Relative Positions of Half-sided Modular Inlcusions

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Starting data:

- Quasi-local C^*-algebra $\mathcal{A}_\infty$ of some QFT;
- A time evolution $\alpha_t : \mathcal{A} \to \mathcal{A}$;
- β-KMS state ω_β w.r.t α_t.

By GNS we get a standard subspace $H_\infty \subset \mathcal{H}_\beta$ with known modular group. (For free theory this reduces to 1-particle level)

Hard question

What are the modular groups of $\mathcal{A}(\mathcal{O})$?

Only known example is in (Borchers and Yngvason, 1999), relies heavily on **Half-sided Modular Inclusion**.

Mathematical questions that came up:

- If $K_1 \subset H_\infty$ and $K_2 \subset H_\infty$ such that $K_1 \cap K_2$ is standard, are $K_1 \cap K_2 \subset K_1$ and $K_1 \cap K_2 \subset K_2$ HSMI?
- If $K_1 \subset H_\infty$ and $K_2 \subset H_\infty$ such that $K_1 \subset K_2$, when is $K_1 \subset K_2$ a HSMI? Can we characterize situations where this is the case?
- How does this relate to the associated standard pairs? Do they commute?
- Can we construct explicit non-trivial examples to check these sorts of questions?

So we investigate **relative positions of Half-sided Modular Inclusions**, both in the abstract and in terms of their representation theory.

Standard Subspaces

Definition

A real subspace $H \subset \mathcal{H}$ of a complex Hilbert space $\mathcal{H}$ is a **standard subspace** if

$$H \cap iH = \{0\} \quad \text{and} \quad \overline{H + iH} = \mathcal{H}.$$

Tomita-Takesaki-modular theory then gives the modular group Δ_H^{it} and modular conjugation J_H that satisfy

$$\Delta_H^{it} H = H, \quad J_H H = H'$$

$$J \Delta_H^{\frac{1}{2}} h = \Delta_H^{-\frac{1}{2}} J_H h = h \quad \text{for all } h \in H$$

Let $\mathcal{H} = L^2(\mathbb{R}, d\theta)$. We define

$$(\Delta_{H_0}^{it}\psi)(\theta) := \psi(\theta - 2\pi t), \quad (J_{H_0}\psi)(\theta) := \overline{\psi(\theta)}.$$

This gives

$$H_0 = \left\{\psi \in \mathbb{H}^2(\mathbb{S}_\pi) \mid \psi(\theta + i\pi) = \overline{\psi(\theta)}\right\}$$

Fourier transforming gives

$$(\Delta_{\widetilde{H_0}}^{it}\psi)(\lambda) := e^{-2\pi t\lambda i}\psi(\lambda), \quad (J_{\widetilde{H_0}}\psi)(\lambda) := \overline{\psi(-\lambda)}$$

and

$$\widetilde{H_0} = \left\{\psi \in L^2(\mathbb{R}, d\lambda) \mid e^{-\pi\lambda}\psi(\lambda) = \overline{\psi(-\lambda)}\right\}$$

Inclusions of standard subspaces

Inclusions of standard subspaces are surprisingly subtle affairs. For example:

Proposition (Borchers, 1999)

Let $K \subset H \subset \mathcal{H}$ be an inclusions of standard subspaces. Then $K \subset H$ if and only if

$$F : \mathbb{R} \to B(\mathcal{H}), \quad t \mapsto \Delta_H^{-it} \Delta_K^{it}$$

extends to a bounded so-continuous function on $\overline{\mathbb{S}_{\frac{1}{2}}}$, analytic in $\mathbb{S}_{\frac{1}{2}}$ such that

$$F(t + \tfrac{i}{2}) = \Delta_H^{-it} J_H J_K \Delta_K^{it}, \quad t \in \mathbb{R}.$$

Proof: $S_H S_K : K + iK \to K + iK$ is equal to the identity.

Half-sided Modular Inclusions and Standard Pairs

Half-sided Modular Inclusion

Recall that $\Delta_H^{it} H = H$. If $K \subset H$ and $\Delta_H^{it} K = K$ for all $t \in \mathbb{R}$, then $K = H$.

Definition

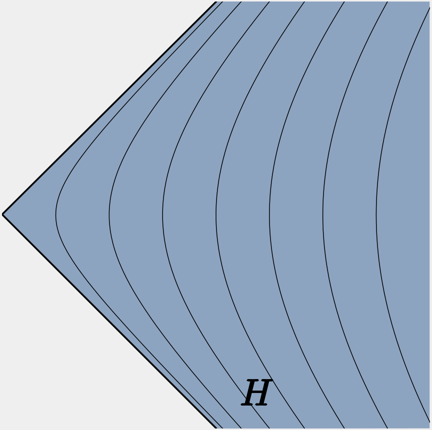
An inclusion $K \subset H$ of standard subspaces is called a **Half-sided Modular Inclusion** (HSMI) if

$$\Delta_H^{-it} K \subset K \quad \text{for all } t \geq 0.$$

We call a Half-sided Modular Inclusion **non-degenerate** if

$$\bigcap_{t \geq 0} \Delta_H^{-it} K = \{0\}$$

Examples of HSMI's (1)



Example 1: Wedge-loclized observables in vacuum Wightman theory.

$$\Delta^{-it}\phi(\vec{x})\Omega = \Delta^{-it}\phi(\vec{x})\Delta^{it}\Omega = \phi(\Lambda_{2\pi t}\vec{x})\Omega.$$

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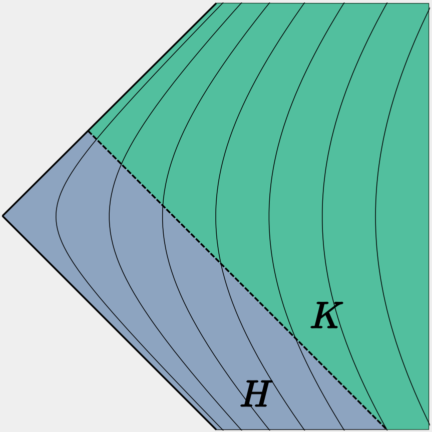

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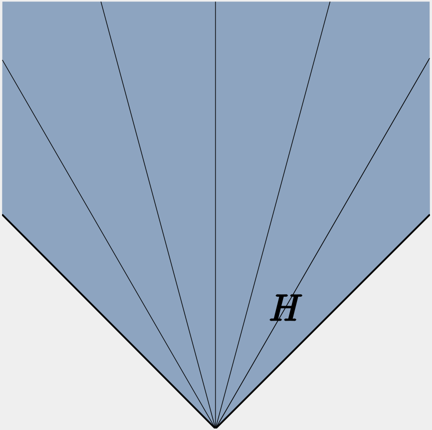
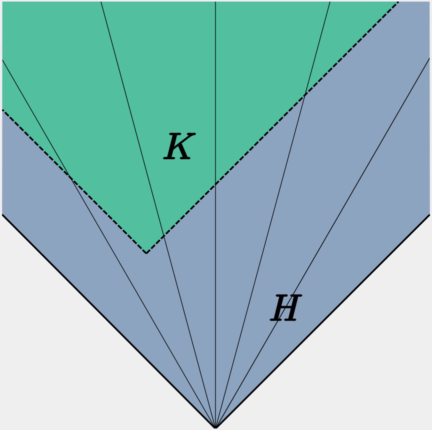


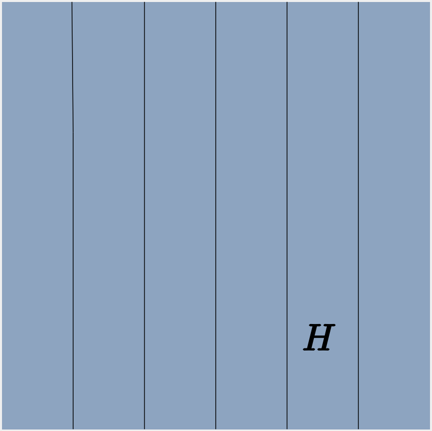
Example 2: Lightcone-localized observables in massless vacuum Wightman theory.

Examples of HSMI's (2)

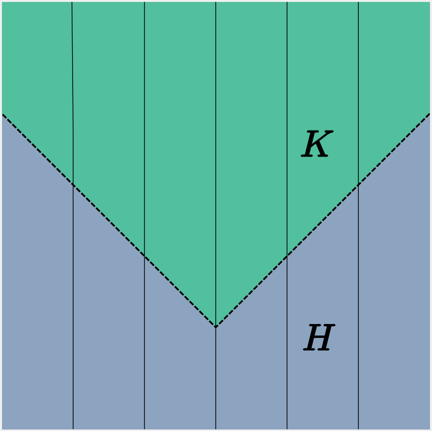


Example 2: Lightcone-localized observables in massless vacuum Wightman theory.

Examples of HSMI's (3)

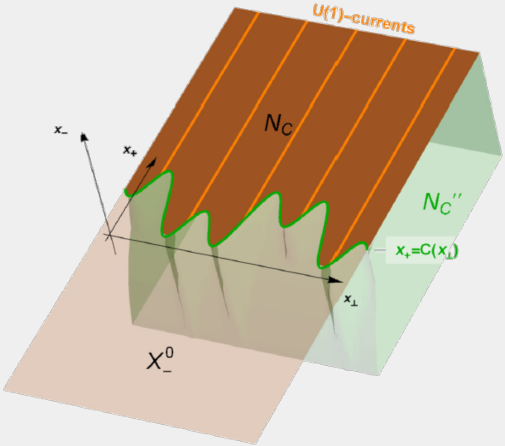


Example 3: Massless thermal field theory (Borchers Yngvason 1999)



Example 3: Massless thermal field theory (Borchers Yngvason 1999)

Examples of HSMI's (4)



Example 4: Standard subspace of a null cut (Morinelli Tanimoto Wegener, 2022).

An easy way of constructing examples of Half-sided Modular Inclusions is trough standard pairs:

Definition

A **standard pair** (H, U) consists of a standard subspace $H \subset \mathcal{H}$ and a positively generated one-parameter group $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ such that

$$U(s)H \subset H \quad \text{for } s \geq 0.$$

We call a standard pair **non-degenerate** if U has no invariant vectors.

We define

$$U_0(s) : L^2(\mathbb{R}, d\theta) \to L^2(\mathbb{R}, d\theta), \quad (U_0(s)\psi)(\theta) = e^{ise^\theta}\psi(\theta)$$

and see that (H_0, U_0) is indeed a standard pair: for $h \in H_0$ and $s \geq 0$ we have

$$(U_0(s)h)(\theta + it) = e^{is\cos(t)e^\theta} e^{-s\sin(t)e^\theta} h(\theta)$$

which is indeed L^2 and

$$(U_0(s)h)(\theta + \pi i) = e^{-ise^\theta} h(\theta + \pi i) = \overline{e^{ise^\theta} h(\theta)}.$$

We can of course Fourier transform to construct a standard pair $(\widetilde{H_0}, \widetilde{U_0})$, with $\widetilde{U_0}$ given by convolution with the distribution

$$\sqrt{\frac{\pi}{2}}\delta + \frac{1}{\sqrt{2\pi}}\mathcal{P}\left(e^{i\lambda\ln(-is)}\Gamma(-i\lambda)\right)$$

Theorem (Borchers 1992)

Let (H, U) be a standard pair. Then

$$\Delta_H^{it} U(s) \Delta_H^{-it} = U(e^{-2\pi t} s) \quad \text{and} \quad J_H U(s) J_H = U(-s)$$

This means that

$$\begin{aligned} \Delta_H^{-it} U(1) H &= U(e^{2\pi t}) \Delta_H^{-it} H \\ &= U(1) U(e^{2\pi t} - 1) H \\ &\subset U(1) H \end{aligned}$$

for $t \geq 0$; so indeed, $U(1)H \subset H$ is a HSMI!

Converse (HSMI → Std. Pair)

Theorem (Wiesbrock 1993, Araki & Zsido 2005)

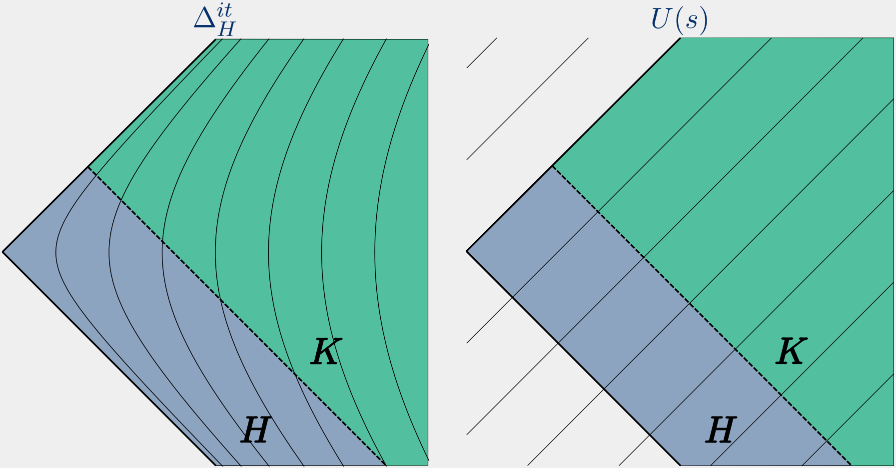
Let $K \subset H$ be a Half-sided Modular Inclusion in $\mathcal{H}$. Then

$$U(1 - e^{-2\pi t}) := \Delta_K^{it} \Delta_H^{-it}$$

can be extended to a one-parameter group $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$.
Furthermore, (H, U) is a **standard pair** and $K = U(1)H$.

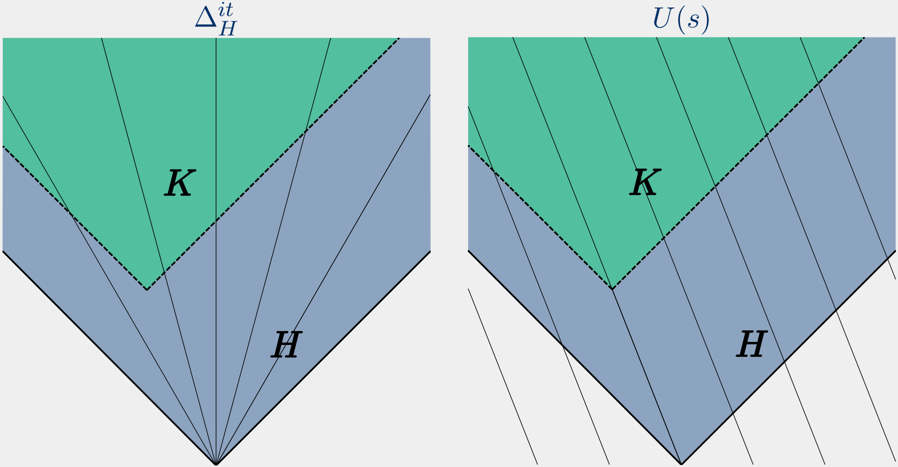
The proof relies on many analytic extension arguments, as is common in modular theory.

HSMI Examples revisited (1)



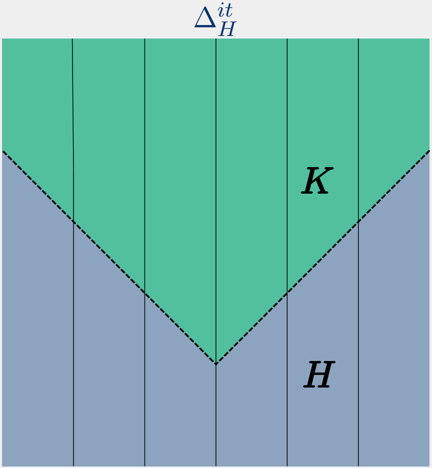
Example 1: Wedge-loclized observables in vacuum Wightman theory.

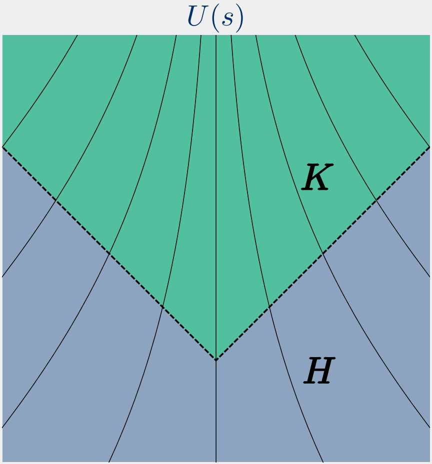
HSMI Examples revisited (2)



Example 2: Lightcone-localized observables in massless vacuum Wightman theory.

HSMI Examples revisited (3)





Example 3: Massless thermal field theory.

Relation between HSMI's and Std. Pairs; Representation theory

Relation between HSMI's and Std. Pairs

Summarizing: for every standard subspace $H \subset \mathcal{H}$ there is a bijection between:

Half-sided Modular Inclusion:
$K \subseteq H \subseteq \mathcal{H}$ standard subspace such that $\Delta_H^{-it} K \subset K$ for $t \geq 0$.

Standard pair:
Positively generated $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ such that $U(s)H \subset H$ for $s \geq 0$.

Proposition

A Half-sided Modular Inclusion is non-degenerate if and only if its associated standard pair is non-degenerate, i.e.

$$\bigcap_{t\geq 0} \Delta_H^{-it} K = \{0\} \Leftrightarrow \ker \partial U = \{0\}$$

Note that it might not be easy to calculate one or the other!

Commutation Relations

These relations are actually just a form of the Canonical Commutation Relations. Compare, for $U(s) = e^{isP}$, the following:

Standard Pair:	$\Delta_H^{it} U(s) \Delta_H^{-it} = U(e^{-2\pi t} s)$
Generator:	$\Delta_H^{it} P \Delta_H^{-it} = e^{-2\pi t} P$
Weyl relations:	$\Delta_H^{it} P^{is} \Delta_H^{-is} = e^{-2\pi tsi} P^{is}$
ax + b group relations:	$[i \ln \Delta_H, iP] = -2\pi i P$
CCR:	$[\ln \Delta_H, \ln P] = 2\pi i$

Our canonical example is simply the Schrödinger representation!

Theorem (Stone - von Neumann)

For every non-degenerate standard pair (H, U) in $\mathcal{H}$ there exists a Hilbert space $\mathcal{K}$, a **maximally abelian** standard subspace $H_{\mathcal{K}} \subset \mathcal{K}$ and a unitary map $\mathbb{V} : \mathcal{H} \to L^2 \otimes \mathcal{K}$ such that

$$\mathbb{V}H = H_0 \otimes_{\mathbb{R}} H_{\mathcal{K}}$$
$$\mathbb{V}U(t)\mathbb{V}^* = U_0(t) \otimes 1_{\mathcal{K}}$$

Note that for H_0, we have

$$\ln \Delta_{H_0} = 2\pi i \partial_\theta \text{ and } P_0 := \partial U_0 = M[e^\theta]$$

For $\widetilde{H_0}$ we have $\ln \Delta_{\widetilde{H_0}} = M[2\pi\lambda]$ and $\widetilde{P_0} := \partial \widetilde{U_0} = e^{i\partial_\lambda}$ (reversed roles/Fourier transformed).

Summary of Prerequisites

- Standard subspaces are real subspaces $H \subset \mathcal{H}$ that 'complexify' to $\mathcal{H}$.
- Inclusions of standard subspaces are in general tricky affairs.
- There is a bijection between Half-sided Modular Inclusions $K \subset H$ (meaning $\Delta_H^{-it} K \subset K$ for $t \geq 0$) and standard pairs (H, U) (meaning $U(s)H \subset H$ for $s \geq 0$).
- The crucial commutation relation for a standard pair is $\Delta_H^{it} U(s) \Delta_H^{-it} = U(e^{-2\pi t} s)$.
- Because of representation theory, every HSMI/Standard Pair is unitarily equivalent to multiples of (H_0, U_0).

Relative Positions

- We fix an 'environment' standard subspace H.
- Suppose we have two HSMI's $K_1 \subset H$ and $K_2 \subset H$, or equivalently, two Standard Pairs (U_1, H) and (U_2, H). We want to know how K_1 and K_2 can relate to each other.
- For concrete examples, it is easier to prescribe U_1 and U_2.
- **Can we read off relative positions of K_1 and K_2 from U_1 and U_2?**

Analytic extension characterization (1)

Recall that $K_1 \subset K_2$ if and only if $\Delta_{K_2}^{-it}\Delta_{K_1}^{it}$ extends boundedly to a strip.
If both are of the form $K_j = U_j(1)H$, one has

$$\begin{aligned}
&\Delta_{K_2}^{-it}\Delta_{K_1}^{it} \\
&= U_1(1)\Delta_H^{-it}U_1(-1)U_2(1)\Delta_H^{it}U_2(-1) \\
&= U_1(1)U_1(-e^{2\pi t})U_2(e^{2\pi t})U_2(-1)
\end{aligned}$$

Lemma

Let (U_j, H), $j = 1, 2$ be standard pairs. Then $U_1(1)H \subset U_2(1)H$ if and only if

$$s \mapsto U_2(-s)U_1(s)$$

extends to a bounded so-continuous function on $\overline{\mathbb{C}_+}$ that is analytic on $\mathbb{C}_+$.

Analytic extension characterization (2)

Lemma

Let (U_j, H), $j = 1, 2$ be standard pairs. Then $U_1(1)H \subset U_2(1)H$ if and only if

$$s \mapsto U_2(-s)U_1(s)$$

extends to a bounded so-continuous function on $\overline{\mathbb{C}_+}$ that is analytic on $\mathbb{C}_+$.

Let $U_j(s) = e^{isP_j}$. Note: because $P_j \geq 0$ we can always extend $s \mapsto U_j(s)$ to $\mathbb{C}_+$. So U_1 has to 'fall off quick enough' to compensate U_2.

So maybe $P_2 \leq P_1$? This is indeed **necessary**, but **not sufficient**.

But bounded + entire analytic = constant, so can we manually bound this function on the lower half plane?

Entire + bounded = constant

Let E_j be the spectral measure for P_j. Then for $\psi_j \in E_j[(a_j, b_j)]$ we see

$$z \mapsto \langle \psi_2, U_2(-z)U_1(z)\psi_1 \rangle = \langle U_2(\overline{z}), U_1(z)\psi_1 \rangle$$

is entire. Also, we have

$$|\langle U_2(\overline{z})\psi_2, U_1(z)\psi_1 \rangle| \leq e^{ya_2} e^{-yb_1} \|\psi_1\| \|\psi_2\|.$$

So if $b_1 \geq a_2$, then $z \mapsto \langle \psi_2, U_2(-z)U_1(z)\psi_1 \rangle$ is entire analytic and bounded, so constant.
If $b_1 > a_2$, then the bound even guarantees that

$$\langle \psi_2, U_2(-z)U_1(z)\psi_1 \rangle = 0$$

for all $z \in \mathbb{C}$. In particular, we have $\langle \psi_2, \psi_1 \rangle = 0$.

Proposition

Let (U_j, H), $j = 1, 2$ be non-degenerate standard pairs, $U_j(t) = e^{itP_j}$, and $E_j[I] = \chi_I(P_j)$. If $U_1(1)H \subset U_2(1)H$, then

$$E_1[(0,a)] \perp E_2[(a,\infty)].$$

for all $a \geq 0$. In particular $E_1[(0,1)]\mathcal{H} \subset E_2[(0,1)]\mathcal{H}$.

Note that because $\Delta_H^{it} P_j \Delta_H^{-it} = e^{-2\pi t} P_j$, we have

$$E_1[(0,1)]\mathcal{H} \subset E_2[(0,1)] \;\Leftrightarrow\; \forall a \geq 0 : E_1[(0,a)]\mathcal{H} \subset E_2[(0,a)]$$

Sufficient condition

For the converse, we assume standard pairs (H, U_j) such that $E_1[(0,a)] \leq E_2[(0,a)]$ for all $a \geq 0$. We want to show that $U_2(-z)U_1(z)$ is bounded for $\mathrm{Im} z \geq 0$; the purely real direction only adds unitaries, so we need to show:

$$\exists C > 0 : \forall y \geq 0 : \|e^{yP_2}e^{-yP_1}\|_\infty \leq C.$$

The idea: for $\lambda > 0$, take $\psi \in E_1[(\lambda-\delta, \lambda+\delta)]\mathcal{H}$. Then in the worst case scenario $e^{-yP_1}\psi \approx e^{-y(\lambda-\delta)}\psi$, and since $E_1[(\lambda-\delta, \lambda+\delta)]\mathcal{H} \subset E_2[(0, \lambda+\delta)]$

$$e^{yP_2}e^{-yP_1}\psi \approx e^{yP_2}e^{-y(\lambda-\delta)}\psi \approx e^{y(\lambda+\delta)}e^{-y(\lambda-\delta)}\psi = e^{2y\delta}\psi.$$

The λ-dependence has disappeared, so this has a chance to be bounded! Also, by choosing δ small enough, we can bound uniformly in y.

Theorem (I.K. 2025)

Let (H, U_j) be non-degenerate standard pairs, $U_j(s) = e^{isP_j}$, and $E_j[I] = \chi_I(P_j)$ for $j = 1, 2$. Then

$$U_1(1)H \subset U_2(1)H \quad \Leftrightarrow \quad E_1[(0,1)]\mathcal{H} \subset E_2[(0,1)]\mathcal{H}$$

In fact we can bound $\|e^{yP_2}e^{-yP_1}\|_\infty \leq 1$, so that

$$e^{-\frac{1}{2}P_1}e^{P_2}e^{-\frac{1}{2}P_1} \leq 1$$

and $e^{P_2} \leq e^{P_1}$. Since $\ln$ is operator monotone, we have $P_2 \leq P_1$.

Examples through representation theory

Canonical example revisited

Consider $H_0 \subset L^2(\mathbb{R}, d\theta)$ given by $(\Delta_{H_0}^{it}\psi)(\theta) = \psi(\theta - 2\pi t)$ and $(J_{H_0}\psi)(\theta) = \overline{\psi(\theta)}$. That came with

$$(U_0(s)\psi)(\theta) = e^{ise^\theta}\psi(\theta)$$

meaning that $U_0(s) = e^{isP_0}$ for $P_0 = M[e^\theta]$. So

$$\chi_{(0,1)}(P_0)L^2(\mathbb{R}, d\theta) = L^2(\mathbb{R}_-, d\theta) = \mathcal{F}\mathbb{H}^2(\mathbb{C}_-)$$

Proposition

Let $H \subset \mathcal{H}$ be a subset such that $\{\Delta_H^{it} \mid t \in \mathbb{R}\}''$ is **maximally abelian**, and let (H, U_1) and (H, U_2) be non-degenerate standard pairs. Then $U_1(1)H \subset U_2(1)H$ if and only if

$$U_1(s) = \varphi(\ln \Delta_H)U_2(s)\varphi(\ln \Delta_H)^*$$

for a **symmetric inner function** $\varphi : \mathbb{C}_- \to \mathbb{C}$.

Nontrivial examples

We take H_0 and U_0 as before, and consider the symmetric inner function $\varphi(\lambda) = \frac{\lambda+i}{\lambda-i}$. Since

$$\Delta_{H_0} = \mathcal{F} M[e^{2\pi\lambda}]\mathcal{F}^*$$

we have

$$\varphi(\ln \Delta_{H_0}) = \mathcal{F} M[\varphi(2\pi\lambda)]\mathcal{F}^*$$

which equals convolution with the Fourier transform of $\lambda \mapsto \varphi(2\pi\lambda)$, which equals

$$\delta(\theta) - \frac{1}{\pi} e^{\theta/2\pi} \chi_{-\infty,0}(\theta).$$

One can explicitly check that $(H, \varphi(\ln \Delta_{H_0}) U_0 \varphi(\ln \Delta_{H_0})^*)$ is a standard pair. However, U_0 and $\varphi(\ln \Delta_{H_0}) U_0 \varphi(\ln \Delta_{H_0})$ do not commute!

Conclusion

- Half-sided Modular Inclusions and Standard Pairs have a rich mathematical structure closely tied to their representation theory.
- Inclusions of standard pairs (i.e. inclusions of the associated HSMI's) is equivalent to inclusions of their spectral subspaces.
- Using this we can construct concrete examples, so we know that:
  - There exists $K_1 \subset H$ and $K_2 \subset H$ HSMI's such that $K_1 \subset K_2$ but **not** as a HSMI.
  - There exist standard pairs (H, U_1) and (H, U_2) such that $P_2 \leq P_1$ but $U_1(1)H \not\subset U_2(1)H$.