Friedrich-Alexander-Universität
Erlangen-Nürnberg

Modular theory in Algebraic Quantum Field Theory

Half-sided Modular Inclusions, Standard Pairs and beyond

Ian Koot

RSME Congress of Young Researchers 2025 January 16, 2025

Algebraic QFT

Mathematical objects (for AQFT in $\mathbb{R}^2$):

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- ω: expected value for observables.

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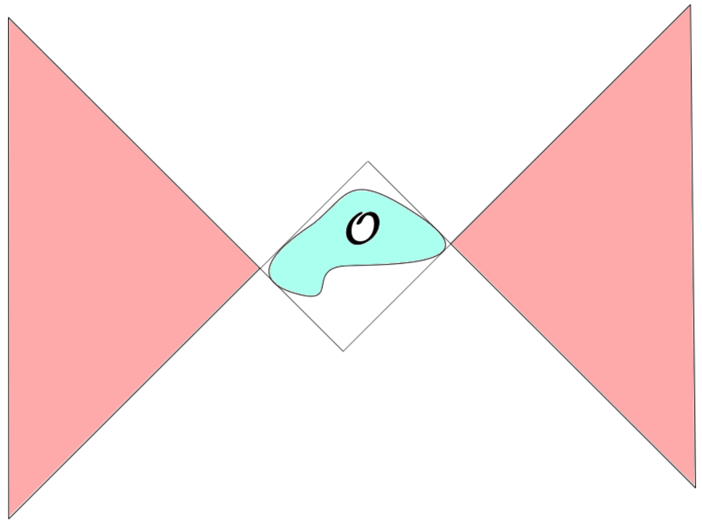

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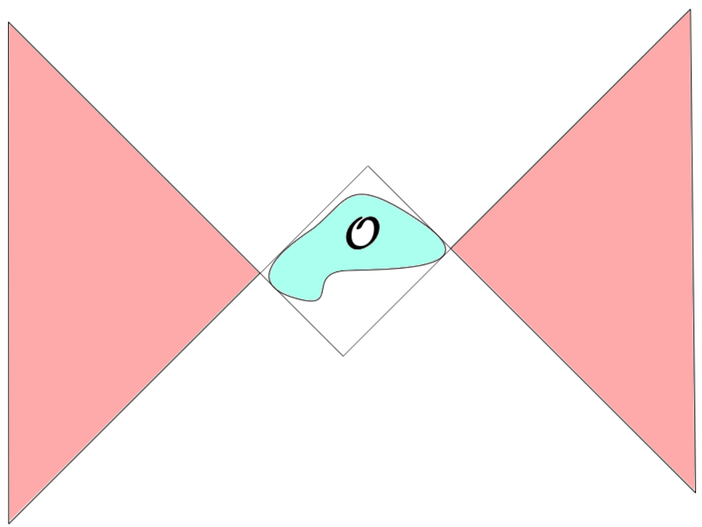

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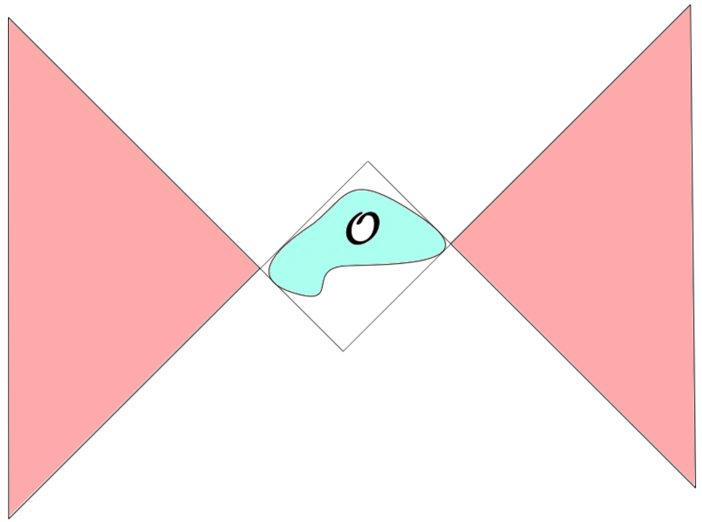

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Where are the 'fields'?

Classically: field is a **value** at each point in spacetime.
Quantum: field is an **observable** at each point in spacetime.
But these are usually singular/unbounded!
$\rightarrow$ we 'smear' them

Modular Theory

Let $\mathcal{A} \subset B(\mathcal{H})$ von Neumann Algebra, $\Omega \in \mathcal{H}$:

$$\text{cyclic: } \overline{\mathcal{A}\Omega} = \mathcal{H}, \qquad \text{separating: } A\Omega = 0 \Rightarrow A = 0$$

Cyclic (resp. separating) for $\mathcal{A}$ is equivalent to separating (resp. cyclic) for $\mathcal{A}'$.

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$$S_{\mathcal{A},\Omega} : \mathcal{A}\Omega \to \mathcal{A}\Omega, \quad A\Omega \mapsto A^*\Omega$$

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Looks innocent, is usually unbounded, with $\overline{S_{\mathcal{A}',\Omega}} = (S_{\mathcal{A},\Omega})^*$. Polar decomposition (with abuse of notation)

$$S_{\mathcal{A},\Omega} = J_{\mathcal{A},\Omega} \Delta^{\frac{1}{2}}_{\mathcal{A},\Omega}$$

Modular Theory

These objects satisfy properties which *a priori* are not obvious at all:

$$\Delta^{it}\Omega = \Omega, \quad \Delta^{it}\mathcal{A}\Delta^{-it} = \mathcal{A}, \quad J\mathcal{A}J = \mathcal{A}'$$

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Example: matrix case

Let $\mathcal{H} := M_n(\mathbb{C})$ with $\langle A, B\rangle :=$ tr(A^*B).
We take $\Omega = \rho^{\frac{1}{2}}$ (from $A \mapsto$ tr(ρA))
Writing $L[A] : B \mapsto AB$ we have $\mathcal{A} := L[M_n(\mathbb{C})] \subset B(M_n(\mathbb{C}))$. Then $\mathcal{A}' = R[M_n(\mathbb{C})]$ (right multipliers).
The modular objects now are:

$$J(A) = A^*, \quad \Delta^{it}(A) = \rho^{it}A\rho^{-it}$$

so that indeed

$$J\Delta^{\frac{1}{2}}(L[A]\rho^{\frac{1}{2}}) = J(\rho^{\frac{1}{2}}(A\rho^{\frac{1}{2}})\rho^{-\frac{1}{2}}) = A^*\rho^{\frac{1}{2}} = L[A]^*\rho^{\frac{1}{2}}$$

How do they combine?

Theorem (Takesaki 1970)

Let $(\mathcal{H}, \mathcal{A}(\cdot), U, \Omega)$ be a QFT. Then the state $\langle \Omega, \cdot \Omega \rangle$ on $\mathcal{A}(\mathcal{O})$ is at "temperature" β w.r.t. the time evolution $U((t,0))$ if and only if

$$\Delta^{it}_{\mathcal{A}(\mathcal{O}),\Omega} = U\left(\left(-\frac{\beta t}{2}, 0\right)\right)$$

Theorem (Borchers 1992)

Let $(\mathcal{H}, \mathcal{A}(\cdot), U, \Omega)$ be a QFT, such that U is a **positive energy representation** and Ω is a **vacuum vector** for U. Then

$$\Delta^{it}_{\mathcal{A}(W),\Omega} U(\vec{x}) \Delta^{-it}_{\mathcal{A}(W),\Omega} = U(\Lambda_{2\pi t}\vec{x})$$

Summary: QFT & Mod. Th.

In Summary:

- QFT: von Neumann Algebras $\mathcal{A}(\mathcal{O})$ with **inclusion** and **commutation** relations $+$ **unitary representation** U of $\mathbb{R}^2$ $+$ a state ω (sometimes given by a vector Ω).

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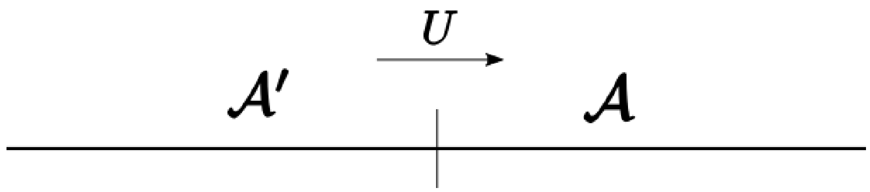
- QFT: von Neumann Algebras $\mathcal{A}(\mathcal{O})$ with **inclusion** and **commutation** relations $+$ **unitary representation** U of $\mathbb{R}^2$ $+$ a state ω (sometimes given by a vector Ω).
- Modular theory: **von Neumann Algebra** and a **standard vector** $\Rightarrow$ the **modular group** Δ^{it} and the **modular conjugation** J 'for free'.
- We're interested in the interaction between QFT and modular theory.

Standard Pair

A simpler situation: suppose $\mathcal{A} \subset B(\mathcal{H})$ VNA, and $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ *positively generated*. Geometrical assumption:

$$U(t)\mathcal{A}U(-t) \subset \mathcal{A} \quad \text{for } t \geq 0$$

The picture we have in mind:



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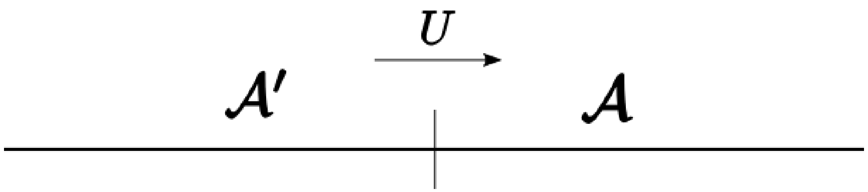


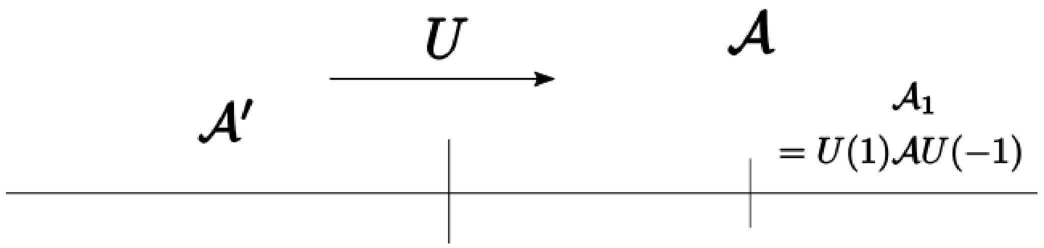
Theorem (Borchers 1992)

If $(\mathcal{A}, U)$ as above, and Ω is standard for $\mathcal{A}$ and invariant for U, then

$$\Delta_{\mathcal{A},\Omega}^{it} U(s) \Delta_{\mathcal{A},\Omega}^{-it} = U(e^{-2\pi t} s)$$

Halfsided Modular Inclusions

Let us now introduce subalgebras; $\mathcal{A}_1 = U(1)\mathcal{A}U(-1)$ is a natural one.



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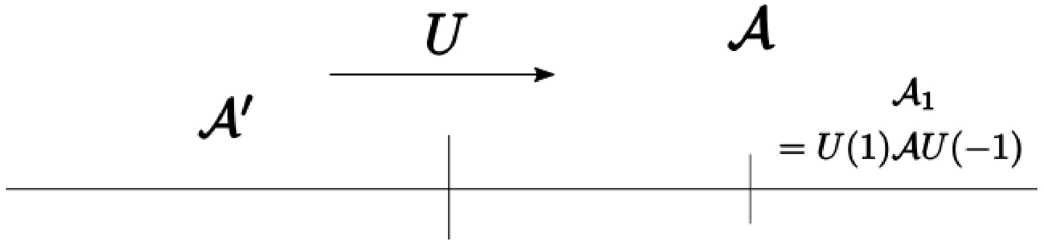

Then $\mathcal{A}_1$ satisfies a property only concerning the modular group:

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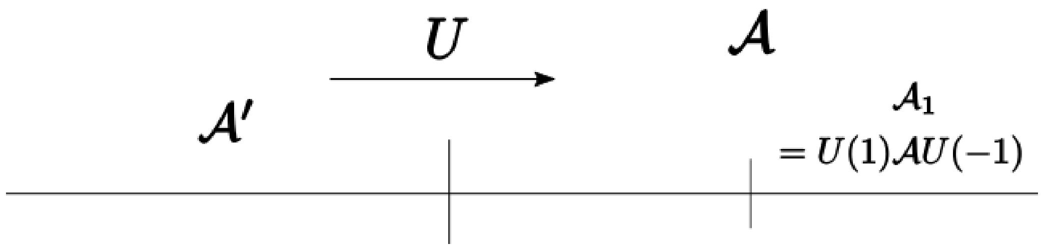

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for $t \leq 0$.
This is special! We call $\mathcal{A}_1 \subset \mathcal{A}$ a **half-sided modular inclusion** (w.r.t. Ω).

Std. Pairs and HSMIs

It is a totally non-obvious fact that *all* half-sided modular inclusions arise this way (Wiesbrock 1993, Araki-Zsido 2004):

$$\left\{ \mathcal{A}_1 \subset \mathcal{A}_2 \,\middle|\, \begin{array}{c} \Omega \text{ std. for } \mathcal{A}_1 \\ \Delta^{it}_{\mathcal{A},\Omega} \mathcal{A}_1 \Delta^{-it}_{\mathcal{A},\Omega} \subset \mathcal{A}_1 \\ \text{for all } t \leq 0 \end{array} \right\} \leftrightarrow \left\{ U : \mathbb{R} \to \mathcal{U}(\mathbb{R}) \,\middle|\, \begin{array}{c} U \text{ pos. gen.} \\ U(t)\mathcal{A}U(-t) \subset \mathcal{A} \\ \text{for all } t \geq 0 \end{array} \right\}$$

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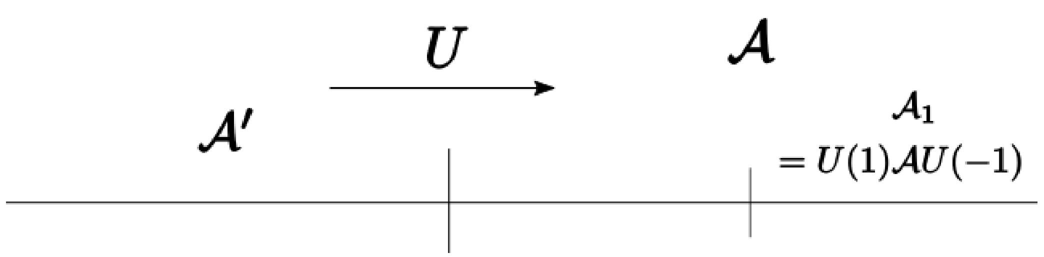
From the perspective of the half-sided modular inclusions, there is a *hidden* geometry.



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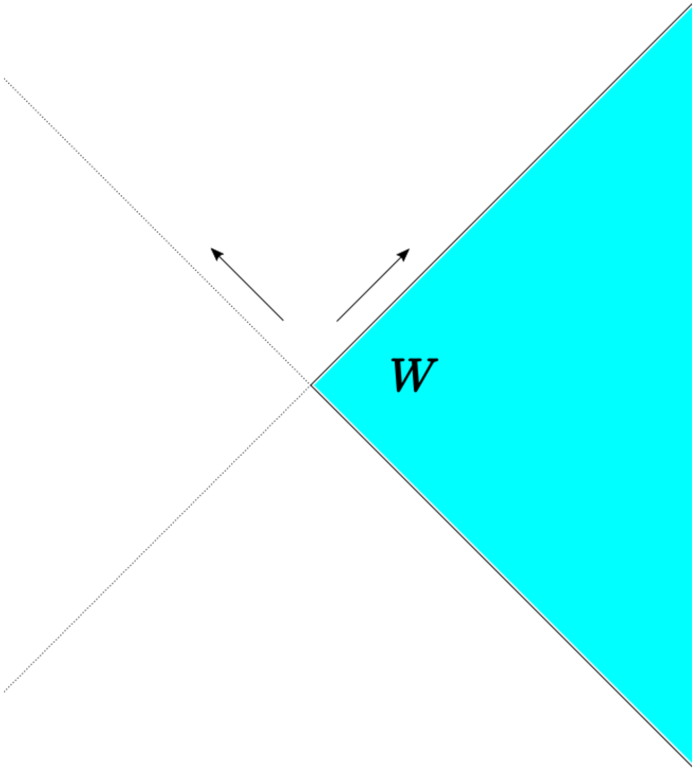

Dimension higher: $\mathcal{A}$ VNA and $U : \mathbb{R}^2 \to \mathcal{U}(\mathcal{H})$ **positive energy** rep. s.t.

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This is for example the case for $\mathcal{A}(W)$ in a vacuum QFT.

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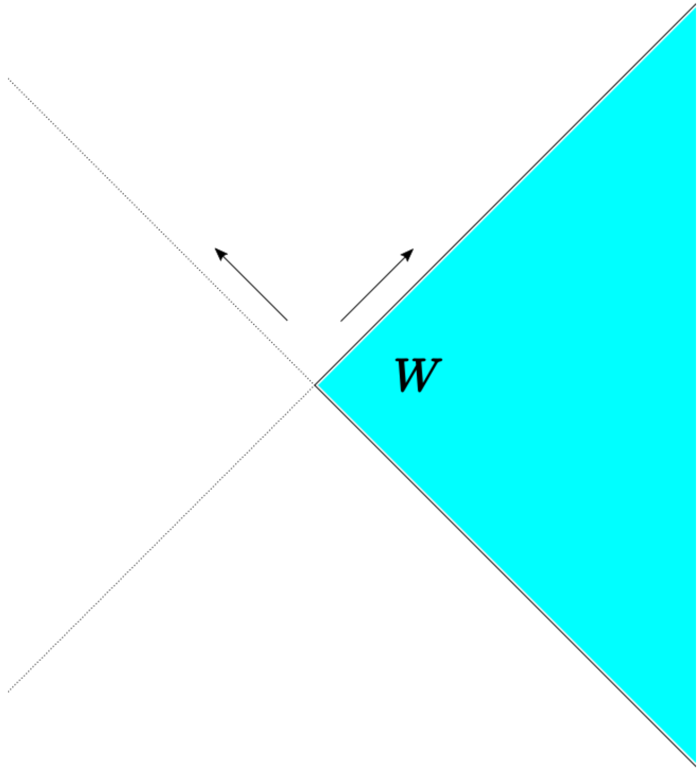

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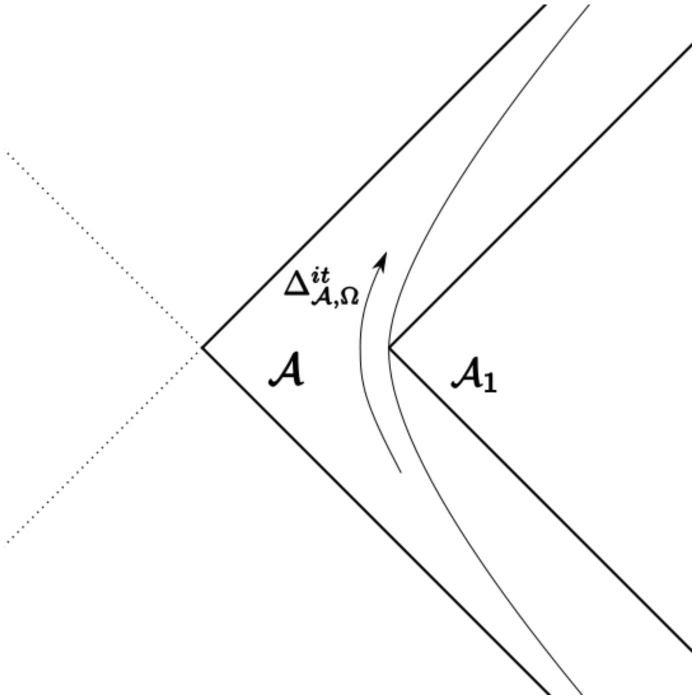
In the above situation we have

$$\Delta^{it}_{\mathcal{A},\Omega} U(\vec{x}) \Delta^{-it}_{\mathcal{A},\Omega} = U(\Lambda_{-2\pi t}\vec{x})$$

where Λ_s are the so called **Lorentz boosts**.

The other direction?



We are working on the following question:

Open Question

When can we reconstruct U from $\mathcal{A}$ and $\mathcal{A}_1$ as in the 1-dimensional case?

The inclusion $\mathcal{A}_1 \subset \mathcal{A}$ is a no longer a HSMI; but with an **extra** direction (corresponding to spatial translations) we can recognize two HSMI's along the lightrays!
We're currently investigating when the two standard pair directions commute.

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- Modular Theory shows up in AQFT (as well as in other places).
- There is a correspondence between standard pairs (VNA + pos. gen. rep.) and half-sided modular inclusions (VNA + subVNA)
- We're researching a generalization to higher dimensions.

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