Relative Positions of Half-sided Modular Inclusions

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Based on [arXiv:2503.18036] + some new stuff.

Modular Theory and Standard Subspaces

Modular Theory: the essentials

Classically, we can summarise modular theory as follows:

- A von Neumann algebra $\mathcal{A} \subset B(\mathcal{H})$ and a vector $\Omega \in \mathcal{H}$ such that the map
$$\mathcal{A} \to \mathcal{H}, \quad A \mapsto A\Omega$$
is **injective** (i.e. Ω is *separating*) and has **dense range** (i.e. Ω is *cyclic*).
- The real subspace $\mathcal{A}_{sa}$ defines a conjugation on (a dense subset of) $\mathcal{H}$. This allows one to define one parameter group Δ_Ω^{it} and anti-unitary involution J_Ω with the properties
$$\Delta_\Omega^{it} \mathcal{A} \Delta_\Omega^{-it} = \mathcal{A}, \quad J\mathcal{A}J = \mathcal{A}',$$
$$\langle \Omega, A\Delta_\Omega B \Delta_\Omega^{-1} \Omega \rangle = \langle \Omega, BA\Omega \rangle$$

We can distil this to the following:

Definition

A real (closed) subspace $H \subset \mathcal{H}$ of a complex Hilbert space $\mathcal{H}$ is a **standard subspace** if

$$H \cap iH = \{0\} \quad \text{and} \quad \overline{H + iH} = \mathcal{H}.$$

This has an 'internal' time evolution Δ_H^{it} and (anti-linear) conjugation J_H, i.e.

$$\Delta_H^{it} H = H, \quad J_H H = H'$$
$$J\Delta_H^{\frac{1}{2}} h = \Delta_H^{-\frac{1}{2}} J_H h = h \quad \text{for all } h \in H$$

Exact connection

- Given a VNA $\mathcal{A}$ and standard vector $\Omega \in \mathcal{H}$, we can always define $H = \overline{\mathcal{A}_{sa}\Omega} \subset \mathcal{H}$. Not every standard subspace comes from a VNA.
- However, given $H \subset \mathcal{H}$, we can construct on $\mathcal{F}_{\pm}(\mathcal{H})$ the CCR and CAR algebra as generated by $\varphi(h) = a(h) + a^*(h)$, for which the vacuum is then again standard.
- One can even generalize this to twisted Fock spaces $\mathcal{F}_T(\mathcal{H})$ to define *twisted Araki-Woods Algebras* $\mathcal{L}_T(H)$, where one can find conditions on the twist T and standard subspace H such that the vacuum is standard. See: [Correa da Silva and Lechner, 2023].
- In both cases, the modular data of the resulting algebra reduces in the appropriate sense to the modular data of $H \subset \mathcal{H}$.

Standard Subspaces (examples)

Let $\mathcal{H} = L^2(\mathbb{R}, d\theta)$. We define

$$(\Delta_{H_0}^{it}\psi)(\theta) := \psi(\theta - 2\pi t), \quad (J_{H_0}\psi)(\theta) := \overline{\psi(\theta)}.$$

This gives

$$H_0 = \left\{\psi \in \mathbb{H}^2(\mathbb{S}_\pi) \mid \psi(\theta + i\pi) = \overline{\psi(\theta)}\right\}$$

Fourier transforming gives

$$(\Delta_{\widetilde{H_0}}^{it}\psi)(\lambda) := e^{-2\pi t\lambda i}\psi(\lambda), \quad (J_{\widetilde{H_0}}\psi)(\lambda) := \overline{\psi(-\lambda)}$$

and

$$\widetilde{H_0} = \left\{\psi \in L^2(\mathbb{R}, d\lambda) \mid e^{-\pi\lambda}\psi(\lambda) = \overline{\psi(-\lambda)}\right\}$$

Half-sided Modular Inclusions and Standard Pairs

Half-sided Modular Inclusion

Inclusions of standard subspaces are subtle affairs. However, with extra assumptions they become more tractable.
Recall that $\Delta_H^{it} H = H$. If $K \subset H$ and $\Delta_H^{it} K = K$ for all $t \in \mathbb{R}$, then $K = H$.

Definition

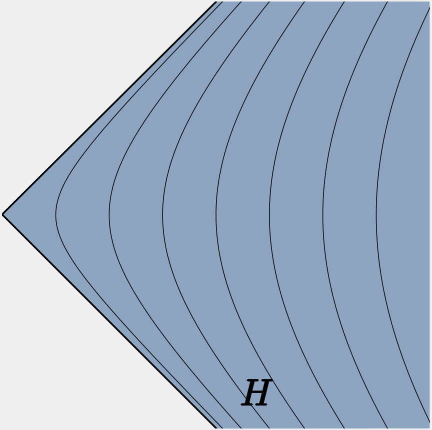
An inclusion $K \subset H$ of standard subspaces is called a **Half-sided Modular Inclusion** (HSMI) if

$$\Delta_H^{-it} K \subset K \quad \text{for all } t \geq 0.$$

We call a Half-sided Modular Inclusion **non-degenerate** if

$$\bigcap_{t \geq 0} \Delta_H^{-it} K = \{0\}$$

Examples of HSMI's (1)



Example 1: Wedge-localized observables in vacuum Wightman theory.

$$\Delta^{-it}\phi(\vec{x})\Omega = \Delta^{-it}\phi(\vec{x})\Delta^{it}\Omega = \phi(\Lambda_{2\pi t}\vec{x})\Omega.$$

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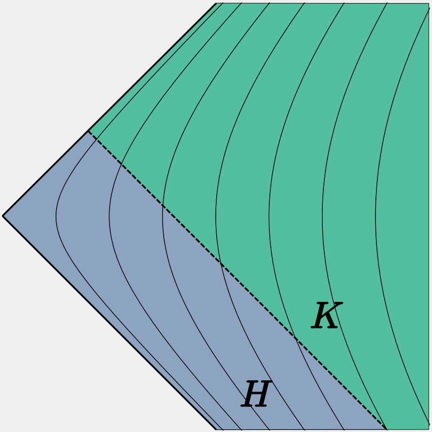

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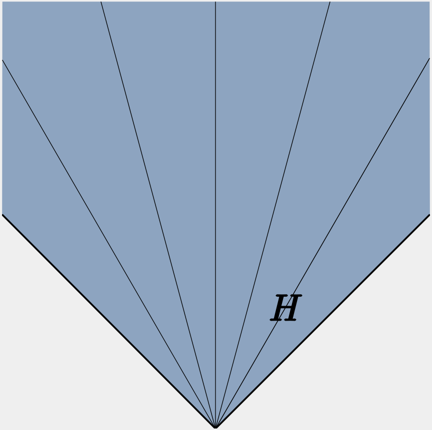
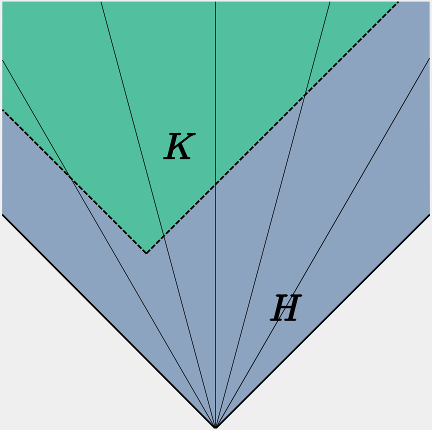


Example 2: Lightcone-localized observables in massless vacuum Wightman theory.

Examples of HSMI's (2)



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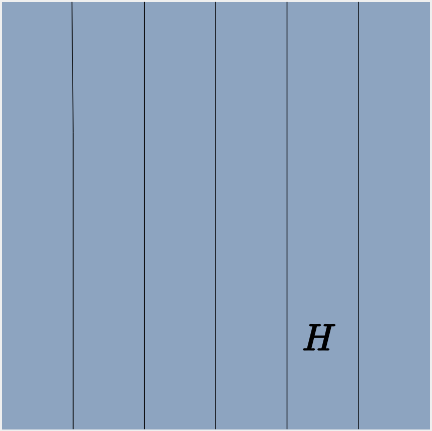
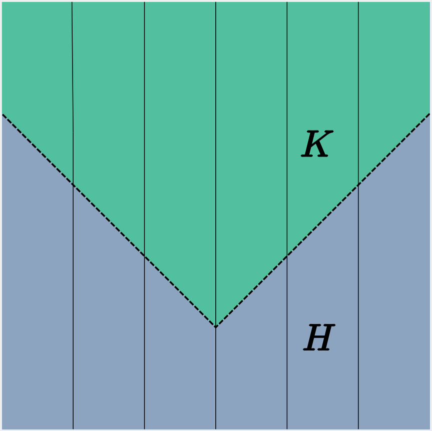
Examples of HSMI's (3)



Example 3: Massless thermal field theory (Borchers Yngvason 1999)

Examples of HSMI's (3)



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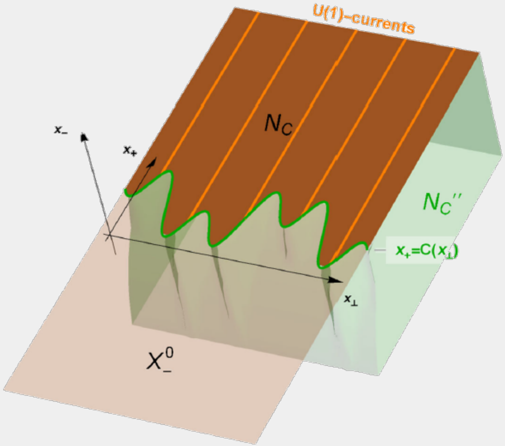
Examples of HSMI's (4)



Example 4: Standard subspace of a null cut (Morinelli Tanimoto Wegener, 2022).

Standard Pairs

An easy way of constructing examples of Half-sided Modular Inclusions is trough standard pairs:

Definition

A **standard pair** (H, U) consists of a standard subspace $H \subset \mathcal{H}$ and a positively generated one-parameter group $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ such that

$$U(s)H \subset H \quad \text{for } s \geq 0.$$

We call a standard pair **non-degenerate** if U has no invariant vectors.

We define

$$U_0(s) : L^2(\mathbb{R}, d\theta) \to L^2(\mathbb{R}, d\theta), \quad (U_0(s)\psi)(\theta) = e^{ise^\theta}\psi(\theta)$$

and see that (H_0, U_0) is indeed a standard pair: for $h \in H_0$ and $s \geq 0$ we have

$$(U_0(s)h)(\theta + it) = e^{is\cos(t)e^\theta} e^{-s\sin(t)e^\theta} h(\theta + it)$$

which is indeed L^2 and

$$(U_0(s)h)(\theta + \pi i) = e^{-ise^\theta} h(\theta + \pi i) = \overline{e^{ise^\theta} h(\theta)}.$$

We can of course Fourier transform to construct a standard pair $(\widetilde{H_0}, \widetilde{U_0})$, with $\widetilde{U_0}$ given by convolution with the distribution

$$\sqrt{\frac{\pi}{2}}\delta + \frac{1}{\sqrt{2\pi}}\mathcal{P}\left(e^{i\lambda\ln(-is)}\Gamma(-i\lambda)\right)$$

Borchers' theorem (Std. Pair → HSMI)

Theorem (Borchers 1992)

Let (H, U) be a standard pair. Then

$$\Delta_H^{it} U(s) \Delta_H^{-it} = U(e^{-2\pi t} s) \quad \text{and} \quad J_H U(s) J_H = U(-s)$$

This means that

$$\begin{aligned} \Delta_H^{-it} U(1) H &= U(e^{2\pi t}) \Delta_H^{-it} H \\ &= U(1) U(e^{2\pi t} - 1) H \\ &\subset U(1) H \end{aligned}$$

for $t \geq 0$; so indeed, $U(1)H \subset H$ is a HSMI!

Converse (HSMI $\to$ Std. Pair)

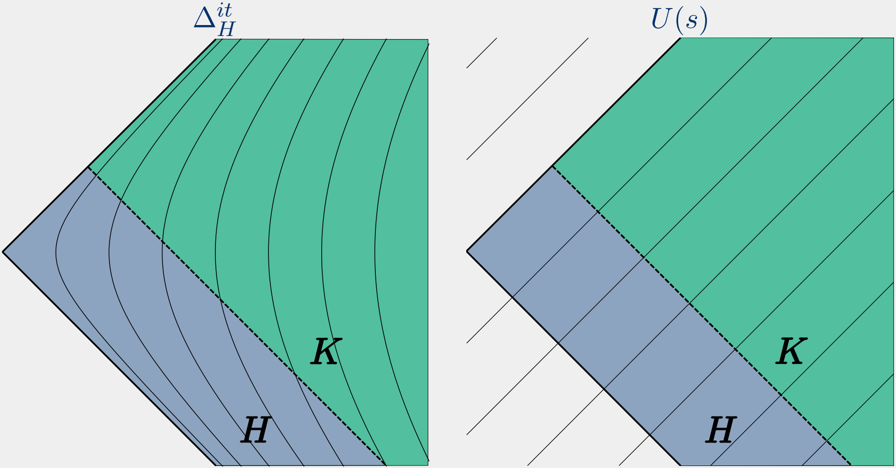
Theorem (Wiesbrock 1993, Araki & Zsido 2005)

Let $K \subset H$ be a Half-sided Modular Inclusion in $\mathcal{H}$. Then

$$U(1 - e^{-2\pi t}) := \Delta_K^{it} \Delta_H^{-it}$$

can be uniquely extended to a one-parameter group $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$. Furthermore, (H, U) is a **standard pair** and $K = U(1)H$.

The proof relies on many analytic extension arguments, as is common in modular theory.



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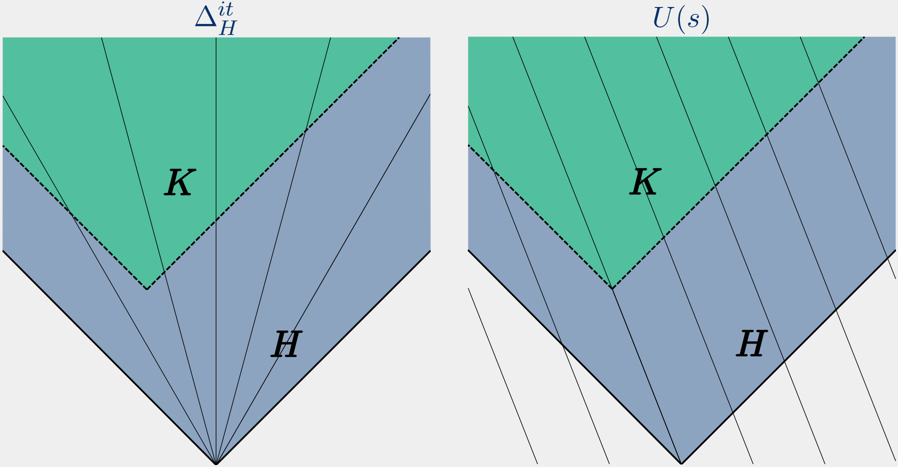
HSMI Examples revisited (2)



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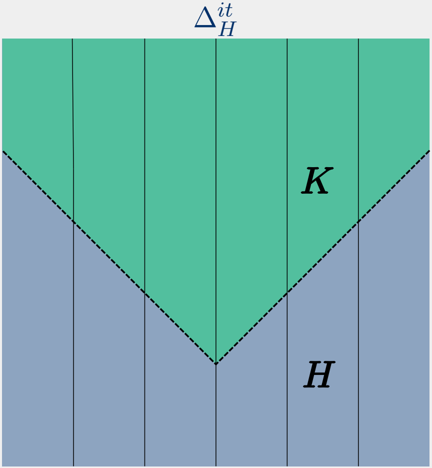
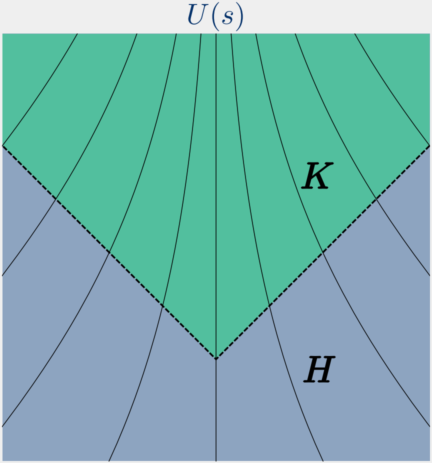




Example 3: Massless thermal field theory.

Relation between HSMI's and Std. Pairs

Summarizing: for every standard subspace $H \subset \mathcal{H}$ there is a bijection between:

Half-sided Modular Inclusion:
$K \subseteq H \subseteq \mathcal{H}$ standard subspace such that $\Delta_H^{-it} K \subset K$ for $t \geq 0$.

Standard pair:
Positively generated $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ such that $U(s)H \subset H$ for $s \geq 0$.

Proposition

A Half-sided Modular Inclusion is non-degenerate if and only if its associated standard pair is non-degenerate, i.e.

$$\bigcap_{t \geq 0} \Delta_H^{-it} K = \{0\} \Leftrightarrow \ker \partial U = \{0\}$$

Note that it might not be easy to calculate one or the other!

Summary of Prerequisites

- Standard subspaces are real subspaces $H \subset \mathcal{H}$ that 'complexify' to $\mathcal{H}$. They have an 'internal' time evolution Δ_H^{it}.
- Inclusions of standard subspaces are in general tricky affairs.
- There is a bijection between Half-sided Modular Inclusions $K \subset H$ (meaning $\Delta_H^{-it} K \subset K$ for $t \geq 0$) and standard pairs (H, U) (meaning $U(s)H \subset H$ for $s \geq 0$).
- The crucial commutation relation for a standard pair is $\Delta_H^{it} U(s) \Delta_H^{-it} = U(e^{-2\pi t} s)$.

Two dimensions

Recall that a standard pair (H, U) consists of a standard subspace $H \subset \mathcal{H}$ and a representation $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ (where $U(t) := e^{itP}$) such that

- $P \geq 0$;
- $U(\mathbb{R}_+)H \subset H$.

We now define a **two-dimensional standard pair** (H, U) to be a standard pair $H \subset \mathcal{H}$ with a representation $U : \mathbb{R}^2 \to \mathcal{U}(\mathcal{H})$ (where $U(\vec{x}) = e^{i(x_0 P_0 + x_1 P_1)}$ such that

- $x_0 P_0 + x_1 P_1 \geq 0$ for $\vec{x} \in V_+$.
- $U(W)H \subset H$.

Two-dimensional Borchers' theorem

For one-dimensional standard pairs we had:

Theorem (Borchers 1992)

Let (H, U) be a standard pair. Then

$$\Delta_H^{it} U(s) \Delta_H^{-it} = U(e^{-2\pi t} s) \quad \text{and} \quad J_H U(s) J_H = U(-s)$$

Now for two-dimensional standard pairs we have:

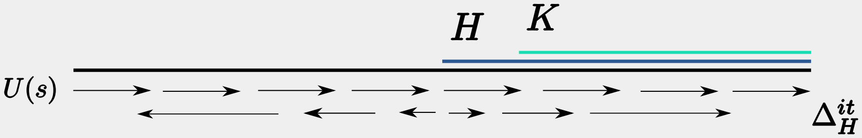
Theorem (Borchers 1992)

Let (H, U) be a two-dimensional standard pair. Then

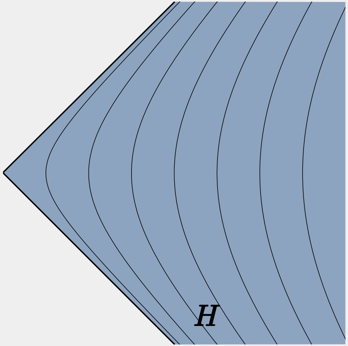
$$\Delta_H^{it} U(\vec{x}) \Delta_H^{-it} = U(\Lambda_{-2\pi t} \vec{x}) \quad \text{and} \quad J_H U(\vec{x}) J_H = U(-\vec{x})$$

Geometrical comparison

One-dimensional situation:



Two-dimensional situation:



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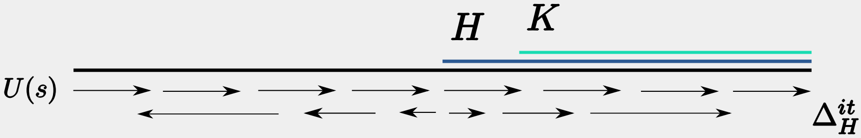
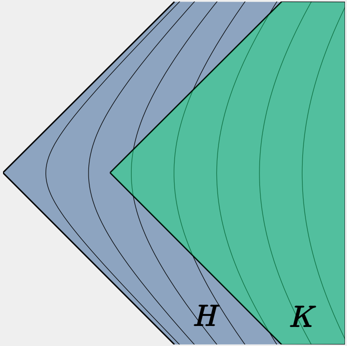


Two-dimensional situation:



2-Dimensional HSMI?

One-dimensional situation: we recognize that $K \subset H$ comes from a 'geometric' situation $K = U(1)H$ by verifying the 'analytical' condition $\Delta_H^{-it} K \subset K$ for all $t \geq 0$.

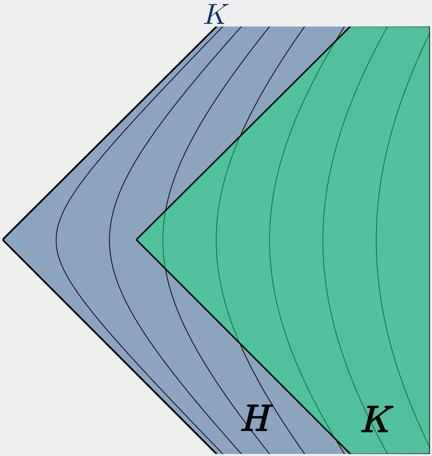
Question

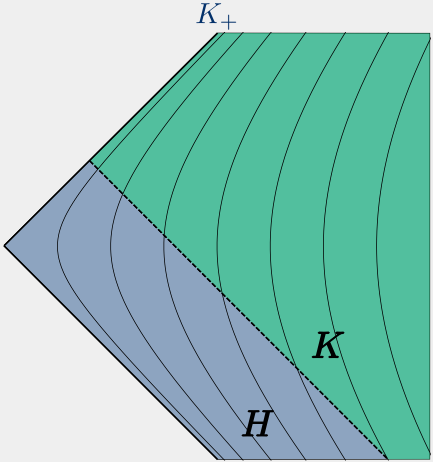
How can we recognise when an inclusion $K \subset H$ of standard subspaces comes from a 2-dimensional standard pair (H, U) as $K = U(0,1)H$?

Plan of attack: suppose $K = U(0,1)H$ for a 2-dimensional standard pair (H, U). Determine properties K satisfies with respect to Δ_H^{it} that we can formulate without referring to U explicitly.

Interpolating HSMI's

Note that $K := U(0,1)H$ is not an HSMI, but $K_+ := U(\frac{1}{2}, \frac{1}{2})H$ is!





We can access the lightlike (= diagonal) directions:

Proposition

Suppose (H, U) is a 2-dimensional standard pair, and $K = U(0,1)H$. Then

$$K_+ := \bigvee_{t \geq 0} \Delta_H^{-it} K = U(\tfrac{1}{2}, \tfrac{1}{2})H, \quad K_- := \bigvee_{t \geq 0} \Delta_H^{it} K = U(-\tfrac{1}{2}, \tfrac{1}{2})H$$

and $K = K_+ \cap K_-$.

Note that we can formulate the requirement $K = K_+ \cap K_-$ without referring to U! What's more, the half-sided modular inclusions $K_+ \subset K$ and $K_- \subset K$ induce the one-parameter groups $t \mapsto U(t,t)$ and $t \mapsto U(-t,t)$.

Are we there yet?

So, given an inclusion of standard subspaces $K \subset H$, we do the following:

- Construct $K_+ := \bigvee_{t\geq 0} \Delta_H^{-it} K$ and $K_- := \bigvee_{t\geq 0} \Delta_H^{it} K$;
- Check that $K = K_+ \cap K_-$.
- Construct the positively generated one-parameter groups U_+ associated to $K_+ \subset H$ and U_- associated to $K_- \subset H$.
- Construct $U(\vec{x}) := U_+(\frac{x_0+x_1}{2})U_-(\frac{x_0-x_1}{2})$

If one started with $K = U(0,1)H$ for some two-dimensional standard pair (H, U), one would recover precisely U.
However, U_+ and U_- are not guaranteed to commute!

Relative Positions

So our scenario is the following:

- We have a standard subspace $H \subset \mathcal{H}$ representing 'the environment'.
- We have two positively generated one-parameter groups U_1, U_2 such that (H, U_1) and (H, U_2) are (one-dimensional) standard pairs.
- We have that $U_1(1)H \cap U_2(1)H$ is standard.

Doest this guarantee that U_1 and U_2 commute? Or, formulated more succinctly, for two Half-sided Modular Inclusions $K_1 \subset H$ and $K_2 \subset H$, what can $K_1 \cap K_2$ look like? Is $K_1 \cap K_2 \subset K_j$ also a HSMI?

Relative Positions of Half-sided Modular Inclusions

If $K_1 \subset H$ and $K_2 \subset H$ are HSMI's, and $K_1 \cap K_2$ is standard, then $K_1 \cap K_2$ is a HSMI. In this case, we can consider $K_1 \cap K_2$ as 'the largest HSMI in H that is contained in both K_1 and K_2'.

Theorem (I.K. 2025)

Let (H, U_j) be non-degenerate standard pairs, $U_j(s) = e^{isP_j}$, and $E_j[I] = \chi_I(P_j)$ for $j = 1, 2$. Then

$$U_1(1)H \subset U_2(1)H \quad \Leftrightarrow \quad E_1[(0,1)]\mathcal{H} \subset E_2[(0,1)]\mathcal{H}$$

Can this happen when U_1 and U_2 do not commute? We will construct specific examples through representation theory.

Canonical example revisited

Consider $H_0 \subset L^2(\mathbb{R}, d\theta)$ given by $(\Delta_{H_0}^{it}\psi)(\theta) = \psi(\theta - 2\pi t)$ and $(J_{H_0}\psi)(\theta) = \overline{\psi(\theta)}$. That came with

$$(U_0(s)\psi)(\theta) = e^{ise^\theta}\psi(\theta)$$

meaning that $U_0(s) = e^{isP_0}$ for $P_0 = M[e^\theta]$. So

$$\chi_{(0,1)}(P_0)L^2(\mathbb{R}, d\theta) = L^2(\mathbb{R}_-, d\theta) = \mathcal{F}\mathbb{H}^2(\mathbb{C}_-)$$

Proposition

Let $H \subset \mathcal{H}$ be a subset such that $\{\Delta_H^{it} \mid t \in \mathbb{R}\}''$ is **maximally abelian**, and let (H, U_1) and (H, U_2) be non-degenerate standard pairs. Then $U_1(1)H \subset U_2(1)H$ if and only if

$$U_1(s) = \varphi(\ln \Delta_H)U_2(s)\varphi(\ln \Delta_H)^*$$

for a **symmetric inner function** $\varphi : \mathbb{C}_- \to \mathbb{C}$.

The relations between $U(s)$ and Δ_H^{it} are actually just a form of the Canonical Commutation Relations. Compare, for $U(s) = e^{isP}$, the following:

Standard Pair:	$\Delta_H^{it} U(s) \Delta_H^{-it} = U(e^{-2\pi t} s)$
Generator:	$\Delta_H^{it} P \Delta_H^{-it} = e^{-2\pi t} P$
Weyl relations:	$\Delta_H^{it} P^{is} \Delta_H^{-is} = e^{-2\pi t s i} P^{is}$
ax + b group relations:	$[i \ln \Delta_H, iP] = -2\pi i P$
CCR:	$[\ln \Delta_H, \ln P] = 2\pi i$

Our canonical example is simply the Schrödinger representation!

Theorem (Stone - von Neumann)

For every non-degenerate standard pair (H, U) in $\mathcal{H}$ there exists a Hilbert space $\mathcal{K}$, a **maximally abelian** standard subspace $H_{\mathcal{K}} \subset \mathcal{K}$ and a unitary map $\mathbb{V} : \mathcal{H} \to L^2 \otimes \mathcal{K}$ such that

$$\mathbb{V}H = H_0 \otimes_{\mathbb{R}} H_{\mathcal{K}}$$
$$\mathbb{V}U(t)\mathbb{V}^* = U_0(t) \otimes 1_{\mathcal{K}}$$

Note that for H_0, we have

$$\ln \Delta_{H_0} = 2\pi i \partial_\theta \text{ and } P_0 := \partial U_0 = M[e^\theta]$$

For $\widetilde{H_0}$ we have $\ln \Delta_{\widetilde{H_0}} = M[2\pi\lambda]$ and $\widetilde{P_0} := \partial\widetilde{U_0} = e^{i\partial_\lambda}$ (reversed roles/Fourier transformed).

Nontrivial example and implications

- So, since

$$U_1(1)H \subset U_2(1)H \Leftrightarrow U_1(s) = \varphi(\ln \Delta_H) U_2(s) \varphi(\ln \Delta_H)^*$$

for φ a symmetric inner function, if we choose φ smartly (e.g. $\varphi(\lambda) = \frac{\lambda+i}{\lambda-i}$), we guarantee U_1 and U_2 do not commute, even though $U_1(1)H \subset U_2(1)H$.

- One can show

$$U_1(1)H \subset U_2(1)H \text{is HSMI} \Leftrightarrow [U_1(s), U_2(s')] = 0$$

So translating to the HSMI-perspective, we know now that there exist HSMI's $K_1 \subset H$ and $K_2 \subset H$ such that $K_1 \subset K_2$ but not as a HSMI.

- 2D standard pair situation: similar techniques can show that the U_+ and U_- do not in general commute.

- Half-sided Modular Inclusions are analytic characterizations of geometric inclusions (i.e. standard pairs).
- One can characterize two-dimensional standard pairs as inclusions $K \subset H$ such that
  - the associated spaces $K_+ := \bigvee_{t \geq 0} \Delta_H^{-it} K$ and $K_- := \bigvee_{t \geq 0} \Delta_H^{it} K$ satisfy $K = K_+ \cap K_-$
  - the one-parameter groups induced by the HSMI's $K_+ \subset K$ and $K_- \subset K$ commute.
- We have counterexamples that show that the last requirement is essential.

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