countable additivity of the net thus implies that

$$\mathsf{H}(\mathcal{O}) = \overline{\sum_{\mathcal{B}\in\mathfrak{B}_{\mathcal{O}}} \mathsf{H}(\mathcal{B})} = \overline{\sum_{j\in J}\sum_{\mathcal{B}\in\mathfrak{B}_j} \mathsf{H}(\mathcal{B})} = \overline{\sum_{j\in J} \mathsf{H}(\mathcal{O}_j)}.$$

Therefore $\mathsf{H}$ is additive. □

*Remark 6.4* Every additive net is isotone because $\mathcal{O}_1 \subseteq \mathcal{O}_2$ implies $\mathcal{O}_2 = \mathcal{O}_1 \cup \mathcal{O}_2$, so that additivity entails

$$\mathsf{H}(\mathcal{O}_2) = \overline{\mathsf{H}(\mathcal{O}_1) + \mathsf{H}(\mathcal{O}_2)} \supseteq \mathsf{H}(\mathcal{O}_1).$$

**Lemma 6.5** *If $\mathsf{H}(\mathcal{O})_{\mathcal{O}\subseteq M}$ is a net on open subsets of the second countable space $M$, each subspace $\mathsf{H}(\mathcal{O})$ is decomposable as*

$$\mathsf{H}(\mathcal{O}) = \int_X^{\oplus} \mathsf{H}_x(\mathcal{O})\, d\mu(x),$$

*and if $\mu$-almost all the nets $(\mathsf{H}_x)_{x\in X}$ are additive, then $\mathsf{H}$ is additive.*

***Proof*** In view of Lemma 6.3, it suffices to show that $\mathsf{H}$ is countably additive. So let $\mathcal{O} = \bigcup_{j\in J} \mathcal{O}_j$ with a countable index set $J$. Then (DI3) in Appendix 7.6 and the additivity of the nets $\mathsf{H}_x$ imply that

$$\overline{\sum_{j\in J} \mathsf{H}(\mathcal{O}_j)} \overset{\text{(DI3)}}{=} \int_X \overline{\sum_{j\in J} \mathsf{H}_x(\mathcal{O}_j)}\, d\mu(x) = \int_X \mathsf{H}_x(\mathcal{O})\, d\mu(x) = \mathsf{H}(\mathcal{O}),$$

which is countable additivity. □

## 6.2 Locality

**Definition 6.6** Let $\mathsf{H}$ be a net of real subspaces on $M = G/H$ that is isotone and covariant. In $M \times M$, we defined the *locality set of* $\mathsf{H}$ by

$$\mathcal{L}_{\mathsf{H}} = \bigcup_{\mathsf{H}(\mathcal{O}_1)\subseteq \mathsf{H}(\mathcal{O}_2)'} \mathcal{O}_1 \times \mathcal{O}_2 \subseteq M \times M.$$

By definition, this is an open subset, and (Cov) implies that it is $G$-invariant. Moreover, it is symmetric in the sense that $(x, y) \in \mathcal{L}_{\mathsf{H}}$ implies $(y, x) \in \mathcal{L}_{\mathsf{H}}$.

The subset $\mathcal{L}_{\mathsf{H}}$ is non-empty if and only if the net $\mathsf{H}$ satisfies the “minimal” locality condition that there exist two non-empty open subsets $\mathcal{O}_1, \mathcal{O}_2 \subseteq M$ with $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)'$.

The subset $\mathcal{L}_{\mathsf{H}} \subseteq M \times M$ completely encodes the locality properties of the net $\mathsf{H}$ in terms of a $G$-invariant subset of the set of pairs in $M$. To connect locality properties of a net $\mathsf{H}$ with the given structures on $M$ therefore reduces to comparing $\mathcal{L}_{\mathsf{H}}$ with the given geometric data.

**Lemma 6.7** *If* $\mathsf{H}$ *is additive and* $\mathcal{O}_1, \mathcal{O}_2$ *are open subsets of* $M$ *with* $\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{L}_{\mathsf{H}}$*, then*

$$\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)'.$$

This corresponds to the locality condition (Loc) in Section 3.

***Proof*** Since $\mathsf{H}$ is additive, it is also isotone, and the real subspace $\mathsf{H}(\mathcal{O}_2)$ is generated by the subspaces $\mathsf{H}(\mathcal{C})$, where $\mathcal{C} \subseteq \mathcal{O}_2$ is a relatively compact open subset of $\mathcal{O}_2$. So it suffices to show that $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{C})'$ for such subsets.

For any $(x, y) \in \mathcal{O}_1 \times \overline{\mathcal{C}} \subseteq \mathcal{L}_{\mathsf{H}}$, there exist open subsets $\mathcal{O}_x^y, \mathcal{O}_y^x \subseteq M$ with

$$x \in \mathcal{O}_x^y \subseteq \mathcal{O}_1, \ y \in \mathcal{O}_y^x \subseteq \mathcal{O}_2 \quad \text{and} \quad \mathsf{H}(\mathcal{O}_x^y) \subseteq \mathsf{H}(\mathcal{O}_y^x)'.$$

Then, for each $x \in \mathcal{O}_1$, the sets $(\mathcal{O}_y^x)_{y \in \overline{\mathcal{C}}}$ form an open covering of the compact subset $\overline{\mathcal{C}} \subseteq \mathcal{O}_2$, so that there exist finitely many points $y_1, \ldots, y_n \in \mathcal{O}_2$ with

$$\mathcal{C} \subseteq \mathcal{O}_{y_1}^x \cup \cdots \cup \mathcal{O}_{y_n}^x.$$

Then

$$\mathcal{O}_x := \mathcal{O}_x^{y_1} \cap \cdots \cap \mathcal{O}_x^{y_n} \subseteq \mathcal{O}_1$$

is an open neighborhood of $x$ for which $\mathsf{H}(\mathcal{O}_x) \subseteq \mathsf{H}(\mathcal{O}_{y_j}^x)'$ for $j = 1, \ldots, n$. Additivity of $\mathsf{H}$ thus implies $\mathsf{H}(\mathcal{O}_x) \subseteq \mathsf{H}(\mathcal{C})'$. Finally, we observe that the $\mathcal{O}_x$ form an open cover of $\mathcal{O}_1$, so that additivity further implies that $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{C})'$. □

*Remark 6.8* If $W$ and $W'$ are open subsets of $M$ with $\mathsf{H}(W) = \mathsf{V}$ and $\mathsf{H}(W') = \mathsf{V}'$, then we have

$$\mathcal{L}_W := G.(W \times W') \cup G.(W' \times W) \subseteq \mathcal{L}_{\mathsf{H}}. \tag{166}$$

If $\mathsf{H}$ is additive, it follows from Lemma 6.7 that $\mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathcal{L}_W$ implies $\mathsf{H}(\mathcal{O}_1) \subseteq \mathsf{H}(\mathcal{O}_2)'$.

**Examples 6.9** (a) If $M = \mathbb{R}^{1,d-1}$ is Minkowski space and $W = W_R$ is the Rindler wedge, then $W' = -W_R$ and

$$\mathcal{L}_W = G.(W \times W') \subseteq M \times M$$

is the set of spacelike pairs (cf. Remark 1.29(d)). For an open subset $\mathcal{O} \subseteq M$, the maximal open subset $\mathcal{O}'$ satisfying

$$\mathcal{O} \times \mathcal{O}' \subseteq \mathcal{L}_W$$

is called the *causal complement* of $\mathcal{O}$ (cf. Definition 5.39(a)).

The same picture prevails for de Sitter space $\mathrm{dS}^d \subseteq \mathbb{R}^{1,d}$.

(b) For $M = \mathbb{S}^1$, a causal flag manifold of $G = \mathrm{SL}_2(\mathbb{R})$, the wedge regions are open non-dense intervals $W \subseteq \mathbb{S}^1$ (Example 3.10). If $W = W_M^+(h)$, then $W' = W_M^+(-h)$ is the interior of the complement of $W$, and

$$\mathcal{L}_W = G.(W \times W') = M^2 \setminus \Delta_M.$$

So $(x, y) \in \mathcal{L}_W$ if and only if $x \neq y$.
(c) In the non-compactly causal symmetric space $M = \mathrm{SL}_4(\mathbb{R})/\mathrm{SO}_{2,2}(\mathbb{R})$ (cf. Example 4.28), not all acausal pairs are contained in $\mathcal{L}_W$, and $M \times M$ contains several open acausal $G$-orbits (cf. [NÓØ21, NÓ26]).

*Remark 6.10* In the context of abstract wedges, represented by elements of the set

$$\mathcal{G}(G_{\tau_h}) := \{(x, \tau) \in \mathfrak{g} \times G\tau_h \colon \mathrm{Ad}(\tau)x = x, \tau^2 = e\}$$

(cf. Exercise 1.51), there is a natural complementation map

$$(x, \tau) \mapsto (x, \tau)' := (-x, \tau).$$

In this context, it is a natural question if $(h, \tau_h)' = (-h, \tau_h)$ is contained in the $G$-orbit of $(h, \tau_h)$. This is equivalent to the symmetry of $h$ and the additional condition that there exists a $g_0 \in G^{\tau_h}$ with $\mathrm{Ad}(g_0)h = -h$ (cf. [MNÓ26a]). If this is the case and $(U, \mathcal{H})$ is an antiunitary representation of $G_{\tau_h}$, then $\mathtt{V}' = U(g_0)\mathtt{V}$ follows from $g_0.(h, \tau_h) = (-h, \tau_h)$. For any net $\mathsf{H}$ satisfying (Cov) and (BW), this implies that

$$W \times g_0.W \subseteq \mathcal{L}_{\mathsf{H}} \tag{167}$$

(cf. (166)).

We refer to [MN26] and [NÓ26] for more detailed discussions of locality properties of nets on causal flag manifolds and non-compactly causal symmetric spaces, respectively. Twisted locality conditions are discussed in [MN21] and [MNÓ26b].

## 6.3 Representations of Lie supergroups

Lie supergroups and their unitary representations arise naturally in Physics in connection with supersymmetry (cf. [Gu75, Gu93, Gu01]). It would be interesting to extend the theory developed in these notes to this context, where the Euler element $h \in \mathfrak{g}$ is supposed to be an even element.

**Definition 6.11** A *Lie supergroup* is a pair $(G, \mathfrak{g})$, where $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ is a finite-dimensional Lie superalgebra and $G$ is a real Lie group with Lie algebra $\mathfrak{g}_{\overline{0}}$, acting smoothly by automorphisms on $\mathfrak{g}$ via $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$, in such a way that the action on $\mathfrak{g}_{\overline{0}}$ is the adjoint action of the Lie group $G$ with Lie algebra $\mathbf{L}(G) = \mathfrak{g}_{\overline{0}}$.

**Definition 6.12** A *unitary representation* of a Lie supergroup $(G, \mathfrak{g})$ is a pair $(U, \beta)$, where $(U, \mathcal{H})$ is a unitary representation of the Lie group $G$ on a graded Hilbert space $\mathcal{H} = \mathcal{H}_{\overline{0}} \oplus \mathcal{H}_{\overline{1}}$, preserving the grading, and

$$\beta\colon \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$$

is a representation of the Lie superalgebra $\mathfrak{g}$ on the space of smooth vectors of $U$ satisfying

$$-i\beta(x) \subseteq \beta(x)^* \quad \text{for} \quad x \in \mathfrak{g}_{\overline{1}}$$

([NS11], [CCTV06]).

**Problem 6.13** One can associate to each real subspace $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ (distribution vectors for $U$) the closed real subspaces $\mathsf{H}^G_{\mathtt{E}}(\mathcal{O})$ generated by

$$\beta(U(\mathfrak{g}))U^{-\infty}(C^\infty_c(\mathcal{O},\mathbb{R}))\mathtt{E}.$$

Does this construction lead to nets that are compatible with fermionic nets in AQFT? Possibly one can develop a "supersymmetric" variant of the theory described in these notes.

Here are some relevant structures and observations.

**Definition 6.14** (The $*$-monoid associated to a Lie supergroup) The antilinear map

$$\mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}\ ,\ x \mapsto x^*, \quad \text{defined by} \quad x^* := \begin{cases} -x & \text{if } x \in \mathfrak{g}_{\overline{0}}\,, \\ -i\,x & \text{if } x \in \mathfrak{g}_{\overline{1}}\,. \end{cases}$$

is an anti-automorphism. It extends to an antilinear anti-automorphism

$$U(\mathfrak{g}_{\mathbb{C}}) \to U(\mathfrak{g}_{\mathbb{C}})\ ,\ D \mapsto D^* \tag{168}$$

in a natural way. Consider the monoid $\mathsf{S}$ with underlying set $G \times U(\mathfrak{g}_{\mathbb{C}})$ and multiplication

$$(D_1, g_1)(D_2, g_2) = (D_1(g_1 \cdot D_2), g_1 g_2),$$

where $g \cdot D$ denotes the adjoint action of $g \in G$ on $D \in U(\mathfrak{g}_{\mathbb{C}})$. The neutral element of $\mathsf{S}$ is $1_{\mathsf{S}} := (1_{U(\mathfrak{g}_{\mathbb{C}})}, e)$. The map

$$\mathsf{S} \to \mathsf{S}\,,\ s \mapsto s^* \quad \text{defined by} \quad (D,g)^* := (g^{-1} \cdot D^*, g^{-1})$$

is an involution of $\mathsf{S}$. Recall that $U(\mathfrak{g}_{\mathbb{C}})$ is an associative superalgebra. An element $(D, g) \in \mathsf{S}$ is called *odd* (resp. *even*) if $D$ is an odd (resp. even) element of $U(\mathfrak{g}_{\mathbb{C}})$.

Replacing in this construction the elements $g \in G$ by compactly supported smooth functions on $G$, one can even construct a graded $*$-algebra $(C^\infty_c(\mathcal{G}), *)$ in such a way that every unitary $\mathcal{G}$-representation integrates to a $*$-representation of $C^\infty_c(\mathcal{G})$ by bounded operators. Thus one even obtains "supergroup $C^*$-algebras". We refer to [NS16] for details.

*Remark 6.15* From the table in [NS11, §2.5] we get some information on which finite-dimensional simple Lie superalgebras $\mathfrak{g}$ have non-trivial unitary representations. According to [NS11, §6], for any unitary representation of $\mathcal{G} = (G, \mathfrak{g})$, we must have the inclusion

$$\mathrm{Cone}(\mathfrak{g}) = \mathrm{cone}\{[x,x] \colon x \in \mathfrak{g}_{\overline{1}}\} \subseteq C_U = \{x \in \mathfrak{g}_{\overline{0}} \colon -i \cdot \partial U(x) \geq 0\}.$$

For a unitary representation with discrete kernel, the cone $C_U$ is pointed, so that the pointedness of the cone generated by the brackets of odd elements is necessary for the existence of non-trivial unitary representations. In this sense [NS11, Thm. 6.2.1] compiles a negative list of simple Lie superalgebras for which this is not the case.

## 6.4 The geometric structure on $M$

Let $(U, \mathcal{H})$ be an antiunitary representation of $G_{\tau_h}$ and $\mathtt{V} := \mathtt{V}(h, U) \subseteq \mathcal{H}$ be the canonical standard subspace from (2). Let $\mathtt{E} \subseteq \mathcal{H}^{-\infty}$ be a finite-dimensional linear subspace, invariant under the subgroup $H \subseteq G$ and $M := G/H$. Then we obtain a net $\mathsf{H}^M_{\mathtt{E}}$ on $M$, satisfying (Iso) and (Cov).

Further

$$W_{\mathtt{E}} := \{gH \in M \colon U^{-\infty}(g)\mathtt{E} \subseteq \mathcal{H}^{-\infty}_{\mathrm{KMS}}\}^\circ$$

specifies an open subset of $M$ that deserves to be called the *wedge region associated to* $\mathtt{E}$, but it may be empty; depending on the subspace $\mathtt{E}$ (cf. Proposition 4.13).

If $W_{\mathtt{E}} \neq \emptyset$, then Proposition 4.13 implies that

$$\mathsf{H}^M_{\mathtt{E}}(W_{\mathtt{E}}) \subseteq \mathtt{V} \quad \text{and} \quad \exp(\mathbb{R}h)W_{\mathtt{E}} \subseteq W_{\mathtt{E}}.$$

For $\xi \in \mathtt{V}^\infty := \mathtt{V} \cap \mathcal{H}^\infty$, we have

$$\begin{aligned}\langle \xi, i\partial U(h)\xi\rangle &= \frac{d}{dt}\Big|_{t=0} \langle \xi, e^{it\partial U(h)}\xi\rangle = \frac{d}{dt}\Big|_{t=0} \langle \xi, \Delta_{\mathtt{V}}^{t/2\pi}\xi\rangle \\ &= \frac{d}{dt}\Big|_{t=0} \|\Delta_{\mathtt{V}}^{t/4\pi}\xi\|^2 \leq 0\end{aligned}$$

because the convex function

$$f \colon [0, 2\pi] \to \mathbb{R}, \quad f(t) := \|\Delta_{\mathtt{V}}^{t/4\pi}\xi\|^2$$

takes its minimal value in $t = \pi$ and has a local maximum in $t = 0$. Here convexity follows from the Spectral Theorem, which implies that $f$ is a Laplace transform, and $\xi \in \mathtt{V} = \mathrm{Fix}(J_{\mathtt{V}}\Delta_{\mathtt{V}}^{1/2})$ implies that it is invariant under reflection in $\pi$.

For $\alpha \in \mathtt{E}$ and $\varphi \in C^\infty_c(W_{\mathtt{E}}, \mathbb{R})$, we have $U(\varphi)\alpha \in \mathtt{V}^\infty$, so that we get

$$\langle U(\varphi)\alpha, i\partial U(h)U(\varphi)\alpha\rangle \leq 0.$$

Letting $\varphi$ tend to a point measure $\delta_g$, $g \in W_{\mathtt{E}}$, we obtain

$$\langle \alpha, i\partial U(\mathrm{Ad}(g)^{-1}h)\alpha\rangle \leq 0$$

in the sense of distributions on $G$. Maybe these inequalities can be related to the generalized positive energy conditions appearing in [JaNi24].

*Remark 6.16* In this context, it becomes apparent that the closed convex cone $C(W_{\mathtt{E}}) \subseteq \mathfrak{g}$, generated by

$$\mathrm{Ad}(g)^{-1}h, \quad g \in W_{\mathtt{E}},$$

plays an important role. It is clearly invariant under $e^{\mathbb{R}\,\mathrm{ad}\,h}$, so that

$$C(W_{\mathtt{E}}) \subseteq C(W_{\mathtt{E}})_+ + \mathfrak{g}_0(h) - C(W_{\mathtt{E}})_-$$

for

$$C(W_{\mathtt{E}})_\pm := \pm C(W_{\mathtt{E}}) \cap \mathfrak{g}_{\pm 1}(h) \subseteq C(W_{\mathtt{E}})$$

(cf. Lemma 3.7).

Here is an alternative approach.

**Proposition 6.17** *Consider an antiunitary representation* $(U, \mathcal{H})$ *of* $G_{\tau_h}$ *and the corresponding standard subspace* $\mathtt{V} := \mathtt{V}(h, U)$. *Assume that the net* $\mathsf{H}$ *on open subsets of* $M = G/H$ *satisfies* (Iso) *and* (Cov). *Suppose further that there exists an open subset* $\emptyset \neq \mathcal{O} \subseteq M$ *for which* $\mathsf{H}(\mathcal{O})$ *is cyclic and contained in* $\mathtt{V}$. *Then* (BW) *holds for the open subset* $W := \exp(\mathbb{R}h).\mathcal{O}$.

***Proof*** Clearly, $W$ is an $\exp(\mathbb{R}h)$-invariant open subset of $M$ and (Cov) and (Iso) imply that $\mathsf{H}(W) \subseteq \mathtt{V}$ is an $U(\exp \mathbb{R}h)$-invariant subspace. As $\mathsf{H}(W)$ contains $\mathsf{H}(\mathcal{O})$, it is cyclic, so that $\mathsf{H}(W) = \mathtt{V}$ follows from the Equality Lemma 1.9. □

**Corollary 6.18** *Assume that the net* $\mathsf{H}$ *on open subsets of* $M = G/H$ *satisfies* (Iso), (Cov), (RS) *and* (Add) *and that there exists an open subset* $\mathcal{O} \subseteq M$ *such that* $\mathsf{H}(\mathcal{O}) \subseteq \mathtt{V} = \mathtt{V}(h, U)$. *Then the union* $W^{\mathsf{H}}$ *of all such open subsets is non-empty, open,* $\exp(\mathbb{R}h)$*-invariant, and satisfies*

$$\mathsf{H}(W^{\mathsf{H}}) = \mathtt{V}.$$

**Problem 6.19** Compare $W^{\mathsf{H}}$ with $W_M^+(h)$ for nets on causal homogeneous spaces $M = G/H$.

### 6.5 Classification of nets of real subspaces

We expect that there are various contexts in which nets on $M = G/H$ could be classified. Specifically, the (BW) property determines the net for a given antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ on all wedge regions $(g.W)_{g\in G}$ in $M$.

For causal flag manifolds, this fact already implies that a net satisfying (Iso), (Cov), (RS), (BW) and (Add) is uniquely determined by the antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$ (see Section 6.1 and [MN26] for details).

**Problem 6.20** Consider $G := \mathrm{Aff}(\mathbb{R})_e$ with the non-symmetric Euler element $h = (0, 1)$ (Example 3.8). Here the intervals $(x, \infty)$, $x \in \mathbb{R}$, are natural wedge regions in $M = \mathbb{R}$. Given an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h} = \mathrm{Aff}(\mathbb{R})$, is it possible to classify all nets on open subsets of $\mathbb{R}$ that satisfy the (BW) condition? Here additivity and locality conditions certainly help to reduce the problem.

For instance, if $\mathsf{H}$ is additive, then it is easy to see that the whole net is determined by the real subspace $\mathsf{H}((0, 1))$, assigned to the open unit interval $(0, 1)$. So one has to determine which real subspaces arise in such nets.

## 7 Appendix

### 7.1 The category of W*-algebras

By the Gelfand–Naimark Theorem, $C^*$-algebras are norm-closed $*$-subalgebras of $B(\mathcal{H})$, $\mathcal{H}$ a complex Hilbert space. On the other hand, we have defined von Neumann algebras directly as $*$-subalgebras $\mathcal{M} \subseteq B(\mathcal{H})$ satisfying $\mathcal{M}'' = \mathcal{M}$. So they are in particular closed with respect to the weak-$*$ topology on $B(\mathcal{H})$, specified by the subspace $B_1(\mathcal{H}) \subseteq B(\mathcal{H})^*$ of trace class operators and the trace pairing $(A, B) \mapsto \mathrm{tr}(AB)$. As $B(\mathcal{H}) \cong B_1(\mathcal{H})^*$, the duality theory of Banach spaces easily implies that $\mathcal{M} \cong Q^*$ for $Q := B_1(\mathcal{H})/\mathcal{M}^\perp$. Hence every von Neumann algebra has a *predual*.

This observation can be used to specify von Neumann algebras axiomatically, independently of an embedding in some $B(\mathcal{H})$.

**Definition 7.1** A $C^*$-algebra $\mathcal{M}$ is called a *$W^*$-algebras* if it has a predual, i.e., there exists a closed subspace $\mathcal{M}_* \subseteq \mathcal{M}^*$ with $\mathcal{M} \cong (\mathcal{M}_*)^*$ as Banach spaces.

This approach has been pursued by S. Sakai, and his monograph [Sa71] is an excellent reference. [Sa71, Cor. 1.13.3] asserts in particular that $W^*$-algebras have a unique predual $\mathcal{M}_*$. Its elements are called *normal linear functionals*. They are the continuous linear functionals for the $\sigma(\mathcal{M}, \mathcal{M}_*)$-topology on $\mathcal{M}$, i.e., the coarsest topology for which all functionals in $\mathcal{M}_*$ are continuous. Any normal selfadjoint functional is a difference of two positive ones, and the positive normal functionals $\varphi$ can also be characterized by the property that, for every uniformly bounded increasing directed subset $(x_j)_{j\in J}$ of $\mathcal{M}$, we have

$$\varphi(\sup x_j) = \sup \varphi(x_j) \tag{169}$$

([Sa71, Thm. 1.13.2]).

We also note that $W^*$-algebras have an identity, which can be derived from the Krein–Milman Theorem because it ensures the existence of extreme points in the unit ball of $\mathcal{M}$, which is compact in the $\sigma(\mathcal{M}, \mathcal{M}_*)$-weak topology ([Sa71, §1.7]).

**Examples 7.2** (a) For every complex Hilbert space $\mathcal{H}$, the full operator algebra $B(\mathcal{H})$ is a $W^*$-algebra with predual $B(\mathcal{H})_* = B_1(\mathcal{H})$ (trace class operators).
(b) For every $\sigma$-finite measures space $(X, \mathfrak{S}, \mu)$, the Banach algebra $L^\infty(X, \mathfrak{S}, \mu)$ is a commutative $W^*$-algebra with $L^\infty(X, \mathfrak{S}, \mu)_* \cong L^1(X, \mathfrak{S}, \mu)$.

The same holds for $\ell^\infty$-direct sums (whose preduals are $\ell^1$-direct sums), and **all commutative $W^*$-algebras are such sums.** More intrinsically, they can be described as the space $L^\infty_{\rm loc}(X, \mathfrak{S}, \mu)$ of bounded, locally measurable functions on a semi-finite measure space $(X, \mathfrak{S}, \mu)$. Here *semi-finite* means that, every $E \in \mathfrak{S}$ with $\mu(E) = \infty$ contains a measurable subset of finite positive measure. A function $f$ is called *locally measurable* if its restriction to all measurable subsets of finite measure is measurable.

**Definition 7.3** A *morphism of $W^*$-algebras* is a complex linear $*$-algebra morphism $\pi\colon \mathcal{M} \to \mathcal{N}$ with $\pi^*\mathcal{N}_* \subseteq \mathcal{M}_*$, i.e., pullbacks of normal functionals are normal. We call these algebra morphisms *normal*. For every complex Hilbert space $\mathcal{H}$, a *normal representation* $(\pi, \mathcal{H})$ of $\mathcal{M}$ is a normal morphism $\pi\colon \mathcal{M} \to B(\mathcal{H})$.

*Remark 7.4* (a) For normal states, the GNS construction produces a normal representation.
(b) This is more generally true for so-called semi-finite weights. A *weight*

$$\omega\colon \mathcal{M}_+ \to [0, \infty]$$

is an additive, positively homogeneous function. It is called *normal* if it is compatible with bounded sup's in the sense of (169). A weight $w$ on $\mathcal{M}$ is called *semi-finite* if the set

$$\{M \in \mathcal{M}_+ \mid w(M) < \infty\}$$

generates a $*$-algebra which is $\sigma(\mathcal{M}, \mathcal{M}_*)$-dense in $\mathcal{M}$.

The GNS construction and the Tomita–Takesaki Theorem extend to normal weights, and faithful normal semi-finite weights always exist ([Bl06, III.2.2.26]). Normal semi-finite weights are sums (in the sense of summability of general families) of normal positive forms (cf. [Haa75]). As a consequence, any von Neumann algebra $\mathcal{M}$ has a standard form representation (cf. [Bl06], [BGN20, §3.1]).

*Remark 7.5* (a) Any $\sigma$-finite measure is semi-finite. If $X$ is a set, then the counting measure

$$\mu\colon \mathbb{P}(X) = 2^X \to \mathbb{N}_0 \cup \{\infty\}, \quad \mu(E) := |E|$$

is semi-finite. It is $\sigma$-finite if and only if $X$ is countable.
(b) If $(X_j, \mathfrak{S}_j, \mu_j)_{j\in J}$ are semi-finite measure spaces, and we put

$$X := \coprod_{j\in J} X_j, \quad \mathfrak{S} := \{E \subseteq X \colon (\forall j \in J)\, E \cap X_j \in \mathfrak{S}_j\}$$

and

$$\mu(E) := \sum_{j\in J} \mu_j(E \cap X_j),$$

then $\mathfrak{S}$ is a $\sigma$-algebra on $X$, $\mu$ is a measure, and $(X, \mathfrak{S}, \mu)$ is a semi-finite measure space. Exercise 7.8 shows that, conversely, up to sets of measure zero, any semi-finite measure space is such a direct sum of finite measure spaces.

**Exercises for Appendix 7.1:**

**Exercise 7.6** (Direct sums of von Neumann algebras) Let $\mathcal{M}_j \subseteq B(\mathcal{H}_j)$ be a family of von Neumann algebras, $\mathcal{H} := \widehat{\bigoplus}_{j\in J} \mathcal{H}_j$ the Hilbert space direct sum of the $\mathcal{H}_j$ and

$$\mathcal{M} := \overline{\bigoplus}_{j\in J} \mathcal{M}_j := \Big\{(M_j)_{j\in J} \in \prod_{j\in J} \mathcal{M}_j \colon \sup_{j\in J} \|M_j\| < \infty\Big\}$$

the $\ell^\infty$-direct sum of the von Neumann algebras $\mathcal{M}_j$ with the norm $\|M\| := \sup_{j\in J} \|M_j\|$. Show that $\mathcal{M}$ can be realized in a natural way as a von Neumann algebra on $\mathcal{H}$.

**Exercise 7.7** (Separability and $\sigma$-finiteness) Let $(X, \mathfrak{S}, \mu)$ be a measure space. Show that:

(a) If $f \in L^p(X, \mu)$, $1 \leq p < \infty$, then the measurable subset $\{f \neq 0\}$ of $X$ is $\sigma$-finite.
(b) If $\mathcal{H} \subseteq L^2(X, \mu)$ is a separable Hilbert subspace, then there exists a $\sigma$-finite measurable subset $X_0 \subseteq X$ with the property that each $f \in \mathcal{H}$ vanishes $\mu$-almost everywhere on $X_0^c = X \setminus X_0$.

**Exercise 7.8** Let $(X, \mathfrak{S}, \mu)$ be a measure space. Show that there exist measurable subsets
$X_j \subseteq X$, $j \in J$, of finite measure such that

$$L^2(X, \mu) \cong \widehat{\bigoplus}_{j\in J} L^2(X_j, \mu|_{X_j}).$$

Hint: Use Zorn's Lemma to find a maximal family $(X_j)_{j\in J}$ of measurable subsets of $X$ of finite positive measure, for which $\mu(X_j \cap X_k) = 0$ for $j \neq k$. Conclude that the corresponding subspaces $L^2(X_j, \mu|_{X_j})$ of $L^2(X, \mu)$ are mutually orthogonal and that the intersection of their orthogonal complements is trivial. For the latter argument, use Exercise 7.7(a).

## 7.2 Background on Lie algebras

In this appendix we collect some relevant facts about Lie algebras, see e.g. [HN12]. In view of the one-to-one correspondence between Lie algebras and simply connected Lie groups, the same terminology (simple, semisimple, compact, hermitian, ...) is used for Lie algebras and Lie groups. Here $G$ denotes a connected Lie group with Lie algebra $\mathfrak{g}$. We denote the centers of $G$ and $\mathfrak{g}$ by $Z(G)$ and $\mathfrak{z}(\mathfrak{g})$, respectively. The adjoint action defines a group homomorphism $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ whose image we denote by $\mathrm{Ad}(G) \cong G/Z(G)$. For a non-connected Lie group $H$, we denote by $H_e$ its identity component.

### 7.2.1 Compact Lie algebras

A Lie algebra $\mathfrak{g}$ is called *compact* if it is the Lie algebra of some compact Lie group. This is equivalent to the existence of an ad-invariant positive definite symmetric bilinear form on $\mathfrak{g}$ ([HN12, Prop. 12.1.4]). It follows that each operator $\mathrm{ad}\, x$ is semisimple (i.e., each $\mathrm{ad}\, x$-invariant subspace has an invariant complement) with purely imaginary spectrum, so $\mathrm{ad}\, x$ can only be nilpotent if $x \in \mathfrak{z}(\mathfrak{g})$.

A subalgebra $\mathfrak{k} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is called *compactly embedded* if the subgroup $\overline{\exp(\mathrm{ad}\, \mathfrak{k})} \subset \mathrm{Aut}(\mathfrak{g})$ is compact. This implies that $\mathfrak{k}$ is compact and the subgroup $K := \overline{\exp(\mathfrak{k})} \subset G$ is closed ([HN12, Lemmas 14.2.3 and 14.2.6]). The group $\mathrm{Ad}(K)$ is compact but $K$ itself need not be compact. For example, the Lie algebra $\mathbb{R}^n$ of the additive Lie group $\mathbb{R}^n$ is compactly embedded in itself (because it is abelian); note that $\mathbb{R}^n$ is also the Lie algebra of the compact Lie group $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

### 7.2.2 Semisimple Lie algebras

A Lie algebra $\mathfrak{g}$ is called *simple* if it is non-abelian and contains no proper ideals. It is called *semisimple* if it is isomorphic to a direct sum of simple ones. By [HN12, Thm. 5.5.9], this is equivalent to nondegeneracy of the *Cartan–Killing form*

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}\, x, \mathrm{ad}\, y).$$

A *Cartan involution* of $\mathfrak{g}$ is an automorphism $\theta$ satisfying $\theta^2 = \mathrm{id}$ and $\kappa(x, \theta x) \leq 0$ for $x \in \mathfrak{g}$. Cartan involutions exist and are unique up to conjugation with inner automorphisms $\mathrm{Ad}_g$, $g \in G$ ([HN12, §§13.2.2/3]). A Cartan involution induces a *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad \text{with} \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}, \tag{170}$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are the $\theta$-eigenspaces to the eigenvalues $+1$ and $-1$, respectively. Conversely, a Cartan decomposition determines a Cartan involution by setting it $\pm\,\mathrm{id}$ on $\mathfrak{k}, \mathfrak{p}$. We have

$$\mathfrak{k} \perp_\kappa \mathfrak{p}, \qquad \kappa < 0 \text{ on } \mathfrak{k}, \qquad \kappa > 0 \text{ on } \mathfrak{p},$$

where $\mathfrak{k} \perp_\kappa \mathfrak{p}$ holds because $\theta$ preserves $\kappa$, and the other two relations follow from this combined with $\kappa(x, \theta x) \leq 0$ and nondegeneracy of $\kappa$. Note that this implies

$$\kappa(x, \theta x) < 0 \quad \text{ for } 0 \neq x \in \mathfrak{g}.$$

On $\mathfrak{gl}_n(\mathbb{C})$ a Cartan involution is given by $\theta(x) = -x^*$, so for a $*$-invariant Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}_n(\mathbb{C})$ the associated Cartan decomposition is given by

$$\mathfrak{k} = \mathfrak{g} \cap \mathfrak{u}_n(\mathbb{C}) \quad \text{ and } \quad \mathfrak{p} = \mathfrak{g} \cap \mathrm{Herm}_n(\mathbb{C})$$

with the skew-Hermitian matrices $\mathfrak{u}_n(\mathbb{C})$ and the hermitian matrices $\mathrm{Herm}_n(\mathbb{C})$.

The subalgebra $\mathfrak{k}$ in a Cartan decomposition is maximal compactly embedded and $\exp(\mathrm{ad}\,\mathfrak{k})$ is a maximal compact subgroup of $\mathrm{Ad}(G)$ ([HN12, Prop. 13.1.5]). Since $\kappa$ is positive definite on $\mathfrak{p}$, it induces a Riemannian metric on $G/K$ making it a Riemannian symmetric space (with geodesic symmetry at the base point $eK$ arising from the involution of $G$ associated to $\theta$).

To develop a finer structure theory of semisimple Lie algebras, we start with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and fix a maximal abelian subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$ and a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$. Then $\mathfrak{t}$ is also maximal abelian in $\mathfrak{g}$ and $\mathfrak{t}_\mathbb{C} \subseteq \mathfrak{g}_\mathbb{C}$ is a Cartan subalgebra (i.e., it is nilpotent and its own normalizer) for which we obtain a root system $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. Moreover, $\mathrm{ad}\,\mathfrak{a}$ is simultaneously diagonalizable, which leads to the *restricted root decomposition*

$$\mathfrak{g} = \mathfrak{g}^0 + \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^\alpha, \quad \text{ where } \quad \mathfrak{g}^\alpha = \{x \in \mathfrak{g} \colon (\forall h \in \mathfrak{a}) \ [h, x] = \alpha(h)x\} \tag{171}$$

are the restricted root spaces and

$$\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) = \{\alpha \in \mathfrak{a}^* \colon \alpha \neq 0, \ \mathfrak{g}^\alpha \neq \{0\}\}$$

is the set of restricted roots.[24] We pick a set

[24] The term "restricted" is due to the fact that one may enlarge $\mathfrak{a}$ to a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, for which we obtain a refined root decomposition and roots $\alpha \colon \mathfrak{h} \to \mathbb{C}$, for which the restricted roots are obtained by restricting them to $\mathfrak{a}$.

$$\Pi := \{\alpha_1, \ldots, \alpha_n\} \subseteq \Sigma$$

of *simple roots*. This is a subset with the property that every root $\alpha \in \Sigma$ is a linear combination $\alpha = \sum_{j=1}^n n_j \alpha_j$, where the coefficients are either all in $\mathbb{Z}_{\geq 0}$ or in $\mathbb{Z}_{\leq 0}$. The convex cone

$$\Pi^\star := \{x \in \mathfrak{a} \colon (\forall \alpha \in \Pi)\ \alpha(x) \geq 0\}$$

is called the *closed positive (Weyl) chamber corresponding to* $\Pi$. We have the *root space decomposition*

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}, \quad \text{where} \quad \mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{k}.$$

Now $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, and for a non-zero element $x_\alpha \in \mathfrak{g}_\alpha$, the 3-dimensional subspace spanned by $x_\alpha, \theta(x_\alpha)$ and $[x_\alpha, \theta(x_\alpha)] \in \mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$ is a Lie subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.

**Lemma 7.9** *For each* $\alpha \in \Sigma$*, there exists a unique element* $\alpha^\vee \in \mathfrak{a}$ *such that* $\alpha(\alpha^\vee) = 2$ *and*

$$[x, \theta(x)] = \frac{2\kappa(x, \theta(x))}{\kappa(\alpha^\vee, \alpha^\vee)} \alpha^\vee \quad \textit{for} \quad x \in \mathfrak{g}_\alpha.$$

***Proof*** We first observe that $0 \neq x \in \mathfrak{g}_\alpha$ implies $\theta(x) \in \mathfrak{g}_{-\alpha}$, hence $[x, \theta(x)] \in \mathfrak{g}_0$. We also have

$$\theta([x, \theta(x)]) = [\theta(x), x] = -[x, \theta(x)],$$

so that $[x, \theta(x)] \in \mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$. As $\kappa$ is positive definite on $\mathfrak{a}$, there exists a unique $h_\alpha \in \mathfrak{a}$ with $\kappa(\cdot, h_\alpha) = \alpha$. For $h \in \mathfrak{a}$, we then have

$$\kappa(h, [x, \theta(x)]) = \kappa([h, x], \theta(x)) = \alpha(h)\kappa(x, \theta(x)) = \kappa(h, \kappa(x, \theta(x))h_\alpha),$$

and this shows that we must have $[x, \theta(x)] = \kappa(x, \theta(x))h_\alpha$. As $\kappa(x, \theta(x)) < 0$ and $\alpha(h_\alpha) = \kappa(h_\alpha, h_\alpha) > 0$, we obtain with

$$\alpha^\vee := \frac{2h_\alpha}{\kappa(h_\alpha, h_\alpha)}$$

that $\alpha(\alpha^\vee) = 2$, and observing that $\kappa(\alpha^\vee, \alpha^\vee) = \frac{4}{\kappa(h_\alpha, h_\alpha)}$ completes the proof. □

Now

$$r_\alpha \colon \mathfrak{a} \to \mathfrak{a}, \quad r_\alpha(x) := x - \alpha(x)\alpha^\vee$$

is a reflection in the hyperplane $\ker \alpha$, and the subgroup

$$\mathcal{W} := \langle r_\alpha \colon \alpha \in \Sigma \rangle \subseteq \mathrm{GL}(\mathfrak{a})$$

is called the *Weyl group*. Its action on $\mathfrak{a}$ provides a good description of the adjoint orbits of hyperbolic elements: Every hyperbolic element in $\mathfrak{g}$ is conjugate to a unique element in the Weyl chamber $\Pi^\star \subseteq \mathfrak{a}$, a fundamental domain for the $G$-action on the subset of hyperbolic elements in $\mathfrak{g}$ and a fundamental domain for the $\mathcal{W}$-action on $\mathfrak{a}$. For $x \in \mathfrak{a}$, the intersection $\mathcal{O}_x \cap \mathfrak{a} = \mathcal{W}x$ is the Weyl group orbit ([KN96, Thm. III.10]).

### 7.2.3 Hermitian Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra with Cartan involution $\theta$ and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We call $\mathfrak{g}$ *hermitian* if $\mathfrak{z}(\mathfrak{k}) \neq \{0\}$ (cf. Definition 3.23). This implies that the center $\mathfrak{z}(\mathfrak{k})$ of $\mathfrak{k}$ is one-dimensional and generated by an element $Z$ with $\mathfrak{k} = \ker(\operatorname{ad} Z)$ for which $\operatorname{ad} Z|_{\mathfrak{p}}$ defines a complex structure on $\mathfrak{p}$ ([Ne00, Thm. A.V.1]). This defines on the Riemannian symmetric space $G/K$ a complex structure, making it a hermitian symmetric space of non-compact type. The term “hermitian” is due to the fact that the tangent space $T_{eK}(G/K) \cong \mathfrak{p}$ carries a $K$-invariant complex structure $I$, so that the invariant scalar product $\kappa(x, y) = \operatorname{tr}(\operatorname{ad} x, \operatorname{ad} y)$ actually extends to a hermitian form $\kappa(x, y) + i\kappa(Ix, y)$.

Actually, for a hermitian Lie algebra $\mathfrak{g}$, the space $G/K$ is biholomorphic to a bounded symmetric domain $\mathcal{D}$ in a finite dimensional complex vector space,[25] and $G/Z(G)$ is the identity component of the group of biholomorphic automorphisms of $\mathcal{D}$ (see [He78, FK94]). If $\mathcal{D}$ is biholomorphic to a *tube domain* $\mathbb{R}^d + i\Omega \subseteq \mathbb{C}^d$, where $\Omega \subseteq \mathbb{R}^d$ is an open convex cone, then we say that $\mathfrak{g}$ is of *tube type*.

The tube type algebras are in one-to-one correspondence with the simple euclidean Jordan algebras that arise as $\mathfrak{g}_1$ for a 3-grading (i.e., a $\mathbb{Z}$-grading supported in degrees $-1, 0$ and $1$)

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \qquad \text{where} \qquad [\mathfrak{g}_j, \mathfrak{g}_k] \subseteq \mathfrak{g}_{j+k}$$

(cf. Rows 7-11 in Table 1 in Section 2.3). To understand the geometric significance of this grading, we first consider the maximal parabolic subgroup

$$Q := \{g \in G \colon \operatorname{Ad}(g)\mathfrak{q} = \mathfrak{q}\} \qquad \text{with Lie algebra} \qquad \mathfrak{q} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_0,$$

specified by the 3-grading and the associated (minimal) flag manifold $M := G/Q$. Then $Z(G) \subseteq Q$ acts trivially on $M$ and we obtain an inclusion of the adjoint group $\operatorname{Ad}(G) \cong G/Z(G) \hookrightarrow \operatorname{Diff}(M)$ and a realization of the Lie algebra $\mathfrak{g}$ by vector fields on $M$. The map

$$\iota \colon \mathfrak{g}_1 \to G/Q, \quad v \mapsto \exp(v)Q$$

[25] This means that in each point $p$ there exists a point symmetry, i.e., a biholomorphic map for which $p$ is an isolated fixed point.

is an open embedding with dense range, and in this chart of $M$, elements of $\mathfrak{g}_j$ correspond to homogeneous vector fields of degree $1-j$, $j=-1,0,1$. Here the abelian subgroup $G_1 := \exp(\mathfrak{g}_1) \cong (\mathfrak{g}_1,+)$ acts by translations and

$$G_0 := \{g \in Q\colon \operatorname{Ad}(g)\mathfrak{g}_0 = \mathfrak{g}_0\}$$

by linear maps. The flows generated by vector fields in $\mathfrak{g}_{-1}$ do not leave the open subset $\iota(\mathfrak{g}_1)$ invariant.

The space $\mathfrak{g}_1$ can be identified with a euclidean Jordan algebra V (see the appendix to Section 3.4 and [FK94] for details), and from this perspective $\mathfrak{g}$ can be identified with the Lie algebra of "conformal vector fields" on V. Accordingly, $\operatorname{Ad}(G)$ is the identifiy component of the conformal group and $M = G/Q$ is called the conformal compactification of V.

If $V = \mathbb{R}^{1,n-1}$ is $n$-dimensional Minkowski space, then $G = \mathrm{SO}_{2,n}(\mathbb{R})_e$[26] and

$$G/Q \cong (\mathbb{S}^1 \times \mathbb{S}^{n-1})/\{\pm \mathbf{1}\}.$$

For $n=4$, this space can be identified with the group $\mathrm{U}_2(\mathbb{C})$, on which $G = \mathrm{SU}_{2,2}(\mathbb{C})$ acts by Möbius transformations.

### 7.2.4 The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$

The simplest hermitian Lie algebra are the trace-free $2\times 2$ matrices $\mathfrak{g} = \mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{R})$, the Lie algebra of the group $G = \mathrm{SL}_2(\mathbb{R})$. It has a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfying the relations

$$[h,e] = e, \quad [h,f] = -f, \quad \text{and} \quad [e,f] = 2h. \tag{172}$$

In the basis $(e,f,h)$, the Killing form is given by the matrix

$$\kappa = \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The Cartan involution $\theta(a) = -a^\top$ gives rise to the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ given by the skew-symmetric and symmetric matrices

$$\mathfrak{k} = \mathfrak{so}_2(\mathbb{R}) = \operatorname{span}\{e-f\}, \qquad \mathfrak{p} = \mathfrak{sl}_2 \cap \operatorname{Herm}_2(\mathbb{R}) = \operatorname{span}\{h, e+f\}$$

The abelian subalgebra $\mathfrak{z}(\mathfrak{k}) = \mathfrak{k} = \mathfrak{so}_2(\mathbb{R})$ is generated by the element

[26] Recall that $H_e$ denotes the identity component of $H$.

$$z_{\mathfrak{k}} = \frac{1}{2}(e-f) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{173}$$

for which $\operatorname{ad} z_{\mathfrak{k}}$ defines on $\mathfrak{p}$ the complex structure

$$\operatorname{ad} z_{\mathfrak{k}}(h) = -\frac{e+f}{2}, \quad \operatorname{ad} z_{\mathfrak{k}}(e+f) = 2h.$$

The form $-\kappa$ is of signature $(1,2)$, which implies that

$$\operatorname{Ad}(G) \cong \mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{SO}_{1,2}(\mathbb{R})_e$$

and that we may consider $\mathfrak{sl}_2(\mathbb{R})$ as Minkowski space $\mathbb{R}^{1,2}$, on which the Lorentz group $\mathrm{SO}_{1,2}(\mathbb{R})$ acts. In this context

$$C_{\mathfrak{sl}_2(\mathbb{R})} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a^2 + bc \leq 0, b \geq c \right\} \tag{174}$$

is a pointed generating closed convex invariant cone. It contains $z_{\mathfrak{k}}$, and the only other invariant cone of this type is $-C$, containing $-z_{\mathfrak{k}}$.

The subalgebra $\mathfrak{k}$ generates the compact subgroup $K = \mathrm{SO}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{R})$. Note that $G$ has center $Z(G) = \{\pm \mathbf{1}\}$, so that $G \to \operatorname{Ad}(G) = G/Z(G)$ and $K \to \operatorname{Ad}(K)$ are 2–1 covers. Since $G$ acts transitively on the upper half plane $\mathbb{C}_+ = \mathbb{R} + i\mathbb{R}_+$ by Möbius transformations via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d},$$

with the stabilizer of $i$ equal to $\mathrm{SO}_2(\mathbb{R})$, we have

$$G/K = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \cong \mathbb{C}_+ \quad \text{and} \quad G/Z(G) = \mathrm{PSL}_2(\mathbb{R}) \cong \operatorname{Aut}(\mathbb{C}_+) = \operatorname{Aut}(\mathbb{C}_+)_e,$$

where $\operatorname{Aut}(\mathbb{C}_+)$ denotes the group of biholomorphisms of $\mathbb{H}$. As $\mathbb{C}_+ = \mathbb{R} + i\mathbb{R}_+$ is a symmetric tube domain, we see that $\mathfrak{sl}_2$ is of tube type. Here $\mathfrak{g}_1 \cong \mathbb{R}$ is the one-dimensional Jordan algebra, with corresponding 3-grading

$$\mathfrak{sl}_2(\mathbb{R}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathbb{R}f \oplus \mathbb{R}h \oplus \mathbb{R}e,$$

and the stabilizer of 0 for the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R} \cup \{\infty\}$ is the parabolic subgroup

$$Q = \left\{ \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} : a \neq 0 \right\}, \qquad G/Q \cong \mathbb{S}^1.$$

The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ here corresponds to the 3-dimensional space of polynomial vector fields on $\mathbb{R}$ of degree $\leq 2$ via $E = z^2\partial_z$, $F = -\partial_z$, $H = z\partial_z$.

The restriction of the Killing form to $\mathfrak{p}$ descends to a $\mathrm{PSL}_2(\mathbb{R})$-invariant metric on $\mathbb{C}_+$, which turns out to be twice the hyperbolic metric.

### 7.2.5 Jordan decomposition in semisimple Lie algebras

An important tool in the structure theory of real semisimple Lie algebras is the real Jordan decomposition. If $A \in \mathrm{End}(V)$ is an endomorphism of the finite-dimensional real vector space $V$, then there exists a unique additive decomposition

$$A = A_h + A_e + A_n,$$

where any two summands commute, $A_n$ is nilpotent, $A_h$ is *hyperbolic*, i.e., diagonalizable over $\mathbb{R}$, and $A_e$ is *elliptic*, i.e., semisimple with purely imaginary eigenvalues. Here $A = A_s + A_n$ with $A_s = A_h + A_e$ is the Jordan decomposition of $A$, where $A_s$ is semisimple, $A_n$ is nilpotent and $[A_s, A_n] = 0$ ([HN12, Def. 5.3.6]). As the complex linear extension $A_{s,\mathbb{C}}$ to $V_{\mathbb{C}}$ is diagonalizable, it has a natural decomposition $A_{s,\mathbb{C}} = X + iY$, where $X$ corresponds to the real part of the eigenvalues and $Y$ to their imaginary parts. Now $X$ and $iY$ preserve the real subspace $V \subseteq V_{\mathbb{C}}$ and $A_h$ and $A_e$ are determined by $A_{h,\mathbb{C}} = X$ and $A_{e,\mathbb{C}} = iY$. The uniqueness of the decomposition follows immdiately from the uniqueness of the corresponding decomposition for $A_{\mathbb{C}}$ on $V_{\mathbb{C}}$.

To apply this to semisimple real Lie algebras, the main observation is that, for any $x \in \mathfrak{g}$, all three Jordan components $(\mathrm{ad}\,x)_h$, $(\mathrm{ad}\,x)_e$ and $(\mathrm{ad}\,x)_n$ are derivations of $\mathfrak{g}$.

**Lemma 7.10** *Let $\mathfrak{g}$ be a semisimple real Lie algebra. Then, for any $x \in \mathfrak{g}$, there exist uniquely determined elements $x_h, x_e, x_n \in \mathfrak{g}$ such that $x = x_h + x_e + x_n$, the summands pairwise commute, and*

$$\mathrm{ad}(x_h) = (\mathrm{ad}\,x)_h, \quad \mathrm{ad}(x_e) = (\mathrm{ad}\,x)_e \quad \textit{and} \quad \mathrm{ad}(x_n) = (\mathrm{ad}\,x)_n. \qquad (175)$$

***Proof*** By [HN12, Prop. 5.3.10], the nilpotent and semisimple Jordan components $(\mathrm{ad}\,x)_n$ and $(\mathrm{ad}\,x)_s$ or the derivation $\mathrm{ad}\,x$ are derivations. To see that $(\mathrm{ad}\,x)_h$ also is a derivation, we first observe that, for eigenvalues $\lambda, \mu \in \mathbb{C}$ of $(\mathrm{ad}\,x)_s$, we have

$$[\mathfrak{g}_{\mathbb{C}}^{\lambda}((\mathrm{ad}\,x)_s), \mathfrak{g}_{\mathbb{C}}^{\mu}((\mathrm{ad}\,x)_s)] \subseteq \mathfrak{g}_{\mathbb{C}}^{\lambda+\mu}((\mathrm{ad}\,x)_s),$$

because $(\mathrm{ad}\,x)_s$ is a derivation. For $\lambda = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$, we have

$$\mathfrak{g}_{\mathbb{C}}^{\alpha}((\mathrm{ad}\,x)_h) = \bigoplus_{\mathrm{Re}\,\lambda = \alpha} \mathfrak{g}_{\mathbb{C}}^{\lambda}((\mathrm{ad}\,x)_s),$$

which immediately leads to $[\mathfrak{g}_{\mathbb{C}}^{\alpha_1}((\mathrm{ad}\,x)_h), \mathfrak{g}_{\mathbb{C}}^{\alpha_2}((\mathrm{ad}\,x)_h)] \subseteq \mathfrak{g}_{\mathbb{C}}^{\alpha_1+\alpha_2}((\mathrm{ad}\,x)_h)$. This implies that $(\mathrm{ad}\,x)_h$ also is a derivation, and finally $(\mathrm{ad}\,x)_e = (\mathrm{ad}\,x)_s - (\mathrm{ad}\,x)_h$ is a derivation.

Next we use that $\mathrm{der}(\mathfrak{g}) = \mathrm{ad}\,\mathfrak{g}$ ([HN12, Thm. 5.5.14]) to find uniquely determined elements $x_h$, $x_e$ and $x_n$ of $\mathfrak{g}$ satisfying (175). We thus obtain a decomposition $x = x_h + x_e + x_n$, where the summands commute pairwise, $x_h$ is hyperbolic, $x_e$ is elliptic and $x_n$ is nilpotent. □

The uniqueness properties of this decomposition imply in particular that, for any automorphism $\varphi \in \mathrm{Aut}(\mathfrak{g})$, we have

$$\varphi(x_h) = \varphi(x)_h, \quad \varphi(x_e) = \varphi(x)_e \quad \text{and} \quad \varphi(x_n) = \varphi(x)_n.$$

**Proposition 7.11** *If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition as in (170), then*

- *$x \in \mathfrak{g}$ is elliptic if and only if its adjoint orbit $\mathcal{O}_x = \mathrm{Inn}(\mathfrak{g}).x$ intersects $\mathfrak{k}$.*
- *$x \in \mathfrak{g}$ is hyperbolic if and only if $\mathcal{O}_x$ intersects $\mathfrak{p}$.*

***Proof*** The form $B(x, y) = -\kappa(x, \theta y)$ is positive definite. If $x \in \mathfrak{k}$, then $(\mathrm{ad}\, x)^\top = -\mathrm{ad}\, x$ with respect to $B$, so that $x$ is elliptic, and if $x \in \mathfrak{p}$, then $(\mathrm{ad}\, x)^\top = \mathrm{ad}\, x$ implies that $x$ is hyperbolic. Conversely, each elliptic element in $\mathfrak{g}$ is conjugate under inner automorphisms to one of $\mathfrak{k}$ ([HN12, Thm. 14.2.7]), and every hyperbolic elements to one in $\mathfrak{p}$ ([KN96, Thm. II.9]). ☐

## 7.3 Polar maps

This is an appendix from [NÓ23b] which discusses polar maps associated to an involution on a symmetric space, resp., to a pair of commuting involutions on a Lie group. Key properties are collected in Lemma 7.15. These results are used to obtain crown domains for Lie groups, and real and complex Olshanski semigroups. They are also relevant for the description of the structure crown domains in complex homogeneous spaces.

### 7.3.1 Some spectral theory

Let $V$ be a finite dimensional complex vector space and $A \in \mathrm{End}(V)$. For a complex power series $f(z) = \sum_{n=0}^\infty a_n z^n$ that converges for all $z \in \mathbb{C}$, we put

$$f(A) := \sum_{n=0}^\infty a_n A^n \in \mathrm{End}(V).$$

In this sense $\frac{\sinh(A)}{A} = \sum_{n=0}^\infty \frac{A^{2n}}{(2n+1)!}$ is defined below.

**Lemma 7.12** $\ker\left(\frac{\sinh(A)}{A}\right) = \bigoplus_{n \neq 0} \ker(A^2 + n^2\pi^2\mathbf{1})$.

***Proof*** We may w.l.o.g. assume that $V$ is complex. Then $B := \frac{\sinh(A)}{A}$ is invertible on all generalized eigenspace corresponding to eigenvalues $\lambda \neq \pi n i$, $n \in \mathbb{Z}\setminus\{0\}$. We may therefore assume that $V$ has only one eigenvalue $\lambda = n\pi i$, $n \neq 0$. Then $A$ is invertible, so that

$$\ker(B) = \ker(\sinh(A)) = \ker(e^A - e^{-A}) = \ker(e^{2A} - \mathbf{1}).$$

Writing $A = A_s + A_n$ for the Jordan decomposition of $A$, it follows that

$$e^{2A} = e^{2A_s} e^{2A_n} = e^{2n\pi i} e^{2A_n} = e^{2A_n}.$$

As $\ker(e^{2A_n} - \mathbf{1}) = \ker(A_n)$ follows from $2A_n = \log(e^{2A_n})$ as a polynomial in $e^{2A_n} - 1$, we see that $\ker(B) = \ker(A_n)$ is the $\lambda$-eigenspace of $A$. □

With similar arguments, or by replacing $A$ by $A - \frac{\pi i}{2}\mathbf{1}$, we get:

**Lemma 7.13** $\ker(\cosh(A)) = \ker(e^{-2A} + \mathbf{1}) = \bigoplus_{n \in \mathbb{N}} \ker\Big(A^2 + \big(n + \frac{1}{2}\big)^2 \pi^2 \mathbf{1}\Big)$.

### 7.3.2 Fine points on polar maps

In this subsection, we consider two commuting involutions $\sigma$ and $\tau$ on a connected, not necessarily reductive, Lie group $G$ and an open $\sigma$-invariant subgroup $H \subseteq G^\tau$. We shall study the polar map

$$\Phi\colon G^\sigma \times_{H^\sigma} \mathfrak{q}^{-\sigma} \to M, \quad [g, x] \mapsto g.\operatorname{Exp}_{eH}(x) = g \exp(x) H \in G/H \tag{176}$$

and its applications.

As $H$ is invariant under $\tau$ and $\sigma$, both define commuting involutions on $M$ and their fixed point manifolds intersect transversally in $eH$. The map

$$\Psi\colon G^\sigma \times_{H^\sigma} \mathfrak{q}^{-\sigma} \to N(G^\sigma/H^\sigma), \qquad [g, x] \mapsto g.x$$

is a diffeomorphism onto the *normal bundle* $N(G^\sigma/H^\sigma)$ of the subspace $G^\sigma.eH \cong G^\sigma/H^\sigma$ and $\Phi = \operatorname{Exp} \circ \Psi$, where $\operatorname{Exp}\colon T(M) \to M$ is the exponential map.

First, we determine the regular points of $\Phi$. As $\Phi$ is $G^\sigma$-equivariant, it suffices to determine for which points $[e, x]$ the tangent map $T_{[e,x]}(\Phi)$ is injective, hence bijective for dimensional reasons. In the following calculation, we shall use the formula

$$T_x(\operatorname{Exp}_{eH})y = \exp(x).\frac{\sinh(\operatorname{ad} x)}{\operatorname{ad} x} y \quad \text{for} \quad x, y \in \mathfrak{q} \tag{177}$$

for the differential of Exp ([DN93, Lemma 4.6]), where

$$\mathfrak{q} \to T_{\operatorname{Exp}_{eH}(x)}, \quad v \mapsto \exp x.v,$$

is the linear isomorphism induced by the action of $\exp x \in G$ on $M$. For $a \in \mathfrak{g}^\sigma$ and $x, b \in \mathfrak{q}^{-\sigma}$, we obtain

$$\begin{aligned} T_{[e,x]}(\Phi)(a, b) &= a.\operatorname{Exp}(x) + T_x(\operatorname{Exp}_{eH})(b) \\ &= \exp(x).\Big(p_{\mathfrak{q}}(e^{-\operatorname{ad} x} a) + \frac{\sinh(\operatorname{ad} x)}{\operatorname{ad} x} b\Big) \end{aligned} \tag{178}$$

Note that $e^{-\operatorname{ad} x}a = \cosh(\operatorname{ad} x)a - \sinh(\operatorname{ad} x)a$. If $a \in \mathfrak{h}^\sigma$ then

$$p_{\mathfrak{q}}(e^{-\operatorname{ad} x}a) = -\sinh(\operatorname{ad} x)a,$$

and if $a \in \mathfrak{q}^\sigma$, then $p_{\mathfrak{q}}(e^{-\operatorname{ad} x}a) = \cosh(\operatorname{ad} x)a$. Writing $a = a_{\mathfrak{h}} + a_{\mathfrak{q}}$ with $a_{\mathfrak{h}} \in \mathfrak{h}^\sigma$ and $a_{\mathfrak{q}} \in \mathfrak{q}^\sigma$, we thus obtain

$$T_{[e,x]}(\Phi)(a,b) = \exp(x).\Big(\underbrace{\cosh(\operatorname{ad} x)a_{\mathfrak{q}}}_{\in \mathfrak{q}^\sigma} + \underbrace{\frac{\sinh(\operatorname{ad} x)}{\operatorname{ad} x}b - \sinh(\operatorname{ad} x)a_{\mathfrak{h}}}_{\in \mathfrak{q}^{-\sigma}}\Big). \tag{179}$$

The following lemma provides a characterization of the regular points.

**Lemma 7.14** *For $x \in \mathfrak{q}$, the following assertions hold:*

(a) $\operatorname{Exp}_{eH}$ *is regular in $x$ if and only if the map $\frac{\sinh(\operatorname{ad} x)}{\operatorname{ad} x} \colon \mathfrak{q} \to \mathfrak{q}$ is invertible, which is equivalent to*

$$\operatorname{Spec}(\operatorname{ad} x|_{\mathfrak{q}_L}) \cap \mathbb{Z}\pi i \subseteq \{0\}, \quad \textit{where} \quad \mathfrak{q}_L := \mathfrak{q} + [\mathfrak{q},\mathfrak{q}]. \tag{180}$$

(b) *If $\operatorname{Exp}_{eH}|_{\mathfrak{q}^{-\sigma}}$ is regular in $x \in \mathfrak{q}^{-\sigma}$, then the polar map $\Phi$ in (177) is regular in $[g,x]$ if and only if, in addition, $\cosh(\operatorname{ad} x) \colon \mathfrak{q}^\sigma \to \mathfrak{q}^\sigma$ is invertible, which is equivalent to*

$$\operatorname{Spec}(\operatorname{ad} x|_{\mathfrak{q}_L}) \cap \Big(\frac{\pi}{2} + \mathbb{Z}\pi\Big)i = \emptyset. \tag{181}$$

***Proof*** (a) follows from the spectral theoretic description of the kernel of $\frac{\sinh(\operatorname{ad} x)}{\operatorname{ad} x}\big|_{\mathfrak{q}}$ as the intersection of $\mathfrak{q}$ with the sum of the eigenspaces of $\operatorname{ad} x$ in $\mathfrak{g}_{\mathbb{C}}$ for the eigenvalues $\lambda \in \pi i\mathbb{Z} \setminus \{0\}$ (Lemma 7.12).
(b) Suppose that the restriction of $\operatorname{Exp}_{eH}$ to $\mathfrak{q}^{-\sigma}$ is regular, i.e., that

$$\frac{\sinh(\operatorname{ad} x)}{\operatorname{ad} x} \colon \mathfrak{q}^{-\sigma} \to \mathfrak{q}^{-\sigma}$$

is invertible. Then (179) shows that $\Phi$ is regular in $[e,x]$ if and only if

$$\cosh(\operatorname{ad} x) \colon \mathfrak{q}^\sigma \to \mathfrak{q}^\sigma$$

is invertible, and this is equivalent to the condition on $\operatorname{Spec}(\operatorname{ad} x|_{\mathfrak{q}_L})$ stated in (b). □

The following lemma contains a wealth of information on singular points of the polar map $\Phi$.

**Lemma 7.15** *Let $\Omega \subseteq \mathfrak{q}^{-\sigma}$ be an open $H^\sigma$-invariant subset consisting of* $\operatorname{Exp}$*-regular elliptic elements, and consider the polar map*

$$\Phi \colon G_e^\sigma \times_{H^\sigma} \Omega \to M, \quad [g,y] \mapsto g.\operatorname{Exp}_{eH}(y).$$

*Let* $x \in \Omega$*, write* $m := \mathrm{Exp}_{eH}(x) \in M = G/H$*,* $\mathcal{O}_m := G_e^\sigma.m$ *for its orbit, and put*

$$\sigma_x := e^{-2\,\mathrm{ad}\,x}, \qquad \zeta_x := e^{-\,\mathrm{ad}\,x} \in \mathrm{Aut}(\mathfrak{g}).$$

*Then the following assertions hold:*

(a) $\mathfrak{g}^{-\sigma_x} := \{y \in \mathfrak{g} \colon \sigma_x(y) = -y\} = \ker(\cosh(\mathrm{ad}\,x))$.
(b) $\mathfrak{q}^{\sigma,-\sigma_x}$ *complements the subspace*

$$\exp(-x).\,\mathrm{im}(T_{[e,x]}(\Phi)) = \cosh(\mathrm{ad}\,x)\mathfrak{q}^\sigma \oplus \mathfrak{q}^{-\sigma}$$

*in* $\mathfrak{q}$*.*
(c) *The eigenspaces* $\mathfrak{g}^{\pm\sigma_x}$ *are* $\tau$*-invariant, and on the Lie subalgebra* $\mathfrak{g}^{\sigma_x^2} = \mathfrak{g}^{\sigma_x} \oplus \mathfrak{g}^{-\sigma_x}$*, the involution* $\tau$ *commutes with* $\sigma_x$*.*
(d) *The eigenspaces* $\mathfrak{g}^{\pm\sigma_x}$ *are* $\zeta_x$*-invariant, on* $\mathfrak{g}^{\sigma_x}$ *the automorphisms* $\zeta_x$ *and* $\tau$ *commute, and on on* $\mathfrak{g}^{-\sigma_x}$ *the complex structure* $\zeta_x$ *and* $\tau$ *anticommute. In particular, we have*

$$\zeta_x(\mathfrak{h}^{\sigma_x}) = \mathfrak{h}^{\sigma_x}, \quad \zeta_x(\mathfrak{q}^{\sigma_x}) = \mathfrak{q}^{\sigma_x}, \quad \zeta_x(\mathfrak{h}^{-\sigma_x}) = \mathfrak{q}^{-\sigma_x}, \quad \zeta_x(\mathfrak{q}^{-\sigma_x}) = \mathfrak{h}^{-\sigma_x}.$$

(e) *The eigenspaces* $\mathfrak{g}^{\pm\sigma_x}$ *are* $\sigma$*-invariant, and on the Lie subalgebra* $\mathfrak{g}^{\sigma_x^2}$*, the involution* $\sigma$ *commutes with* $\sigma_x$*. On* $\mathfrak{g}^{\sigma_x}$*, the automorphisms* $\zeta_x$ *and* $\sigma$ *commute, and on on* $\mathfrak{g}^{-\sigma_x}$ *the complex structure* $\zeta_x$ *anticommute with* $\sigma$*. In particular, we have*

$$\zeta_x(\mathfrak{g}^{\sigma,\sigma_x}) = \mathfrak{g}^{\sigma,\sigma_x}, \quad \zeta_x(\mathfrak{g}^{-\sigma,\sigma_x}) = \mathfrak{g}^{-\sigma,\sigma_x}, \quad \zeta_x(\mathfrak{g}^{\pm\sigma,-\sigma_x}) = \mathfrak{g}^{\mp\sigma,-\sigma_x}.$$

(f) *The stabilizer Lie algebra of* $m$ *in* $\mathfrak{g}$ *is*

$$\mathfrak{g}_m = \mathfrak{g}^{\tau\sigma_x} = \{y \in \mathfrak{g} \colon \sigma_x(y) = \tau(y)\} \quad \text{and} \quad \mathfrak{g}_m^{\sigma_x^2} = \mathfrak{h}^{\sigma_x} \oplus \mathfrak{q}^{-\sigma_x}.$$

*The stabilizer Lie algebra in* $\mathfrak{g}^\sigma$ *is*

$$\mathfrak{g}_m^\sigma = \mathfrak{h}^{\sigma,\sigma_x} \oplus \mathfrak{q}^{\sigma,-\sigma_x}. \tag{182}$$

*The stabilizer group* $G_m$ *acts on* $T_m(M) = \exp x.\mathfrak{q}$ *by* $g.(\exp x.y) = \exp x.\big(\mathrm{Ad}(\zeta_x^G(g))y\big)$*.*
(g) *The tangent space of the orbit* $\mathcal{O}_m$ *is*

$$T_m(\mathcal{O}_m) = \exp(x).\big(\cosh(\mathrm{ad}\,x)\mathfrak{q}^\sigma + [x, \mathfrak{h}^\sigma]\big).$$

(h) *Let* $\mathfrak{q}_x := \mathfrak{q}^{\sigma,-\sigma_x} + \mathfrak{q}^{-\sigma,\sigma_x}$ *and* $\mathfrak{h}_x := [\mathfrak{q}_x, \mathfrak{q}_x]$*. If the group* $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}_x)$ *acts as a relatively compact group on* $\mathfrak{q}_x$ *and* $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{h}_x)\mathfrak{q}^{-\sigma,\sigma_x} = \mathfrak{q}_x$*, then* $m$ *is an interior point of* $\mathrm{im}(\Phi)$*.*

***Proof*** (a) follows directly from $2\cosh(\mathrm{ad}\,x) = e^{\mathrm{ad}\,x} + e^{-\,\mathrm{ad}\,x}$.
(b) With (178) we see that the image of $T_{[e,x]}(\Phi)$ is the subspace

$$\exp(x).\Big(\underbrace{\cosh(\operatorname{ad} x)\mathfrak{q}^\sigma}_{\subseteq\mathfrak{q}^\sigma} + \underbrace{\frac{\sinh(\operatorname{ad} x)}{\operatorname{ad} x}\mathfrak{q}^{-\sigma}}_{\subseteq\mathfrak{q}^{-\sigma}}\Big).$$

As $\operatorname{ad} x$ is semisimple, a complement of $\exp(-x).\operatorname{im}(T_{[e,x]}(\Phi))$ in $\mathfrak{q}$ is

$$\ker\big(\cosh(\operatorname{ad} x)|_{\mathfrak{q}^\sigma}\big) = \mathfrak{q}^{\sigma,-\sigma_x}. \tag{183}$$

(c) In view of $\tau\sigma_x^2\tau = \sigma_x^{-2}$, the fixed point space $\mathfrak{g}^{\sigma_x^2}$ is $\tau$-invariant. On this subspace $\sigma_x = \sigma_x^{-1} = \tau\sigma_x\tau$, so that $\tau$ preserves the two eigenspaces $\mathfrak{g}^{\pm\sigma_x}$ of $\sigma_x$ on $\mathfrak{g}^{\sigma_x^2}$.
(d) As $\zeta_x$ commutes with $\sigma_x$, the eigenspaces $\mathfrak{g}^{\pm\sigma_x}$ are $\zeta_x$-invariant. Further, (c) implies that, on $\mathfrak{g}^{\sigma_x}$, we have $\tau\zeta_x\tau = \zeta_x^{-1} = \zeta_x$.
(e) is shown with similar arguments as (c) and (d).
(f) In $M = G/H$, the point $m = \operatorname{Exp}_{eH}(\exp x) = \exp xH$ is obtained by acting with $\exp x$ on the base point $eH$. Therefore its stabilizer group is $G_m = \exp xH\exp(-x)$ with the Lie algebra

$$\mathfrak{g}_m = e^{\operatorname{ad} x}\mathfrak{h} = \operatorname{Fix}(e^{\operatorname{ad} x}\tau e^{-\operatorname{ad} x}) = \operatorname{Fix}(\tau e^{-2\operatorname{ad} x}) = \operatorname{Fix}(\tau\sigma_x).$$

Now the $\tau$-invariance of $\mathfrak{g}^{\sigma_x^2}$ implies that

$$\mathfrak{g}_m^{\sigma_x^2} = \mathfrak{g}_m \cap \mathfrak{g}^{\sigma_x^2} = \mathfrak{g}^{\sigma_x,\tau} \oplus \mathfrak{g}^{-\sigma_x,-\tau} = \mathfrak{h}^{\sigma_x} \oplus \mathfrak{q}^{-\sigma_x}.$$

To verify (182), let $y = y_\mathfrak{h} + y_\mathfrak{q} \in \mathfrak{g}^\sigma$ with $y_\mathfrak{h} \in \mathfrak{h}^\sigma$ and $y_\mathfrak{q} \in \mathfrak{q}^\sigma$. Then the corresponding vector field $X_y^M$ on $M$ satisfies

$$\begin{aligned} X_y^M(m) = y.m = \exp(x).p_\mathfrak{q}(e^{-\operatorname{ad} x}y) &= \exp(x).\Big(\frac{1}{2}(e^{-\operatorname{ad} x}y - \tau(e^{-\operatorname{ad} x}y)\Big) \\ &= \exp(x).\Big(\frac{1}{2}(e^{-\operatorname{ad} x}y - e^{\operatorname{ad} x}\tau(y))\Big) \\ &= \exp(x).\big(\cosh(\operatorname{ad} x)y_\mathfrak{q} - \sinh(\operatorname{ad} x)y_\mathfrak{h}\big). \end{aligned} \tag{184}$$

Therefore $X_y^M(m) = 0$ is equivalent to

$$0 = \underbrace{\cosh(\operatorname{ad} x)y_\mathfrak{q}}_{\in\mathfrak{q}^\sigma} - \underbrace{\sinh(\operatorname{ad} x)y_\mathfrak{h}}_{\in\mathfrak{q}^{-\sigma}}.$$

Thus both summands have to vanish, which is equivalent to

$$e^{-2\operatorname{ad} x}y_\mathfrak{q} = -y_\mathfrak{q} \quad \text{and} \quad e^{-2\operatorname{ad} x}y_\mathfrak{h} = y_\mathfrak{h}.$$

This implies (182).

To complete the proof of (f), we note that, for $v \in \mathfrak{q}$ and $g \in G_m$, we have

$$g.(\exp(x).y) = \exp(x).\big(\operatorname{Ad}(\zeta_x^G(g))y\big), \tag{185}$$

where $\zeta_x^G(G_m) = H$ acts on $T_{eH}(M) \cong \mathfrak{q}$ by the adjoint representation.
(g) From (184) it follows that

$$\begin{aligned}\exp(-x).T_m(\mathcal{O}_m) &= \cosh(\operatorname{ad} x)\mathfrak{q}^\sigma + \sinh(\operatorname{ad} x)\mathfrak{h}^\sigma \\ &= \cosh(\operatorname{ad} x)\mathfrak{q}^\sigma + \frac{\sinh(\operatorname{ad} x)}{\operatorname{ad} x}[x, \mathfrak{h}^\sigma] \\ &\overset{!}{=} \cosh(\operatorname{ad} x)\mathfrak{q}^\sigma + [x, \mathfrak{h}^\sigma].\end{aligned}$$

Here the last equality follows from $(\operatorname{ad} x)^2\mathfrak{h}^\sigma \subseteq \mathfrak{h}^\sigma$ and the Exp-regularity of $x$, which, by Lemma 7.14, is equivalent to the invertibility of $\frac{\sinh(\operatorname{ad} x}{\operatorname{ad} x}$ on $\mathfrak{q}$.
(h) By (183), a natural complement of

$$\exp(-x).T_m(\mathcal{O}_m) = \cosh(\operatorname{ad} x)\mathfrak{q}^\sigma + [x, \mathfrak{h}^\sigma]$$

in $\mathfrak{q}$ is the subspace

$$\mathfrak{q}_x := \mathfrak{q}^{\sigma,-\sigma_x} \oplus (\mathfrak{q}^{-\sigma} \cap \ker(\operatorname{ad} x)) = \mathfrak{q}^{\sigma,-\sigma_x} \oplus \mathfrak{q}^{-\sigma,\sigma_x},$$

where the last equality follows from

$$\mathfrak{q}^{\sigma_x} = \bigoplus_{n\in\mathbb{Z}} \ker((\operatorname{ad} x)^2 + \pi^2 n^2 \operatorname{id}_{\mathfrak{q}}) \quad \text{and} \quad \operatorname{Spec}(\operatorname{ad} x) \cap \mathbb{Z}\pi i \subseteq \{0\}.$$

As $x$ is $\operatorname{Exp}_{eH}$-regular and

$$\operatorname{Exp}_{eH}(x + y) = \exp(x).\operatorname{Exp}_m(y) \quad \text{for} \quad y \in \mathfrak{q}^{-\sigma} \cap \ker(\operatorname{ad} x) = \mathfrak{q}^{-\sigma,\sigma_x},$$

the subset

$$\Omega := \{y \in \mathfrak{q}_x \colon \operatorname{Exp}_m(\exp(x).y) \in \operatorname{im}(\Phi)\}$$

contains a 0-neighborhood $U_0$ in $\mathfrak{q}^{-\sigma,\sigma_x}$. Now our assumption $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h}_x)\mathfrak{q}^{-\sigma,\sigma_x} = \mathfrak{q}_x$ and the relative compactness of the group $\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h}_x)$ on $\mathfrak{q}_x$ imply that $U_1 := \operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h}_x)U_0$ is a 0-neighborhood in $\mathfrak{q}_x$.

Finally, we obtain from (182)

$$\zeta_x^{-1}(\mathfrak{h}_x) \subseteq \zeta_x^{-1}(\mathfrak{h}^{\sigma,\sigma_x} + \mathfrak{h}^{-\sigma,-\sigma_x}) \overset{(d),(e)}{\subseteq} \mathfrak{h}^{\sigma,\sigma_x} + \mathfrak{q}^{\sigma,-\sigma_x} \overset{(182)}{=} \mathfrak{g}_m^\sigma,$$

so that

$$\begin{aligned}\operatorname{Exp}_m(\exp(x).U_1) &= \operatorname{Exp}_m(\exp(x).\operatorname{Inn}_{\mathfrak{g}}(\mathfrak{h}_x)U_0) \\ &= (G_e^\sigma)_m.\operatorname{Exp}_m(\exp(x).U_0) \subseteq \operatorname{im}(\Phi).\end{aligned}$$

This means that $\Omega$ is a 0-neighborhood in $\mathfrak{q}_x$, and hence that $\operatorname{im}(\Phi)$ is a neighborhood of $m$ because the map $G_e^\sigma \times \mathfrak{q}_x \to M, (g, y) \mapsto g.\operatorname{Exp}_m(\exp(x).y)$ has surjective differential in $(e, 0)$. □

Below we write

$$r_{\mathrm{Spec,im}}(A) := \sup\{|\operatorname{Im}\lambda| \colon \lambda \in \operatorname{Spec}(A)\}$$

for the *imaginary spectral radius* of a linear map $A \in \operatorname{End}(V)$, $\dim V < \infty$.

*Remark 7.16* If $r_{\mathrm{Spec,im}}(\operatorname{ad} x) < \pi/2$, then the polar map $\Phi$ in the preceding lemma is regular in $[e, x]$ (Lemma 7.14). In this case $\ker(\cosh(\operatorname{ad} x)) = \mathfrak{g}^{-\sigma_x}$ is trivial, so that

$$\mathfrak{g}^\sigma_m = \mathfrak{h}^{\sigma,\sigma_x} = \mathfrak{h}^\sigma \cap \ker(\operatorname{ad} x) = \mathfrak{z}_{\mathfrak{h}^\sigma}(x)$$

follows from Lemma 7.15(f).

The next lemma is useful to verify condition (h) in the preceding lemma.

**Lemma 7.17** *Suppose that $(\mathfrak{g}, \tau)$ is a reductive symmetric Lie algebra such that $\mathfrak{q}$ consist of elliptic elements, $\sigma\tau = \tau\sigma$, and that $\mathfrak{a} \subseteq \mathfrak{q}^{-\sigma}$ is a maximal abelian subspace in $\mathfrak{q}$. Then*

$$\mathfrak{q}^\sigma = [\mathfrak{h}^{-\sigma}, \mathfrak{q}^{-\sigma}], \qquad \operatorname{Inn}_{\mathfrak{g}}([\mathfrak{q}, \mathfrak{q}])\mathfrak{q}^{-\sigma} = \mathfrak{q},$$

*and the group $\operatorname{Inn}_{\mathfrak{g}}([\mathfrak{q}, \mathfrak{q}])$ is compact.*

***Proof*** Let $\mathfrak{a} \subset \mathfrak{q}$ be maximal abelian in $\mathfrak{q}$. Then $\mathfrak{a}$ contains $\mathfrak{z}(\mathfrak{q})$. We then note that $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{a} \oplus \mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})$ is $\tau$- and $\sigma$-invariant. Further $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}) \oplus [\mathfrak{a}, \mathfrak{g}]$ implies that $\mathfrak{q} = \mathfrak{a} \oplus [\mathfrak{a}, \mathfrak{h}]$. As

$$[\mathfrak{a}, \mathfrak{h}] = [\mathfrak{a}, \mathfrak{h}^\sigma \oplus \mathfrak{h}^{-\sigma}] \subseteq \mathfrak{q}^{-\sigma} \oplus \mathfrak{q}^\sigma,$$

this leads to

$$\mathfrak{q}^\sigma = [\mathfrak{a}, \mathfrak{h}^{-\sigma}] \subseteq [\mathfrak{q}^{-\sigma}, \mathfrak{h}^{-\sigma}] \subseteq \mathfrak{q}^\sigma.$$

The second assertion follows from $\operatorname{Inn}_{\mathfrak{g}}([\mathfrak{q}, \mathfrak{q}])\mathfrak{a} = \mathfrak{q}$, and the third from the fact that the reductive Lie algebra $\mathfrak{q} + [\mathfrak{q}, \mathfrak{q}]$ (it is an ideal of $\mathfrak{g}$) is compact. □

### 7.3.3 Fibers of the polar map

For the polar map we have to analyze the relation

$$\operatorname{Exp}(x) = g.\operatorname{Exp}(y).$$

Applying the quadratic representation yields

$$\exp(2x) = g\exp(2y)g^\sharp = \exp(2\operatorname{Ad}(g)y)gg^\sharp.$$

For $x, y \in \mathfrak{q}^{-\sigma}$ and $g \in G^\sigma$, we also have $gg^\sharp \in G^\sigma$, so that

$$\exp(4x) = \exp(2x)\sigma(\exp(2x))^{-1} = \exp(4\operatorname{Ad}(g)y).$$

If $x$ and $y$ are sufficiently small (imaginary spectral radius $< \frac{\pi}{4}$), we thus obtain $\mathrm{Ad}(g)y = x$, and thus $gg^\sharp = e$, i.e., $\tau(g) = g$.

See [HN93, Cor. 7.35] for similar arguments.

### 7.3.4 Fibers of Exp

Suppose that $x, y \in \mathfrak{q}$ have the same exponential image $\mathrm{Exp}(x) = \mathrm{Exp}(y)$ in $M$. We further assume that $\mathrm{Spec}(\mathrm{ad}\, x) \cap i\pi\mathbb{Z} \subseteq \{0\}$, so that Exp is regular in $x$. Then we obtain in $G$ the identity

$$\exp(2x) = Q(\mathrm{Exp}\, x) = Q(\mathrm{Exp}\, y) = \exp(2y),$$

and since $\mathrm{Spec}(\mathrm{ad}(2x)) \cap 2\pi i\mathbb{Z} \subseteq \{0\}$, exp is regular in $x$. Therefore [HN12, Lemma 9.2.31] implies that

$$[x, y] = 0 \quad \text{and} \quad \exp(2x - 2y) = e.$$

We conclude that $\exp(x - y) = \exp(y - x)$, which leads to

$$\mathrm{Exp}(y - x) = \tau_M(\mathrm{Exp}(x - y)) = \exp(y - x)H = \exp(x - y)H = \mathrm{Exp}(x - y),$$

so that $\mathrm{Exp}(y - x) \in M^\tau$, and $\mathrm{Exp}(\mathbb{R}(x - y)) \subseteq M$ is a closed geodesic.

We also conclude that $x - y$ is elliptic with $\mathrm{Spec}(\mathrm{ad}(x - y)) \subseteq \pi i\mathbb{Z}$.

**Lemma 7.18** *If* $r_{\mathrm{Spec,im}}(\mathrm{ad}\, x), r_{\mathrm{Spec,im}}(\mathrm{ad}\, y) < \pi$, *then* $\exp(x) = \exp(y)$ *implies that* $x - y \in \mathfrak{z}(\mathfrak{g})$. *If, in addition,* $G$ *is simply connected or* $\mathfrak{g}$ *is semisimple, then* $x = y$.

***Proof*** The preceding discussion implies that $[x, y] = 0$. Now

$$r_{\mathrm{Spec,im}}(\mathrm{ad}(x - y)) < 2\pi$$

leads to $\mathrm{ad}(x - y) = 0$, i.e., to $x - y \in \mathfrak{z}(\mathfrak{g})$. □

## 7.4 From unitary to antiunitary representations

Antiunitary representations are somewhat harder to deal with when it comes to direct integrals. In addition, their restriction to $G$ may have more invariant subspaces. To deal with these issues in the context of standard subspaces, the following lemma is a useful tool.

**Lemma 7.19** (The antiunitary extension) *Let* $(U, \mathcal{H})$ *be a unitary representation of* $G$ *and write* $\overline{\mathcal{H}}$ *for the Hilbert space* $\mathcal{H}$, *endowed with the opposite complex structure. Then the following assertions hold:*

(a) *On* $\widetilde{\mathcal{H}} := \mathcal{H} \oplus \overline{\mathcal{H}}$ *we obtain by* $\widetilde{U}(g) := U(g) \oplus U(\tau_h(g))$ *a unitary representation which extends by* $\widetilde{U}(\tau_h)(v,w) := \widetilde{J}(v,w) := (w,v)$ *to an antiunitary representation of* $G_{\tau_h}$*. The corresponding standard subspace* $\widetilde{\mathtt{V}} := \mathtt{V}(h,\widetilde{U})$ *coincides with the graph*

$$\widetilde{\mathtt{V}} = \Gamma(\Delta^{1/2}), \tag{186}$$

*and its modular operator is* $\widetilde{\Delta} := \Delta \oplus \Delta^{-1}$*.*

(b) *If* $U$ *extends to an antiunitary representation of* $G_{\tau_h}$ *by* $J = U(\tau_h)$ *on* $\mathcal{H}$*, then the following assertions hold:*

(1) $\Phi\colon \mathcal{H}^{\oplus 2} \to \widetilde{\mathcal{H}}, \Phi(v,w) = (v, Jw)$ *is a unitary intertwiner of* $\widetilde{U}$ *and the antiunitary representation* $U^\sharp$ *of* $G_{\tau_h}$ *on* $\mathcal{H}^{\oplus 2}$*, given by*

$$U^\sharp|_G = U^{\oplus 2} \quad \text{and} \quad U^\sharp(\tau_h)(v,w) := J^\sharp(v,w) := (Jw, Jv).$$

(2) *The standard subspace* $\mathtt{V}^\sharp := \mathtt{V}(h, U^\sharp)$ *coincides with the graph* $\Gamma(T_{\mathtt{V}})$ *of the Tomita operator* $T_{\mathtt{V}} = J\Delta^{1/2}$ *of* $\mathtt{V}$*.*

(3) *The antiunitary representation* $\widetilde{U}$ *is equivalent to the antiunitary representation* $U^{\oplus 2}$ *of* $G_{\tau_h}$ *on* $\mathcal{H}^{\oplus 2}$*.*

(4) *If* $A \subseteq G$ *is a subset, then* $\widetilde{\mathtt{V}}_A$ *is cyclic in* $\widetilde{\mathcal{H}}$ *if and only if* $\mathtt{V}_A$ *is cyclic in* $\mathcal{H}$*.*

***Proof*** ([MN24, Lemma 2.22]) (a) The first assertion is a direct verification (cf. [NÓ17, Lemma 2.10]). Since

$$\widetilde{\Delta} = e^{2\pi i \cdot \partial \widetilde{U}(h)} = \Delta \oplus \Delta^{-1},$$

the description of the standard subspace $\widetilde{\mathtt{V}} = \mathrm{Fix}(\widetilde{J}\widetilde{\Delta}^{1/2})$ follows immediately.
(b) (1) Clearly, $\Phi$ is a complex linear isometry that intertwines the antiunitary representation $\widetilde{U}$ with the antiunitary representation $U^\sharp$.

(2) As $\Delta^\sharp = \Phi^{-1}\widetilde{\Delta}\Phi = \Delta \oplus \Delta$, the relation

$$(v,w) = J^\sharp(\Delta^\sharp)^{1/2}(v,w) = (J\Delta^{1/2}w, J\Delta^{1/2}v) = (T_{\mathtt{V}}w, T_{\mathtt{V}}v)$$

is equivalent to $w = T_{\mathtt{V}}v$. Hence $\mathtt{V}^\sharp = \Gamma(T_{\mathtt{V}})$.

(3) As the restrictions of $U^{\oplus 2}$ and $U^\sharp$ to $G$ coincide, [NÓ17, Thm. 2.11] implies their equivalence as antiunitary representations. However, in the present concrete case, it is easy to see an intertwining operator. The matrix

$$A := \frac{1}{2}\begin{pmatrix} (1+i)\mathbf{1} & (1-i)\mathbf{1} \\ (1-i)\mathbf{1} & (1+i)\mathbf{1} \end{pmatrix} \quad \text{with} \quad A^2 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}$$

defines a unitary operator on $\mathcal{H}^{\oplus 2}$, commuting with $U^\sharp(G)$. It satisfies $J^{\oplus 2}AJ^{\oplus 2} = A^* = A^{-1}$, so that

$$AJ^{\oplus 2}A^{-1} = A^2 J^{\oplus 2} = J^\sharp.$$

(4) If $U|_G$ extends to an antiunitary representation $U$ of $G_{\tau_h}$ on $\mathcal{H}$, then (3) implies that $\widetilde{U} \cong U^{\oplus 2}$, and any equivalence $\Psi\colon (\widetilde{U}, \widetilde{\mathcal{H}}) \to (U^{\oplus 2}, \mathcal{H}^{\oplus 2})$ maps $\widetilde{\mathtt{V}}_A$ to $(\mathtt{V} \oplus \mathtt{V})_A = \mathtt{V}_A \oplus \mathtt{V}_A$. Therefore $\widetilde{\mathtt{V}}_A$ is cyclic if and only if $\mathtt{V}_A$ is cyclic in $\mathcal{H}$. □

The following definition extends the classical type of irreducible complex representations to the case where the involution on $G$ is non-trivial. For a unitary representation $(U, \mathcal{H})$, we write $(\overline{U}, \overline{\mathcal{H}})$ for the unitary representation on the complex conjugate space $\overline{\mathcal{H}}$ by $\overline{U}(g) = U(g)$. We observe that, for an antiunitary representation $(U, \mathcal{H})$ of $G_{\tau_h}$, its *commutant*

$$\begin{aligned} U(G_{\tau_h})' &= \{A \in B(\mathcal{H})\colon (\forall g \in G_{\tau_h})\, AU(g) = U(g)A\} \\ &= \{A \in U(G)'\colon U(\tau_h)A = AU(\tau_h)\} \end{aligned}$$

is only a real subalgebra of $B(\mathcal{H})$ because $U(\tau_h)$ is antilinear.

**Definition 7.20** ([NÓ17, Def. 2.12]) Let $(U, \mathcal{H})$ be an irreducible unitary representation of $G$. We say that $U$ is (with respect to $\tau_h$), of

- *real type* if there exists an antiunitary involution $J$ on $\mathcal{H}$ such that $U^\sharp(\tau_h) := J$ extends $U$ to an antiunitary representation $U^\sharp$ of $G_{\tau_h}$ on $\mathcal{H}$, i.e., $JU(g)J = U(\tau_h(g))$ for $g \in G$. Then the commutant of $U^\sharp(G_{\tau_h})$ is $\mathbb{R}$.
- *quaternionic type* if there exists an antiunitary complex structure $I$ on $\mathcal{H}$ satisfying $IU(g)I^{-1} = U(\tau_h(g))$ for $g \in G$. Then $\overline{U} \circ \tau_h \cong U$, $U$ has no extension on the same space, and the antiunitary representation $(\widetilde{U}, \widetilde{\mathcal{H}})$ of $G_{\tau_h}$ with $\widetilde{U}|_G \cong U \oplus (\overline{U} \circ \tau_h)$ is irreducible with commutant $\mathbb{H}$.
- *complex type* if $\overline{U} \circ \tau_h \not\cong U$. This is equivalent to the non-existence of $V \in \mathrm{AU}(\mathcal{H})$ such that $U(\tau_h(g)) = VU(g)V^{-1}$ for all $g \in G$, i.e., to the non-existence of an antiunitary extension of $U$ to $G_{\tau_h}$ on $\mathcal{H}$. Then $(\widetilde{U}, \widetilde{\mathcal{H}})$ is an irreducible antiunitary representation of $G_{\tau_h}$ with commutant $\mathbb{C}$.

*Remark 7.21* (Antiunitary tensor products) Let $G = G_1 \times G_2$ be a product of type I groups and $\tau$ an involutive automorphism of $G$ preserving both factors, i.e., $\tau = \tau_1 \times \tau_2$. We want to describe irreducible antiunitary representations $(U, \mathcal{H})$ of the group $G_\tau = G \rtimes \{\mathrm{id}_G, \tau\}$ using [NÓ17, Thm. 2.11(d)].
(a) The first possibility is that $U|_G$ is irreducible, so that $U(G)' \cong \mathbb{R}$. Then

$$(U|_G, \mathcal{H}) \cong (U_1, \mathcal{H}_1) \otimes (U_2, \mathcal{H}_2)$$

with irreducible unitary representations $(U_j, \mathcal{H}_j)$ of $G_j$ both extending to antiunitary representations $U_j^\sharp$ of $G_j$. Hence both $U_1$ and $U_2$ are of real type.
(b) The second possibility is that $U|_G$ is reducible with $U(G)' \cong \mathbb{C}$ or $\mathbb{H}$, so that

$$U|_G \cong V \oplus (\overline{V} \circ \tau),$$

where $(V, \mathcal{K})$ is an irreducible unitary representation of $G$ of complex or quaternionic type. Now $V = U_1 \otimes U_2$, and thus

$$\mathcal{H} \cong (\mathcal{H}_1 \otimes \mathcal{H}_2) \oplus (\overline{\mathcal{H}_1} \otimes \overline{\mathcal{H}_2}), \quad U|_G \cong (U_1 \otimes U_2) \oplus (\overline{U_1} \circ \tau_1 \otimes \overline{U_2} \circ \tau_2).$$

If $U_j$ is of complex type, then $\overline{U_j} \circ \tau_j \not\cong U_j$ implies that $V$ is of complex type. If both $U_1$ and $U_2$ are of quaternionic type, then $\overline{U_j} \circ \tau_j \cong U_j$ for $j = 1, 2$ implies $\overline{V} \circ \tau \cong V$, so that $V$ is of quaternionic type.

## 7.5 Smooth and analytic vectors

In this appendix we collect some material on distribution vectors and hyperfunction vectors of unitary representations $U \colon G \to \mathrm{U}(\mathcal{H})$.

### 7.5.1 The integrated representation

**Definition 7.22** Let $G$ be a Lie group. We fix a left-invariant Haar measure $\mu_G$ on $G$ and we often write $dg$ for $d\mu_G(g)$. This measure defines on $L^1(G) := L^1(G, \mu_G)$ the structure of a Banach-$*$ algebra by the *convolution product* and

$$(\varphi * \psi)(x) = \int_G \varphi(g)\psi(g^{-1}x)\, d\mu_G(g), \quad \text{and} \quad \varphi^*(g) = \overline{\varphi(g^{-1})}\Delta_G(g)^{-1} \tag{187}$$

is the involution, where $\Delta_G : G \to \mathbb{R}_+$ is the *modular function* determined by

$$\int_G \varphi(y)\, d\mu_G(y) = \int_G \varphi(y^{-1})\Delta_G(y)^{-1}\, d\mu_G(y) \quad \text{and} \tag{188}$$

$$\Delta_G(x) \int_G \varphi(yx)\, d\mu_G(y) = \int_G \varphi(y)\, d\mu_G(y) \quad \text{for} \quad \varphi \in C_c(G). \tag{189}$$

We put $\varphi^\vee(g) = \varphi(g^{-1}) \cdot \Delta_G(g)^{-1}$, so that

$$\int_G \varphi(g)\, d\mu_G(g) = \int_G \varphi^\vee(g)\, d\mu_G(g). \tag{190}$$

The formulas above show that we have two isometric actions of $G$ on $L^1(G)$, given by

$$(\lambda_g f)(x) = f(g^{-1}x) \quad \text{and} \quad (\rho_g f)(x) = f(xg)\Delta_G(g). \tag{191}$$

Note that

$$(\lambda_g f)^* = \rho_g f^* \quad \text{and} \quad (\lambda_g f)^\vee = \rho_g f^\vee. \tag{192}$$

Now let $(U, \mathcal{H})$ be a continuous unitary representation of the Lie group $G$, i.e., a homomorphism $U : G \to \mathrm{U}(\mathcal{H}), g \mapsto U(g)$ such that, for each $\eta \in \mathcal{H}$, the orbit map $U^\eta(g) = U(g)\eta$ is continuous. For $\varphi \in L^1(G)$ the operator-valued integral

$$U(\varphi) := \int_G \varphi(g)U(g)\,dg$$

exists and is uniquely determined by

$$\langle \eta, U(\varphi)\zeta\rangle = \int_G \varphi(g)\langle \eta, U(g)\zeta\rangle\,dg \quad \text{for} \quad \eta,\zeta \in \mathcal{H}. \tag{193}$$

Then $\|U(\varphi)\| \leq \|\varphi\|_1$, and the so-obtained continuous linear map $L^1(G) \to B(\mathcal{H})$ is a representation of the Banach-$*$ algebra $L^1(G)$, i.e., $U(\varphi * \psi) = U(\varphi)U(\psi)$ and $U(\varphi^*) = U(\varphi)^*$. We also note that, for $g \in G$ and $\varphi \in L^1(G)$

$$U(g)U(\varphi) = U(\lambda_g\varphi) \quad \text{and} \quad U(\varphi)U(g) = U(\rho_g^{-1}\varphi). \tag{194}$$

For $\varphi_g(x) := \varphi(xg)$, we then have $\varphi_g = \Delta_G(g)^{-1}\rho_g\varphi$ by (191), and thus by (194)

$$U(\varphi_g) = \Delta_G(g)^{-1}U(\varphi)U(g^{-1}) \quad \text{for} \quad g \in G. \tag{195}$$

### 7.5.2 The space of smooth vectors and its dual

A *smooth vector* is an element $\eta \in \mathcal{H}$ for which the orbit map

$$U^\eta : G \to \mathcal{H}, \quad g \mapsto U(g)\eta$$

is smooth. We write $\mathcal{H}^\infty = \mathcal{H}^\infty(U)$ for the space of smooth vectors. It carries the *derived representation* $\mathtt{d}U$ of the Lie algebra $\mathfrak{g}$ given by

$$\mathtt{d}U(x)\eta = \lim_{t\to 0} \frac{U(\exp tx)\eta - \eta}{t}. \tag{196}$$

For $x \in \mathfrak{g}$, we write $\partial U(x)$ for the infinitesimal generator of the one-parameter group $U(\exp tx)$, so that $U(\exp tx) = e^{t\partial U(x)}$. As $\mathcal{H}^\infty$ is dense and $U(G)$-invariant, $\partial U(x)$ is the closure of the operator $\mathtt{d}U(x)$ ([RS73, Thm. VIII.10]).

We extend the representation $\mathtt{d}U$ to a homomorphism $\mathtt{d}U : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(\mathcal{H}^\infty)$, where $\mathcal{U}(\mathfrak{g})$ is the complex enveloping algebra of $\mathfrak{g}$. This algebra carries an involution $D \mapsto D^*$ determined uniquely by $x^* = -x$ for $x \in \mathfrak{g}$. For $D \in \mathcal{U}(\mathfrak{g})$, we obtain a seminorm on $\mathcal{H}^\infty$ by

$$p_D(\eta) = \|\mathtt{d}U(D)\eta\| \quad \text{for} \quad \eta \in \mathcal{H}^\infty.$$

These seminorms define a topology on $\mathcal{H}^\infty$ which turns the injection

$$\eta \colon \mathcal{H}^\infty \to \mathcal{H}^{\mathcal{U}(\mathfrak{g}_{\mathbb{C}})}, \quad \xi \mapsto (\mathtt{d}U(D)\xi)_{D\in\mathcal{U}(\mathfrak{g}_{\mathbb{C}})} \tag{197}$$

into a topological embedding, where the right-hand side carries the product topology (cf. [Mag92, 3.19]). It turns $\mathcal{H}^\infty$ into a complete locally convex space for which the linear operators $\mathtt{d}U(D)$, $D \in \mathcal{U}(\mathfrak{g})$, are continuous. Since $\mathcal{U}(\mathfrak{g})$ has a countable basis, countably many such seminorms already determine the topology, so that $\mathcal{H}^\infty$ is metrizable. As it is also complete, it is a Fréchet space. We also observe that the inclusion $\mathcal{H}^\infty \hookrightarrow \mathcal{H}$ is continuous.

The space $\mathcal{H}^\infty$ of smooth vectors is $G$-invariant and we denote the corresponding representation by $U^\infty$. We thus obtain a smooth action of $G$ on this Fréchet space ([Ne10]). We have the intertwining relation

$$\mathtt{d}U(\mathrm{Ad}(g)x) = U(g)\mathtt{d}U(x)U(g)^{-1} \quad \text{for} \quad g \in G, x \in \mathfrak{g}. \tag{198}$$

If $\varphi \in C_c^\infty(G)$ and $\xi \in \mathcal{H}$, then $U(\varphi)\xi \in \mathcal{H}^\infty$, and differentiation under the integral sign shows that

$$\mathtt{d}U(x)U(\varphi)\xi := U(-x^R\varphi)\xi, \quad \text{where} \quad (x^R\varphi)(g) = \frac{d}{dt}\Big|_{t=0}\varphi((\exp tx)g). \tag{199}$$

**Definition 7.23** A sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c^\infty(G)$ is called a *$\delta$-sequence* if $\int_G \varphi_n(g)\,dg = 1$ for every $n \in \mathbb{N}$ and, for every $e$-neighborhood $U \subseteq G$, we have $\mathrm{supp}(\varphi_n) \subseteq U$ if $n$ is sufficiently large.

If $(\varphi_n)_{n\in\mathbb{N}}$ is a $\delta$-sequence, then $U(\varphi_n)\xi \to \xi$, so that $\mathcal{H}^\infty$ is dense in $\mathcal{H}$.

We write $\mathcal{H}^{-\infty}$ for the space of continuous antilinear functionals on $\mathcal{H}^\infty$. Its elements are called *distribution vectors*. The group $G$, $\mathcal{U}(\mathfrak{g})$ and $C_c^\infty(G)$ act on $\eta \in \mathcal{H}^{-\infty}$ by

- $(U^{-\infty}(g)\eta)(\xi) := \eta(U(g^{-1})\xi)$, $g \in G, \xi \in \mathcal{H}^\infty$.
  If $U\colon G \to \mathrm{AU}(\mathcal{H})$ is an antiunitary representation and $U(g)$ is antiunitary, then we have to modify this definition slightly by $(U^{-\infty}(g)\eta)(\xi) := \overline{\eta(U(g^{-1})\xi)}$.
- $(\mathtt{d}U^{-\infty}(D)\eta)(\xi) := \eta(\mathtt{d}U(D^*)\xi)$, $D \in \mathcal{U}(\mathfrak{g}), \xi \in \mathcal{H}^\infty$.
- $U^{-\infty}(\varphi)\eta = \eta \circ U^\infty(\varphi^*)$, $\varphi \in C_c^\infty(G)$.

We have natural $G$-equivariant linear embeddings

$$\mathcal{H}^\infty \hookrightarrow \mathcal{H} \xrightarrow{\xi\mapsto\langle\cdot,\xi\rangle} \mathcal{H}^{-\infty}. \tag{200}$$

It is an important feature of (200) that the representation of $\mathcal{U}(\mathfrak{g})$ on $\mathcal{H}^{-\infty}$ provides an embedding of the whole Hilbert space $\mathcal{H}$ into a larger space on which the Lie algebra acts. The following lemma shows that $\mathcal{H}^\infty$ is the maximal $\mathfrak{g}$-invariant subspace of $\mathcal{H} \subseteq \mathcal{H}^{-\infty}$ and that the subspace $\mathcal{H}$ generates $\mathcal{H}^{-\infty}$ as a $\mathfrak{g}$-module.

**Lemma 7.24** *The following assertions hold:*

(a) $\mathcal{H}^\infty = \{\xi \in \mathcal{H} \subseteq \mathcal{H}^{-\infty} \colon (\forall D \in \mathcal{U}(\mathfrak{g}))\ \mathtt{d}U^{-\infty}(D)\xi \in \mathcal{H}\}$.
(b) $\mathcal{H}^{-\infty} = \mathrm{span}\big(\mathtt{d}U^{-\infty}(\mathcal{U}(\mathfrak{g}))\mathcal{H}\big)$.

***Proof*** (a) This follows by combining [Oeh21, Prop. A.1], asserting that

$$\mathcal{D}(\partial U(x)) = \{\xi \in \mathcal{H} \colon \mathtt{d}U^{-\infty}(x)\xi \in \mathcal{H}\},$$

with the fact that

$$\mathcal{H}^\infty = \bigcap \{\mathcal{D}(\partial U(x_1) \cdots \partial U(x_n)) \colon n \in \mathbb{N}, x_1, \ldots, x_n \in \mathfrak{g}\}$$

([Ne10, Lemma 3.4]).
(b) Let $\eta \in \mathcal{H}^{-\infty}$ and consider $\mathcal{H}^\infty$ as a subspace of the topological product $\mathcal{H}^{\mathcal{U}(\mathfrak{g})}$ as in (197). By the Hahn–Banach Extension Theorem, $\eta$ extends to a continuous antilinear functional $\widetilde{\eta}$ on $\mathcal{H}^{\mathcal{U}(\mathfrak{g})}$. Since the dual of a direct product is the direct sum of the dual spaces, there exist $D_1, \ldots, D_n \in \mathcal{U}(\mathfrak{g})$ and $\xi_1, \ldots, \xi_n \in \mathcal{H}$, such that

$$\eta(\xi) = \sum_{j=1}^n \langle \xi_j, \mathtt{d}U(D_j)\xi \rangle = \sum_{j=1}^n \langle \mathtt{d}U^{-\infty}(D_j^*)\xi_j, \xi \rangle \quad \text{for} \quad \xi \in \mathcal{H}^\infty,$$

which means that $\eta = \sum_{j=1}^n \mathtt{d}U^{-\infty}(D_j^*)\xi_j$. □

For each $\varphi \in C_c^\infty(G)$, the map $U(\varphi) \colon \mathcal{H} \to \mathcal{H}^\infty$ is continuous, so that its adjoint defines a weak-$*$ continuous map $U^{-\infty}(\varphi^*) \colon \mathcal{H}^{-\infty} \to \mathcal{H}$. We actually have $U^{-\infty}(\varphi)\mathcal{H}^{-\infty} \subseteq \mathcal{H}^\infty$ as a consequence of the Dixmier–Malliavin Theorem [DM78, Thm. 3.1], which asserts that every $\varphi \in C_c^\infty(G)$ can be written as a finite sum of functions of the form $\varphi_1 * \varphi_2$ with $\varphi_j \in C_c^\infty(G)$.

### 7.5.3 The space of analytic vectors and its dual

In this subsection, we briefly discuss the space of analytic vectors of a unitary representation of a Lie group. Let $(U, \mathcal{H})$ be a unitary representation of the connected real Lie group $G$. We write

$$\mathcal{H}^\omega = \mathcal{H}^\omega(U) \subseteq \mathcal{H}$$

for the space of *analytic vectors*, i.e., those $\xi \in \mathcal{H}$ for which the orbit map $U^\xi \colon G \to \mathcal{H}, g \mapsto U(g)\xi$, is analytic.

To endow $\mathcal{H}^\omega$ with a locally convex topology, we specify subspaces $\mathcal{H}_V^\omega$ by open convex 0-neighborhoods $V \subseteq \mathfrak{g}$ as follows. Let $\eta_G \colon G \to G_\mathbb{C}$ denote the universal complexification of $G$ and assume that $\eta_G$ has discrete kernel (this is always the case if $G$ is semisimple or 1-connected). We assume that $V$ is so small that the map

$$\eta_{G,V} \colon G_V := G \times V \to G_\mathbb{C}, \quad (g, x) \mapsto \eta_G(g)\exp(ix) \qquad (201)$$

is a covering. Then we endow $G_V$ with the unique complex manifold structure for which $\eta_{G,V}$ is holomorphic.

We now write $\mathcal{H}_V^\omega$ for the set of those analytic vectors $\xi$ for which the orbit map $U^\xi \colon G \to \mathcal{H}$ extends to a holomorphic map

$$U_V^\xi \colon G_V \to \mathcal{H}.$$

As any such extension is $G$-equivariant by uniqueness of analytic continuation, it must have the form

$$U_V^\xi(g,x) = U(g)e^{i\cdot\partial U(x)}\xi \quad \text{for} \quad g \in G, x \in V, \tag{202}$$

so that $\mathcal{H}_V^\omega \subseteq \bigcap_{x\in V} \mathcal{D}(e^{i\cdot\partial U(x)})$.

The following lemma shows that we have equality.

**Lemma 7.25** *If $V \subseteq \mathfrak{g}$ is an open convex 0-neighborhood for which (201) is a covering, then $\mathcal{H}_V^\omega = \bigcap_{x\in V} \mathcal{D}(e^{i\cdot\partial U(x)})$.*

***Proof*** ([FNÓ25a, Lemma 1]) It remains to show that each $\xi \in \bigcap_{x\in V} \mathcal{D}(e^{i\cdot\partial U(x)})$ is contained in $\mathcal{H}_V^\omega$. For that, we first observe that the holomorphy of the functions $z \mapsto e^{iz\partial U(x)}v$ on a neighborhood of the closed unit disc in $\mathbb{C}$ implies that the $\mathcal{H}$-valued power series

$$f_\xi(x) := \sum_{n=0}^\infty \frac{i^n}{n!}\partial U(x)^n\xi$$

converges for each $x \in V$. Further, [Go69, Thm. 1.1] implies that $\xi \in \mathcal{H}^\infty$, so that the functions $x \mapsto \partial U(x)^n\xi = \mathtt{d}U(x)^n\xi$ are homogeneous $\mathcal{H}$-valued polynomials (cf. [BS71]). Thus [BS71, Thm. 5.2] shows that the above series defines an analytic function $f_\xi \colon V \to \mathcal{H}$. It follows in particular that $\xi$ is an analytic vector, and the map

$$U_V^\xi \colon G_V \to \mathcal{H}, \quad (g,x) \mapsto U^\xi(g,x) := U(g)e^{i\partial U(x)}\xi$$

is defined. It is clearly equivariant. We claim that it is holomorphic. As it is locally bounded, it suffices to show that, for each $\eta \in \mathcal{H}^\omega$, the function

$$f \colon G_V \to \mathbb{C}, \quad f(g,x) := \langle \eta, U^\xi(g,x)\rangle$$

is holomorphic ([Ne00, Cor. A.III.3]). We have

$$f(g,x) = \langle U(g)^{-1}\eta, e^{i\partial U(x)}\xi\rangle,$$

and the orbit map of $\eta$ is analytic, therefore $f$ is real analytic. Therefore it suffices to show that it is holomorphic on some 0-neighborhood. This follows from the fact that it is $G$-equivariant and coincides on some 0-neighborhood with the local holomorphic extension of the orbit map of $\xi$. Here we use that, for $x, y \in \mathfrak{g}$ sufficiently small, the holomorphic extension $U^\xi$ of the $\xi$-orbit map satisfies

$$U^\xi(\exp(x * iy)) = U(\exp x)U^\xi(\exp iy) = U(\exp x)f_\xi(y) = U^\xi_V(\exp x, y),$$

where $a * b = a + b + \frac{1}{2}[a, b] + \cdots$ denotes the Baker–Campbell–Hausdorff series. □

We topologize the space $\mathcal{H}^\omega_V$ by identifying it with $\mathcal{O}(G_V, \mathcal{H})^G$, the Fréchet space of $G$-equivariant holomorphic maps $F\colon G_V \to \mathcal{H}$, endowed with the Fréchet topology of uniform convergence on compact subsets. Now $\mathcal{H}^\omega = \bigcup_V \mathcal{H}^\omega_V$, and we topologize $\mathcal{H}^\omega$ as the locally convex direct limit of the Fréchet spaces $\mathcal{H}^\omega_V$ (cf. [GN26], [Tr67]). If the universal complexification $\eta_G\colon G \to G_\mathbb{C}$ is injective, we thus obtain the same topology as in [GKS11]. Note that, for any monotone basis $(V_n)_{n\in\mathbb{N}}$ of convex 0-neighborhoods in $\mathfrak{g}$, we then have

$$\mathcal{H}^\omega \cong \varinjlim \mathcal{H}^\omega_{V_n},$$

so that $\mathcal{H}^\omega$ is a countable locally convex limit of Fréchet spaces. As the evaluation maps

$$\mathcal{O}(G_V, \mathcal{H})^G \to \mathcal{H}, \quad F \mapsto F(e, 0)$$

are continuous, the inclusion $\iota\colon \mathcal{H}^\omega \to \mathcal{H}$ is continuous.

We write $\mathcal{H}^{-\omega}$ for the space of continuous antilinear functionals $\eta\colon \mathcal{H}^\omega \to \mathbb{C}$ (called *hyperfunction vectors*) and

$$\langle \cdot, \cdot \rangle\colon \mathcal{H}^\omega \times \mathcal{H}^{-\omega} \to \mathbb{C}$$

for the natural sesquilinear pairing that is linear in the second argument. We endow $\mathcal{H}^{-\omega}$ with the weak-$*$ topology. We then have natural continuous inclusions

$$\mathcal{H}^\omega \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}^{-\omega}.$$

Our specification of the topology on $\mathcal{H}^\omega$ differs from the one [GKS11] because we do not want to assume that the universal complexification $\eta_G\colon G \to G_\mathbb{C}$ is injective, but both constructions define the same topology. Moreover, the arguments in [GKS11] apply with minor changes to general Lie groups.

We actually have the following chain of complex linear embeddings

$$\mathcal{H}^\omega \subseteq \mathcal{H}^\infty \subseteq \mathcal{H} \subseteq \mathcal{H}^{-\infty} \subseteq \mathcal{H}^{-\omega}, \tag{203}$$

where all inclusions are continuous and $G$ acts on all spaces by representations denoted $U^\omega$, $U^\infty$, $U$, $U^{-\infty}$, and $U^{-\omega}$, respectively. These representations can be integrated to the convolution algebra $C^\infty_c(G) := C^\infty_c(G, \mathbb{C})$ of test functions (cf. (187)), for instance

$$U^{-\infty}(\varphi) := \int_G \varphi(g) U^{-\infty}(g)\, dg, \tag{204}$$

where $dg$ stands for a left Haar measure on $G$.

### 7.6 Direct integral techniques

Here we collect some material from [MN24] and [BN25]. We refer to [BR87] for the basics on direct integrals; see also [DD63].

Let $\mathcal{H} = \int_X^\oplus \mathcal{H}_m \, d\mu(m)$ be a direct integral of Hilbert spaces on a standard measure space $(X, \mu)$. We call a closed real subspace $\mathsf{H} \subseteq \mathcal{H}$ *decomposable* if it is of the form

$$\mathsf{H} = \int_X^\oplus \mathsf{H}_m \, d\mu(m), \tag{205}$$

where $(\mathsf{H}_m)_{m \in X}$ is a measurable field of closed real subspaces. Now let $(\mathsf{H}^k)_{k \in K}$ be an at most countable family of decomposable real subspaces. Then we have ([MT19, Lemma B.3]):

(DI1) $\mathsf{H}' = \int_X^\oplus \mathsf{H}'_m \, d\mu(m)$.
(DI2) $\bigcap_{k \in K} \mathsf{H}^k = \int_X^\oplus \bigcap_{k \in K} \mathsf{H}^k_m \, d\mu(m)$ for any at most countable set $K$.
(DI3) $\overline{\sum_k \mathsf{H}^k} = \int_X^\oplus \overline{\sum_k \mathsf{H}^k_m} \, d\mu(m)$.

**Lemma 7.26** *The subspace* $\mathsf{H}$ *as in* (205) *is cyclic/separating/standard if and only if* $\mu$*-almost all* $\mathsf{H}_m$ *have this property.*

***Proof*** (a) First we deal with the separating property. By (DI2) we have

$$\mathsf{H} \cap i\mathsf{H} = \int_X^\oplus (\mathsf{H}_m \cap i\mathsf{H}_m)\, d\mu(m),$$

and this space is trivial if and only if $\mu$-almost all spaces $\mathsf{H}_m \cap i\mathsf{H}_m$ are trivial, which means that $\mathsf{H}_m$ is separating.
(b) The subspace $\mathsf{H}$ is cyclic if and only if $\mathsf{H}'$ is separating. By (DI1) and (a) this means that $\mu$-almost all $\mathsf{H}'_m$ are separating, i.e., that $\mathsf{H}_m$ is cyclic.
(c) By (a) and (b) $\mathsf{H}$ is standard if and only if $\mu$-almost all $\mathsf{H}_m$ are cyclic and separating, i.e., standard. □

**Lemma 7.27** *For a countable family* $(\mathsf{H}^k)_{k \in K}$ *of decomposable cyclic closed real subspaces, the intersection* $\mathtt{V} := \bigcap_{k \in K} \mathsf{H}^k$ *is cyclic if and only if, for* $\mu$*-almost every* $m \in X$*, the subspace* $\mathtt{V}_m := \bigcap_{k \in K} \mathsf{H}^k_m$ *is cyclic.*

***Proof*** By (DI2), we have $\mathtt{V} = \int_X^\oplus \mathtt{V}_m \, d\mu(m)$, so that the assertion follows from Lemma 7.26. □

For a direct integral

$$(U,\mathcal{H}) = \int_X^{\oplus} (U_m,\mathcal{H}_m)\, d\mu(m)$$

of antiunitary representations of $G_{\tau_h}$, the canonical standard subspace $\mathtt{V} = \mathtt{V}(h,U) \subseteq \mathcal{H}$ from (2) is specified by the decomposable operator $J\Delta^{1/2} = U(\tau_h)e^{\pi i\, \partial U(h)}$, hence decomposable:

$$\mathtt{V} = \int_X^{\oplus} \mathtt{V}_m\, d\mu(m). \tag{206}$$

**Lemma 7.28** *Assume that $G$ has at most countably many components. For any subset $A \subseteq G$ and a real subspace $\mathsf{H} \subseteq \mathcal{H}$, we put*

$$\mathsf{H}_A := \bigcap_{g\in A} U(g)\mathsf{H}. \tag{207}$$

*Then the following assertions hold:*

(a) *If $\mathsf{H}$ is decomposable, then $\mathsf{H}_A = \int_X^{\oplus} \mathsf{H}_{m,A}\, d\mu(m)$.*
(b) *$\mathsf{H}_A$ is cyclic if and only if $\mu$-almost all $\mathsf{H}_{m,A}$ are cyclic.*

***Proof*** (a) As $G$ has at most countably many components, it carries a separable metric, so that there exists a countable subset $B \subseteq A$ which is dense in $A$. For $\xi \in \mathcal{H}$, we have

$$\xi \in \mathsf{H}_A \quad \text{if and only if} \quad U(A)^{-1}\xi \subseteq \mathsf{H}.$$

Now the closedness of $\mathsf{H}$ and the density of $B$ in $A$ show that this is equivalent to $U(B)^{-1}\xi \subseteq \mathsf{H}$, i.e., to $\xi \in \mathsf{H}_B$. This shows that $\mathsf{H}_A = \mathsf{H}_B$. We likewise obtain $\mathsf{H}_{m,A} = \mathsf{H}_{m,B}$ for every $m \in X$. Hence the assertion follows by applying (DI2) to the real subspace $\mathsf{H}_B = \mathsf{H}_A$.
(b) follows from (a) and Lemma 7.26. □

**Lemma 7.29** *Let $\mathcal{H} = \int_X^{\oplus} \mathcal{H}_x d\mu(x)$, a direct integral von Neumann algebra $\mathcal{A} = \int_X^{\oplus} \mathcal{A}_x d\mu(x)$ and a strongly continuous, unitary, direct integral representation of a Lie group $G$ with countably many connected components, $(U,\mathcal{H}) = \int_X^{\oplus}(U_x,\mathcal{H}_x)d\mu(x)$. Then, for any subset $N \subset G$, we have*

$$\bigcap_{g\in N} \mathcal{A}_g = \int_X^{\oplus} \bigcap_{g\in N} (\mathcal{A}_g)_x d\mu(x) \quad \textit{where} \quad \mathcal{A}_g = U(g)\mathcal{A}U(g)^*.$$

***Proof*** As $G$ has at most countably many components, it carries a separable metric. Hence there exists a countable subset $N_0 \subseteq N$ which is dense in $N$. For $A \in B(\mathcal{H})$, the map

$$F\colon G \to B(\mathcal{H}), \quad F(g) = U(g)AU(g)^*,$$

is weak operator continuous, so that the set of all $g \in G$ with $F(g) \in \bigcap_{g \in N_0} \mathcal{A}_g$ is a closed subset, hence contains $N$. We conclude that

$$\bigcap_{g \in N_0} \mathcal{A}_g = \bigcap_{g \in N} \mathcal{A}_g.$$

We likewise obtain for every $x \in X$ the relation

$$\bigcap_{g \in N_0} \mathcal{A}_{x,g} = \bigcap_{g \in N} \mathcal{A}_{x,g} \quad \text{for} \quad \mathcal{A}_{x,g} = U_x(g)\mathcal{A}_x U_x(g)^*.$$

From [BR87, Prop. 4.4.6(b)] we thus obtain

$$\bigcap_{g \in N} \mathcal{A}_g = \bigcap_{g \in N_0} \mathcal{A}_g = \int_X^{\oplus} \bigcap_{g \in N_0} \mathcal{A}_{x,g} \, d\mu(x) = \int_X^{\oplus} \bigcap_{g \in N} \mathcal{A}_{x,g} \, d\mu(x).$$

Finally, we observe that, for every $g \in G$

$$\mathcal{A}_g = \int_X^{\oplus} (\mathcal{A}_g)_x \, d\mu(x) = \int_X^{\oplus} \mathcal{A}_{x,g} \, d\mu(x)$$

follows by the uniqueness of the direct integral decomposition. □

### 7.7 Some facts on convex cones

**Lemma 7.30** ([MNÓ23, Lemma B.1]) *Let $E$ be a finite-dimensional real vector space, $C \subseteq E$ a closed convex cone and $E_1 \subseteq E$ a linear subspace. If the interior $C^\circ$ of $C$ intersects $E_1$, then $C^\circ \cap E_1$ coincides with the relative interior $C_1^\circ$ of the cone $C_1 := C \cap E_1$ in $E_1$.*

**Lemma 7.31** *Let $V$ be a finite-dimensional real vector space, $A \in \mathrm{End}(V)$ diagonalizable, and let $C \subseteq V$ be a closed convex cone invariant under $e^{\mathbb{R}A}$. Let $\lambda_{\min}$ and $\lambda_{\max}$ be the minimal/maximal eigenvalues of $A$. For an eigenvalue $\lambda$ of $A$ we write $V_\lambda(A)$ for the corresponding eigenspace and $p_\lambda \colon V \to V_\lambda(A)$ for the projection along all other eigenspaces. Then*

$$p_{\lambda_{\min}}(C) = C \cap V_{\lambda_{\min}}(A) \quad \textit{and} \quad p_{\lambda_{\max}}(C) = C \cap V_{\lambda_{\max}}(A).$$

*If $A$ has only two eigenvalues, it follows that $C = p_{\lambda_{\min}}(C) \oplus p_{\lambda_{\max}}(C)$.*

***Proof*** Since we can replace $A$ by $-A$, it suffices to verify the second assertion. So let $v \in C$ and write it as a sum $v = \sum_\lambda v_\lambda$ of $A$-eigenvectors. Then

$$v_{\lambda_{\max}} = \lim_{t \to \infty} e^{-t\lambda_{\max}} e^{tA} v \in C$$

implies that $p_{\lambda_{\max}}(C) \subseteq C \cap V_{\lambda_{\max}}(A)$, and the other inclusion is trivial. $\square$

**Lemma 7.32** *Let $E$ be a finite-dimensional real vector space and $C \subseteq E$ be a closed convex cone. In the affine group $G := \mathrm{Aff}(E) \cong E \rtimes \mathrm{GL}(E)$, we then have*

$$S_C := \{g \in G\colon gC \subseteq C\} = C \rtimes \{g \in \mathrm{GL}(E)\colon gC \subseteq C\}. \qquad (208)$$

*If $C$ has interior points, then $S_{C^\circ} = S_C$ and $C = \overline{C^\circ}$.*

***Proof*** We write $g = (b, a)$ with $gx = b + ax$. Then $g.C \subseteq C$ implies $b = g.0 \in C$.

Moreover, for the recession cone

$$\lim(C) := \{x \in E\colon x + C \subseteq C\} = \{x \in E\colon (\exists c \in C)\, c + \mathbb{R}_+ x \subseteq C\}$$

([Ne00, Prop. V.1.6]), the relation $g.C \subseteq C$ implies

$$aC = \lim(b + aC) = \lim(g.C) \subseteq \lim(C) = C,$$

and this implies (208).

If $C$ has interior points, then $g.C^\circ \subseteq C^\circ$ and $C = \overline{C^\circ}$ imply $g.C \subseteq C$, so that $S_{C^\circ} \subseteq S_C$. Conversely, $C + C^\circ \subseteq C^\circ$ implies that $S_C \subseteq S_{C^\circ}$. $\square$

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# Index