Standard Subspaces and Twisted Araki-Woods Subfactors

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The purpose of this talk was to describe a large family of interesting subfactors that are often of type III, without normal conditional expectation (infinite index), but despite these differences have various analogies with the more familiar finite index type II_1-subfactors. For instance, they require an underlying braiding and a modular theory version of the subfactor-theoretic Fourier transform. These subfactors go by the name of twisted Araki-Woods subfactors and have been introduced in [CdSL23]. They are based on two data: an inclusion of standard subspaces and a twist.

Inclusions of standard subspaces. A *standard subspace* H of a complex Hilbert space $\mathcal{H}$ is a closed real subspace $H \subset \mathcal{H}$ such that $H \cap iH = \{0\}$ and $\overline{H + iH} = \mathcal{H}$. Specific examples of standard subspaces arise from von Neumann algebras $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ with cyclic separating vector Ω as $H := \overline{\mathcal{M}_{\mathrm{sa}}\Omega}$. Although not all standard subspaces are of this form, the lattice $\mathrm{Std}(\mathcal{H})$ of of all standard subspaces of $\mathcal{H}$ has interesting structural similarities to the lattice of von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$ and subfactors:

(i) Symplectic complementation $H \mapsto H' := \{v \in \mathcal{H} : 0 = \mathrm{Im}\langle v, h\rangle \, \forall h \in H\}$ is an order-reversing involution on $\mathrm{Std}(\mathcal{H})$, resembling the commutant of von Neumann algebras and the Bicommutant Theorem,
(ii) there is a natural notion of factor subspace, namely $H \in \mathrm{Std}(\mathcal{H})$ with $H \cap H' = \{0\}$,
(iii) proper irreducible inclusions $K \subsetneq H$ of factor subspaces $K, H \in \mathrm{Std}(\mathcal{H})$ exist (i.e. $K' \cap H = \{0\}$), resembling irreducible subfactors,
(iv) any inclusion $H_0 \subset H_1$ of standard subspaces $H_0, H_1 \in \mathrm{Std}(\mathcal{H})$ naturally extends to a tower and tunnel

$$\ldots \subset H_{-1} \subset H_0 \subset H_1 \subset H_2 \subset \ldots \, , \tag{1}$$

resembling iterations of Jones' basic construction.

While inclusions of standard subspaces do not come with an index, and are basically incompatible with (analogues of) conditional expectations, a good replacement for these missing tools is modular theory: Any $H \in \mathrm{Std}(\mathcal{H})$ defines a Tomita operator $S_H : H + iH \to H + iH$, given by $S_H(h_1 + ih_2) := h_1 - ih_2$, and the polar decomposition of this closed involution defines a one-parameter group of unitaries Δ_H^{it}, $t \in \mathbb{R}$, preserving H, and an antiunitary J_H mapping H onto H'. With this technique one for instance quickly checks that $H_2 := J_{H_1} J_{H_0} H$ and $H_{-1} := J_{H_0} J_{H_1} H_0$ are standard subspaces satisfying (1). One also checks that proper inclusions $K \subsetneq H$ can only exist for $\dim \mathcal{H} = \infty$ because $K \subset H$ is equivalent to an extension $S_K \subset S_H$ of Tomita operators.

Inclusions of standard subspaces can be seen as a spatial analogue of subfactors, and are of interest in their own right [**?**]. No canonical map from inclusions of standard subspaces to inclusions of von Neumann algebras exists, which is why we have to introduce more data to define twisted Araki-Woods subfactors.

Twisted Araki-Woods von Neumann algebras. Given a complex Hilbert space $\mathcal{H}$, an operator $T = T^* \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ with $\|T\| \leq 1$ is called a *twist* if the operators $P_{T,n} \in \mathcal{B}(\mathcal{H}^{\otimes n})$, $n \in \mathbb{N}$, iteratively defined by

$$P_{T,1} = 1, \quad P_{T,2} := 1 + T, \quad P_{T,n+1} = (1 \otimes P_{T,n})(1 + T_1 + T_1 T_2 + \ldots + T_1 \cdots T_n)$$

(in standard tensor leg notation), are all *positive*. In case T satisfies the Yang-Baxter equation, $P_{T,n}$ is the corresponding quantum symmetrizer.

Given a twist T, we consider the tensor algebra $\bigoplus_{n\geq 0} \mathcal{H}^{\otimes n}$ and the quotient by its left ideal $\bigoplus_{n\geq 0} \ker P_{T,n}$. Completed in the scalar product given by $\langle [\Psi], [\Phi] \rangle_T = \sum_{n\geq 0} \langle [\Psi]_n, P_{T,n} [\Phi]_n \rangle$, it becomes a Hilbert space (the T-twisted Fock space $\mathcal{F}_T(\mathcal{H})$), on which left tensor multiplication by $\xi \in \mathcal{H}$ defines an operator $a^*_{T,L}(\xi)$. With these definitions, the left twisted Araki-Woods von Neumann algebra with twist T and standard subspace $H \in \mathrm{Std}(\mathcal{H})$ is

$$\mathcal{L}_T(H) := \{a^*_{T,L}(h) + a_{T,L}(h) \,:\, h \in H\}''. \tag{2}$$

Denoting by F the tensor flip, the von Neumann algebras $\mathcal{L}_{qF}(H)$ are second quantization factors for $q = 1$, generated by CAR algebras for $q = -1$, free group factors for $q = 0$ and H maximally abelian [Voi85], variations of free group factors for $q = 0$ and general standard subspace [Shl97]. So even in this very restricted class of examples one sees type I, II, and III von Neumann algebras, commutative and noncommutative ones, hyperfinite and non-hyperfinite ones, showing that $\mathcal{L}_T(H)$ depends crucially on H and T.

From the point of view of modular theory, it is most important to understand when the Fock vacuum $\Omega \in \mathcal{F}_T(\mathcal{H})$ is cyclic and separating for $\mathcal{L}_T(H)$.

Theorem 1. *Let $H \in \mathrm{Std}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ be a twist such that $[T, \Delta_H^{it} \otimes \Delta_H^{it}] = 0$ for all $t \in \mathbb{R}$. Then Ω is cyclic and separating for $\mathcal{L}_T(H)$ if and only if T satisfies the Yang-Baxter equation and is* crossing-symmetric *w.r.t. H (explained below).*

Crossing-Symmetry. In order to define crossing-symmetry, we begin by saying that an operator $T \in \mathcal{B}(\mathcal{H}^{\otimes 2})$ is crossable if the equation given in terms of matrix-coefficients

$$\langle \psi_1 \otimes \psi_2, \mathrm{Cr}_H(T) \psi_3 \otimes \psi_4 \rangle = \langle \psi_2 \otimes S_H^* \psi_4, T S_H \psi_1 \otimes \psi_3 \rangle \tag{3}$$

defines a bounded operator $\mathrm{Cr}_H(T)$. In case the Hilbert space is infinite dimensional, one has to take into consideration that the vectors ψ_1 and ψ_4 must lie in the domain of S_H and S_H^*, respectively. A crossable operator is called crossing-symmetric if $\mathrm{Cr}_H(T) = T^*$.

The Yang-Baxter equation and crossing-symmetry, in the light of the theorem above, are equivalent to the KMS condition and, in particular, the crossing-symmetry carries all the analytic content of the KMS condition [CGL24]. Furthermore, one can immediately recognize the connection between the crossing map and the subfactor-theoretic Fourier transform when representing the map defined in (3) in graphical notation, where we highlight the dependence of the standard subspace through S_H and S_H^*, in contrast with the subfactor-theoretic Fourier transform.

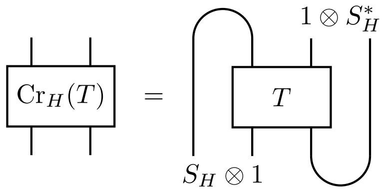

FIGURE 1. Graphical representation of the crossing map.

The behaviour of two examples of twists under the crossing map is worth mentioning: The tensor flip F is always crossing-symmetric independent of H; The identity operator is crossable if and only if $\dim \mathcal{H} < \infty$, in which case $\mathrm{Cr}_H(1) = \mathrm{Tr}(\Delta)P_\xi$, where P_ξ is the orthogonal projection in the direction of the vector $\xi = \sum_{n=1}^{\dim \mathcal{H}} e_n \otimes S_H e_n$, which is a Temperley-Lieb projection, *i.e.* $(P_\xi \otimes 1)(1 \otimes P_\xi)(P_\xi \otimes 1) = \mathrm{Tr}(\Delta)^{-1} P_\xi \otimes 1$, as one could expected from the connection between the subfactor-theoretic Fourier transform and the Temperley–Lieb algebra.

Inclusions of Twisted Araki-Woods von Neumann algebras. Turns out that if T satisfies the Yang-Baxter equation, right tensor multiplication by $\xi \in \mathcal{H}$ also defines an operator denoted $a^*_{T,R}(\xi)$ and, similarly, the right twisted Araki-Woods von Neumann algebra $\mathcal{R}_T(H)$. Under the hypotheses of Theorem 1, namely, in case T is also crossing-symmetric w.r.t. H and satisfies $[T, \Delta_H^{it} \otimes \Delta_H^{it}] = 0$ for all $t \in \mathbb{R}$, we can also determine the commutant of the twisted Araki-Woods algebras $\mathcal{L}_T(H)' = \mathcal{R}_T(H')$.

Given an inclusion of standard subspaces $K \subset H$, we have the correspondent inclusion of the von Neumann algebras $\mathcal{L}_T(K) \subset \mathcal{L}_T(H)$. We are interested in knowing when such inclusion is irreducible, *i.e.* when the relative commutant satisfy $\mathcal{C}(K,H) := \mathcal{L}_T(K)' \cap \mathcal{L}_T(H) = \mathbb{C} \cdot 1$, [CdSL23, CdSL25].

Theorem 2. *Let $H \in \mathrm{Std}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ be a twist such that, $\|T\| < 1$, $[T, \Delta_H^{it} \otimes \Delta_H^{it}] = 0$ for all $t \in \mathbb{R}$, T satisfies the Yang-Baxter equation and is crossing-symmetric w.r.t. H. Then, if $\Delta_H^{\frac{1}{4}} E_K$ is non-compact, $\mathcal{C}(K,H) = \mathbb{C} \cdot 1$.*

It follows that, if we have a twist T satisfying the hypothesis of the theorem above for the standard subspace H and another standard subspace $K \subset H$ such that $\Delta_H^{\frac{1}{4}} E_K$ is non-compact. Then, $\Delta_H^{\frac{1}{4}} E_H$ is also non-compact and

$$\mathcal{Z}(\mathcal{L}_T(H)) = \mathcal{C}(H,H) = \mathbb{C} \cdot 1 = \mathcal{C}(K,H) \supset \mathcal{Z}(\mathcal{L}_T(K)),$$

meaning, in particular, that $\mathcal{L}_T(K)$ is a subfactor of $\mathcal{L}_T(H)$.

We remark that, from the point of view of Algebraic Quantum Field Theory, one is often interested in having a large relative commutant. In that direction, we can say that, if in addition to all assumptions of Theorem 2, but $\|T\| < 1$, one also have that $\Delta_H^{\frac{1}{4}} E_K$ is trace class and its trace norm is less than 1. Then, in case $\mathcal{L}_T(H)$ is a type III factor, the relative commutant $\mathcal{C}(K,H)$ is also type III.

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