Friedrich-Alexander-Univ
Erlangen-Nürnberg

Thermal States in Algebraic Quantum Field Theory

Promotionsprogramm Mathematik

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What is Algebraic Quantum Field Theory?

A **Quantum Field Theory** $(\mathcal{H}, \Omega, \mathcal{A}, U)$ consists of

- A Hilbert space $\mathcal{H}$ and a vector $\Omega \in \mathcal{H}$,
- A net of Von Neumann Algebras $\mathcal{A}$, i.e. $\mathcal{A}(O) \subseteq B(\mathcal{H})$ is a Von Neumann algebra for each open subset $O \subset \mathbb{R}^4$.
- A (strongly continuous) representation

$$U : \mathbb{R}^4 \rtimes O(1,3) \to \mathcal{U}(\mathcal{H})$$

such that

- Ω is standard for all $\mathcal{A}(O)$ with O bounded.
- Ω is invariant under U.
- if $O_1 \subseteq O_2$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
- if $O_1 \subseteq O_2'$, then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)'$.
- for all $g \in \mathbb{R}^4 \rtimes O(1,3)$ we have

$$U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO)$$

1. Quantum Mechanics and Non-commutative Probability

2. Relativistic Field Theory

3. Thermal states in Free Fermionic Klein Gordon Theory

Measurements

- Suppose we represent each **possible measurement** by a symbol. Let $\mathcal{A}_o$ be the collection of all those symbols ('observables').
- Suppose we represent each **way of preparing the system** by another symbol. Let $\mathcal{S}(\mathcal{A})$ be the collection of all these symbols ('states').
- We do not assume that for a fixed preparation and a fixed measurement we get the same outcome every time. If $a \in \mathcal{A}_o$ and $\omega \in \mathcal{S}(\mathcal{A})$ we write $\omega(a)$ for the **expected** outcome of measurement a when the system is prepared in state ω.
- For every observable $a \in \mathcal{A}_o$, there is an observable $a^2 \in \mathcal{A}_o$ created by taking the measurement a and squaring the result. This measurement has a positive outcome **independent** of how we prepared the system; it is a **positive** observable.
- We now want to describe correlations between different measurements. In classical probability theory, we could for example calculate

  $$\text{Cov}_\mu(X, Y) := \mathbb{E}_\mu[(X - \mathbb{E}_\mu[X])(Y - \mathbb{E}_\mu[Y])] = \mathbb{E}_\mu[X \cdot Y] - \mathbb{E}_\mu[X]\mathbb{E}_\mu[Y].$$

  In order to describe correlations, we need a product.

However, quantum correlations are special. We need a **non-commutative** product.

Algebras and States

Definition

A $*$-**algebra** is a $\mathbb{C}$-vector space $\mathcal{A}$ with bilinear multiplication $\cdot$ (with unit $1 \in \mathcal{A}$) and antilinear involution $x \mapsto x^*$ such that $(xy)^* = y^* x^*$.
An element $x \in \mathcal{A}$ is called **observable** if $x^* = x$ and **positive** if $x = y^* y$ for some $y \in \mathcal{A}$.
A **state** is a linear map $\omega : \mathcal{A} \to \mathbb{C}$ that is unital and positive.

More specifically: $\mathcal{A}$ with norm such that $\|xy\| \leq \|x\|\|y\|$, $\|x^*\| = \|x\|$ and $\|x^* x\| = \|x^2\|$ is a C^*-**algebra**. States are then continuous.

Theorem

Commutative C^*-algebras are isomorphic to $C_0(\Omega)$ for some L.C. Hausdorff space Ω.
A state on a commutative C^*-algebra is given by a measure.

Example: uncertainty relation

$$\omega((x-\omega(x))^2)\omega((y-\omega(y))^2) \geq \left|\frac{1}{2}\omega(\{x,y\}) - \omega(x)\omega(y)\right|^2 + \left|\frac{1}{2}\omega([x,y])\right|^2$$

Important Example: Operator Algebras!

An example of a noncommutative C^*-algebra are the matrices over $\mathbb{C}$ (more specifically, bounded operators on a Hilbert space $B(\mathcal{H})$).

- Multiplication is composition/matrix multiplication.
- The involution is the adjoint:

$$(A^*)_{ij} = \overline{A_{ji}} \quad \langle v, Aw \rangle = \langle A^* v, w \rangle$$

- Self-ajoint is equivalent to diagonalizable with real eigenvalues, positive is equivalent to diagonalizable with non-negative eigenvalues.
- Unit vectors define states through $A \mapsto \langle v, Av \rangle$ ('expectation values'), *but these are not all the states!*.

Note that if $[A, B] = 0$ (that is to say, $AB = BA$), then A and B are simultaneously diagonalizable: they can be simultaneously well-defined by an eigenvector state (there are 'no quantum-correlations').

Von Neumann Algebras

We saw that C^*-algebras are 'non-commutative topological spaces'. But we want states to be similar to integration, i.e. should be closed under weaker limits then uniform limits, and states should be like integrals, i.e. satisfy dominated convergence:

Definition

The **commutant** of a set $\mathcal{A} \subset B(\mathcal{H})$ is defined as

$$\mathcal{A}' = \{B \in B(\mathcal{H}) : \forall A \in \mathcal{A} : [A, B] = 0\}.$$

A **von Neumann** algebra is a C^*-subalgebra $\mathcal{A} \subset B(\mathcal{H})$ such that $\mathcal{A}$ is closed in the *weak operater topology*, or equivalently, that satisfies $\mathcal{A}'' = \mathcal{A}$.
A **normal** state is a weakly continuous state, or equivalently, a state ω such that for all H_α with $\sup_\alpha H_\alpha = H$, we have $\lim \omega(H_\alpha) = \omega(H)$.

Advantages: nice structure theory, Tomita-Takesaki Theory, measurable Functional Calculus, etc.

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Very Fast Physics History

What is the universe?

- Newton: particles, i.e. curves $\vec{x}_i : \mathbb{R} \to \mathbb{R}^3$ or $\vec{x}_i : \mathbb{R} \to \mathbb{R}^4$
- Maxwell: fields, i.e. functions $\phi : \mathbb{R} \times \mathbb{R}^3 \to V$ (here V is might be a vector space, or something more general)

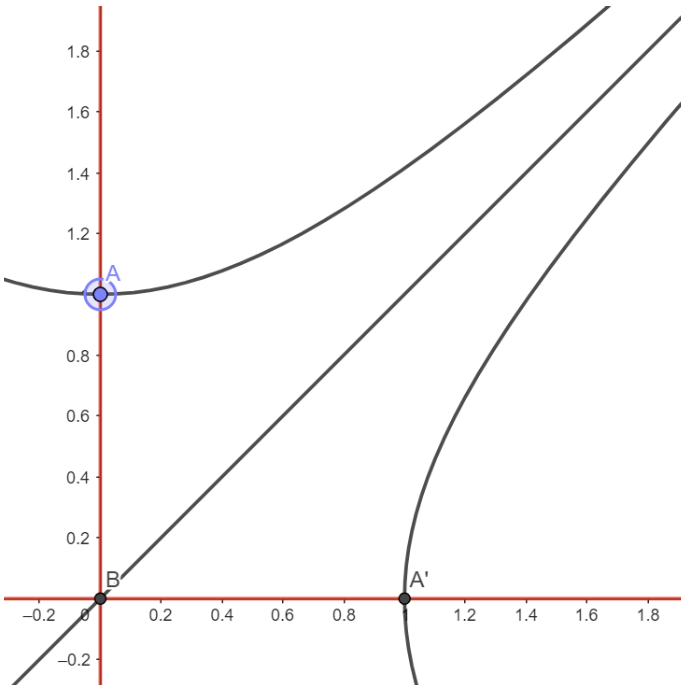
What are the laws of Physics? (what is a Physical model?)

- Newton: forces, i.e. $\vec{F} = m\vec{a}$.
- Lagrange: principal of least action, i.e. $\mathcal{L}[\vec{x}_i]$ or $\mathcal{L}[\phi]$ is minimal.

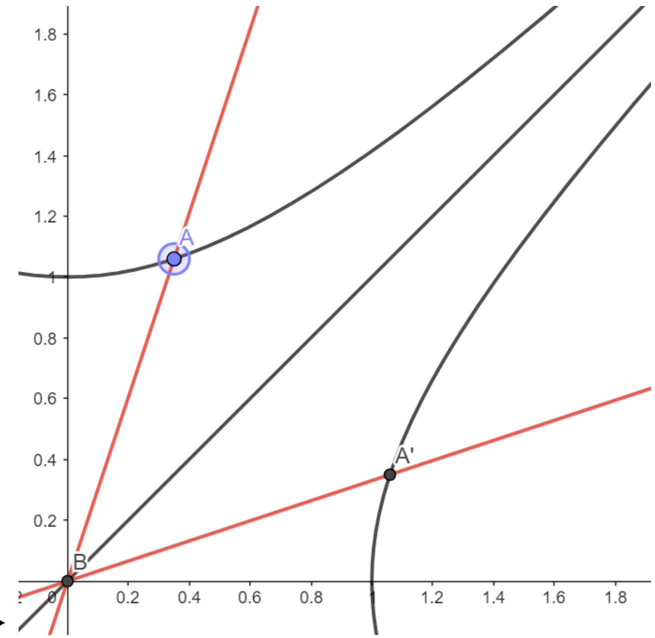
The laws of physics should be the same for everyone; how does another observer see the universe?

- Galilei: just add positions and velocities.
- Einstein: positions add, but velocities **boost**.

Special Relativity



→ Lorentz Boost →



The Problem with QFT

So if we combine all these views with the Quantum Mechanics we had before, we would look for:

- An observable $\phi(x) \in \mathcal{A}$ at each point in space;
- that minimizes some Lagrangian $\mathcal{L}[\phi] \to$ satisfies the equations of motion;
- A state for $\mathcal{A}$.

However, two problems:

- We don't know of any such constructions for any nontrivial $\mathcal{L}$ (in 4 dimensions).
- Even in the trivial case, the $\phi(x)$ are often unbounded.

So instead of taking a specific model, we are looking for any example of...

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Free Theory

Definition

Let $\mathcal{K}$ be a Hilbert space and let Γ be a anti-unitary involution. The **Self-Dual CAR-Algebra** $\text{CAR}(\mathcal{K}, \Gamma)$ is the *-algebra generated by $\phi(k)$ for $k \in \mathcal{K}$ such that

- $k \mapsto \phi(k)$ is $\mathbb{C}$-linear
- $\phi(\Gamma k) = \phi(k)^*$
- $\phi(k_1)\phi(k_2) + \phi(k_2)\phi(k_1) = \langle \Gamma k_1, k_2 \rangle$

It turns out you can define a state (called a quasi free state with covariance C) by setting $\omega(\phi(k_1)^*\phi(k_2)) = \langle k_1, Ck_2 \rangle$ for C a positive operator such that $0 \leq C \leq 1$ and $\Gamma C \Gamma = 1 - C$.

This is in some sense the 'simplest' set of relations that observables in a QFT can have (together with CCR).

Thermal Free Klein Gordon

We let $\mathcal{K}$ be $C_c^\infty(\mathbb{R}^2, \mathbb{C})$, quotiented such that

$$\phi((\partial_t^2 - \partial_x^2 - m^2)f) = 0$$

with Γ the complex conjugation.
The state we examine is a thermal state:

Definition

Let α_t be a one-parameter group of Automorphisms on a C^*-algebra $\mathcal{A}$. A KMS-state ω at inverse-temperature β is a state such that for all $A, B \in \mathcal{A}$ the function

$$t \to \omega(A\alpha_t(B))$$

has an analytic extension to the strip of width β, with

$$t + i\beta \to \omega(\alpha_t(B)A).$$

This exists in this 'trivial model'. What are the properties that models need to have for it to exist? Can we calculate it? What are its properties?

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