Relative Positions of Half-sided Modular Inlcusions

AQFTUK 10, Cardiff

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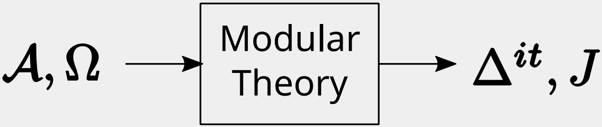
Based on [arXiv:2503.18036]



We consider the **standard subspace** $\overline{\mathcal{A}_{sa}\Omega} \subset \mathcal{H}$.

Definition

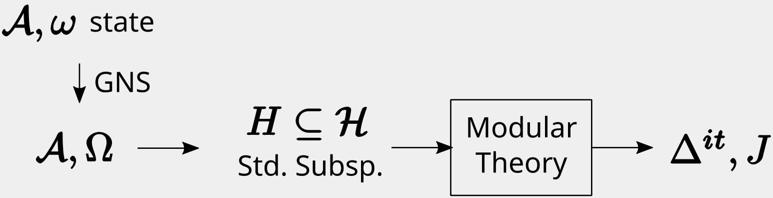
A real subspace $H \subset \mathcal{H}$ is called a **standard subspace** if

$$H \cap iH = \{0\}, \quad \overline{H + iH} = \mathcal{H}$$

Then $S : h_1 + ih_2 \mapsto h_1 - ih_2$ gives $S =: J\Delta^{\frac{1}{2}}$, and

$$\Delta^{it} H = H, \quad JH = H'$$

Modular Theory



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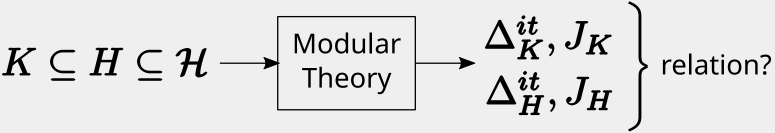
Inclusions of standard subspaces

In AQFT we are interested in inclusions. What do inclusions do in this machine?



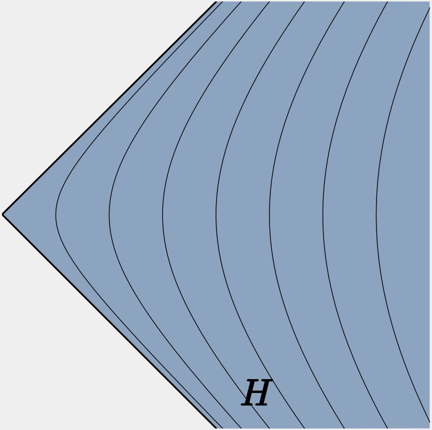
In general there is no clear relation. However, there is an 'easier' situation:

Definition

An inclusion of standard subspaces $K \subset H \subset \mathcal{H}$ is called a **Half-sided Modular Inclusion** (HSMI) if

$$\Delta_H^{-it} K \subset K \quad \text{for } t \geq 0.$$

Examples of HSMI's (1)



Example 1: Wedge-loclized observables in vacuum Wightman theory.

$$\Delta^{-it}\phi(\vec{x})\Omega = \Delta^{-it}\phi(\vec{x})\Delta^{it}\Omega = \phi(\Lambda_{2\pi t}\vec{x})\Omega.$$

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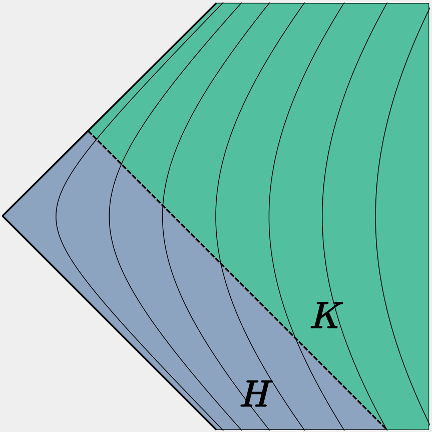

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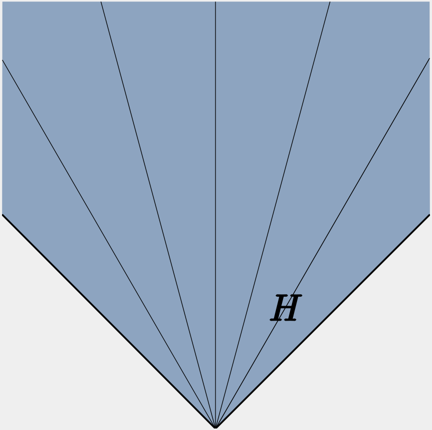
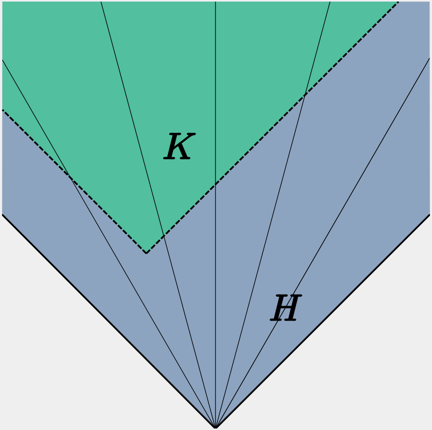
$$\Delta^{-it}\phi(\vec{x})\Omega = \Delta^{-it}\phi(\vec{x})\Delta^{it}\Omega = \phi(\Lambda_{2\pi t}\vec{x})\Omega.$$

Examples of HSMI's (2)



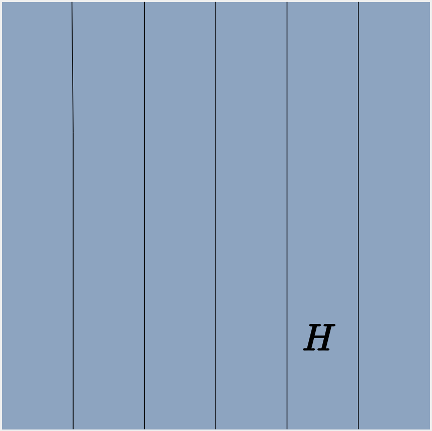
Example 2: Lightcone-localized observables in massless vacuum Wightman theory.

Examples of HSMI's (2)

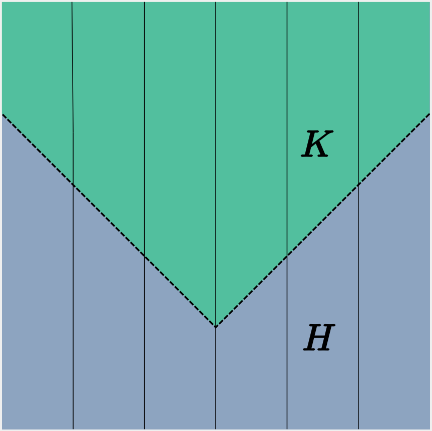


Example 2: Lightcone-localized observables in massless vacuum Wightman theory.

Examples of HSMI's (3)



Example 3: Massless thermal field theory (Borchers Yngvason 1999)



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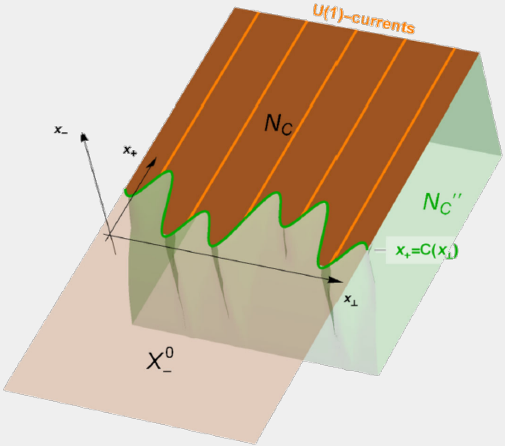
Examples of HSMI's (4)



Example 4: Standard subspace of a null cut (Morinelli Tanimoto Wegener, 2022).

$$K \subseteq H \subseteq \mathcal{H}$$

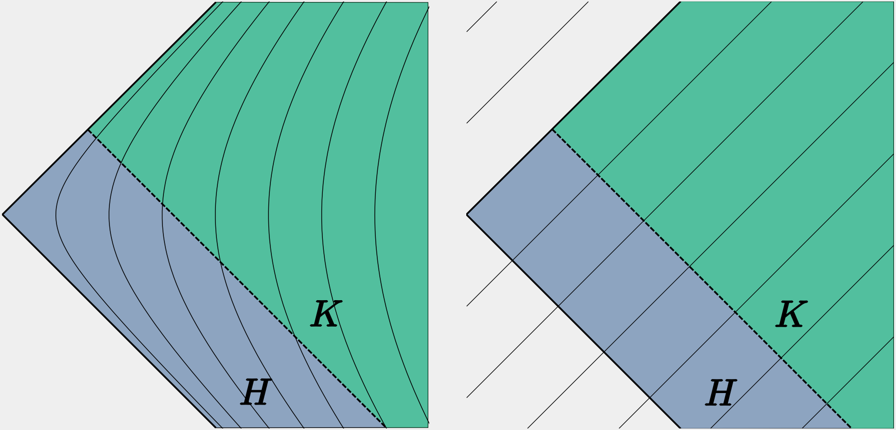
H.S.M.I. → Modular Theory →

$$U(t) := e^{itP}, P \geq 0$$
$$U(s)H \subseteq H \text{ for } s \geq 0$$
$$U(1)H = K$$
$$\Delta_K^{it} = U(1)\Delta_H^{it}U(-1)$$

So what does a HSMI actually bring?

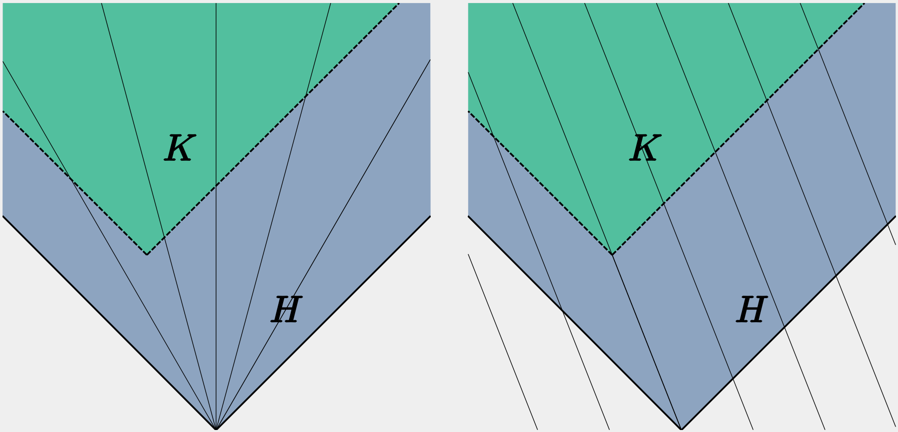
Theorem

Let $K \subseteq H \subseteq \mathcal{H}$ be a HSMI. Then there exists a positively generated one-parameter group U such that $U(s)H \subseteq H$ for all $s \geq 0$ and $U(1)H = K$.



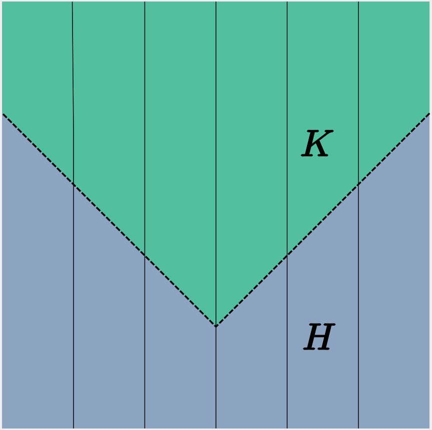
Example 1: Wedge-loclized observables in vacuum Wightman theory.

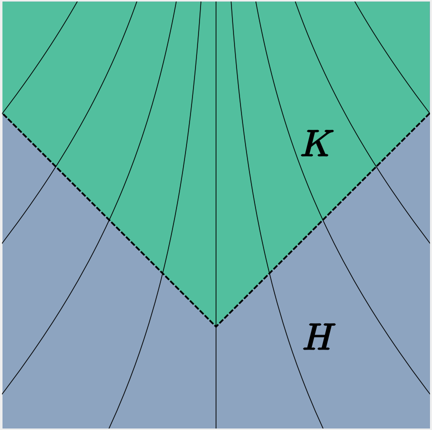
HSMI Examples revisited (2)



Example 2: Lightcone-localized observables in massless vacuum Wightman theory.

HSMI Examples revisited (3)





Example 3: Massless thermal field theory.

By Borchers' Theorem, for a given standard subspace $H \subset \mathcal{H}$ there is a bijection between the following concepts:

Half-sided Modular Inclusion:
$K \subseteq H \subseteq \mathcal{H}$ standard subspace such that $\Delta_H^{-it} K \subset K$ for $t \geq 0$.

Standard pair:
Positively generated $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ such that $U(s)H \subset H$ for $s \geq 0$.

These concepts are related by the following commutation relations:

$$\Delta_H^{it} U(s) \Delta_H^{-it} = \Delta_K^{it} U(s) \Delta_K^{-it} = U(e^{-2\pi t} s)$$

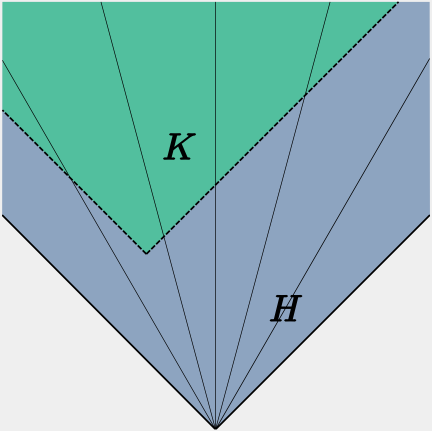
These relations are actually just a form of the Canonical Commutation Relations. Compare, for $U(s) = e^{isP}$, the following:

Standard Pair:	$\Delta_H^{it} U(s) \Delta_H^{-it} = U(e^{-2\pi t} s)$
Generator:	$\Delta_H^{it} P \Delta_H^{-it} = e^{-2\pi t} P$
Weyl relations:	$\Delta_H^{it} P^{is} \Delta_H^{-is} = e^{-2\pi tsi} P^{is}$
ax + b group relations:	$[i \ln \Delta_H, iP] = -2\pi i P$
CCR:	$[\ln \Delta_H, \ln P] = 2\pi i$

The Question

For a given standard subspace $H \subseteq \mathcal{H}$, what does the set of all Half-sided Modular Inclusions in H look like?

- Characterize lattice structure?
- Can a HSMI be 'partly localized'?
- Good way to construct non-trivial examples?



The Tool

It turns out, in this case we can relate inclusions of standard subspaces to inclusions of complex subspaces.

Theorem (IK '25)

Let (U_1, H) and (U_2, H) be two n.d. standard pairs, and let $U_j(t) =: e^{itP_j}$ for $j = 1, 2$. Furthermore, let E^{P_j} be the spectral measure associated to P_j. Then

$$U_1(1)H \subset U_2(1)H \quad \Leftrightarrow \quad E^{P_1}[(0,1)] \leq E^{P_2}[(0,1)].$$

The proof is based on anlysis of the extension of $\Delta_2^{it}\Delta_1^{-it}$ to a strip in the complex plane.

We further use the representation theory of CCR to conclude:

Stone-Von Neumann Theorem

If (U, H) is a n.d. standard pair, then there is a Hilbert space $\mathcal{K}$ and a unitary map $\mathbf{V} : \mathcal{H} \to L^2(\mathbb{R}, \mathcal{K})$ such that

$$(\mathbf{V} \ln P \mathbf{V}^* \psi)(\theta) = i\psi'(\theta), \quad (\mathbf{V} \ln \Delta_H \mathbf{V}^* \psi)(\theta) = 2\pi\theta\psi(\theta)$$

In particular, this means that

$$\mathbf{V} E^P[(0,1)] = \mathbb{H}^2(\mathbb{C}_+, \mathcal{K})$$

Using this we can draw some conclusions:

- If the standard pairs (U_1, H) and (U_2, H) are *irreducible*, then $U_1(1)H \subset U_2(1)H$ if and only if $U_1 = \varphi(\ln \Delta_H) U_2 \overline{\varphi}(\ln \Delta_H)$ for a **symmetric inner function** $\varphi : \mathbb{C}_+ \to \mathbb{C}$.
- There exist standard subspaces $K_1, K_2, H \subseteq \mathcal{H}$ such that $K_1 \subseteq H$ and $K_2 \subseteq H$ are HSMI, and $K_1 \subset K_2$, but $K_1 \subset K_2$ is not a HSMI.
- There exist standard pairs (U_1, H) and (U_2, H) such that $U_1(1)H \subset U_2(1)H$ but $[U_1(t), U_2(t)] \neq 0$.