Friedrich-Alexander-Universität
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Thermal Field theories and Wedge-Modular Inclusions

LQP49, Erlangen

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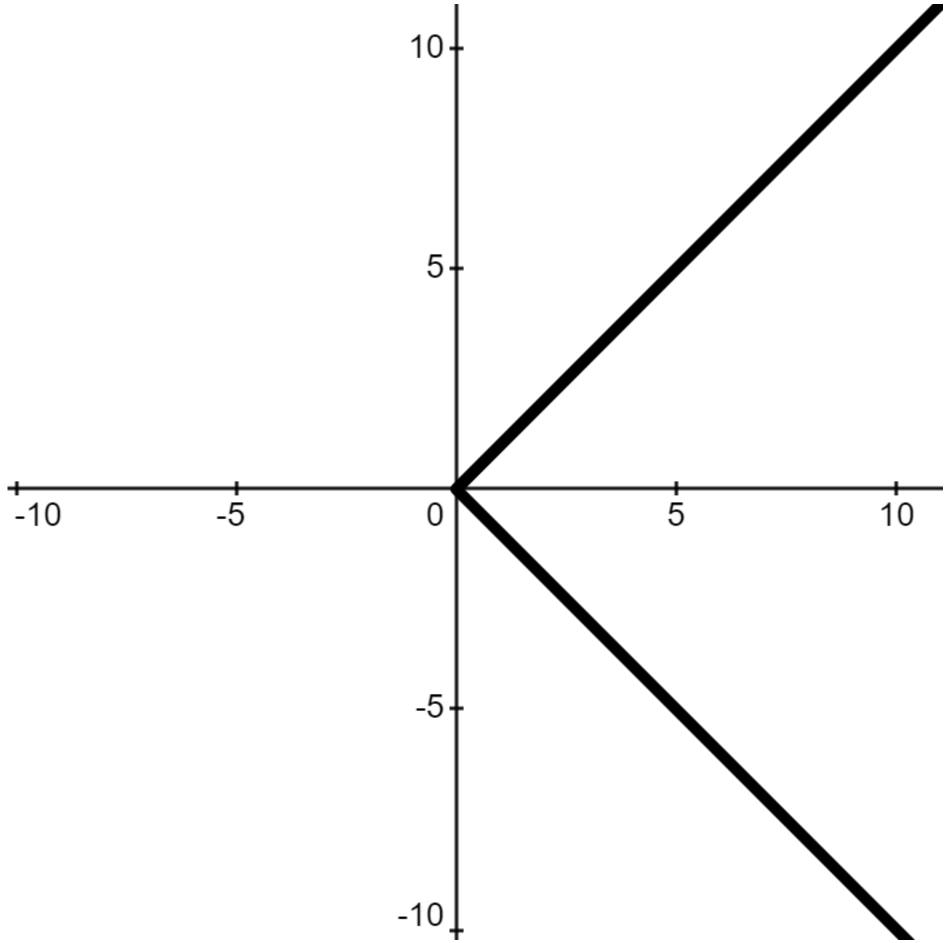
1. The Setting
2. Standard Pairs and HSMI's
3. Generalization to 2 Dimensions
4. Wedge-Modular Inclusions

The Setting



- $\mathcal{A}_\infty$ a C^*-algebra: **quasi-local algebra**
- α_t a one-parameter group of automorphisms: **time evolution**
- ω a KMS-state on $\mathcal{A}_\infty$ for α_t : **thermal state**
- $\mathcal{A} \subset \mathcal{A}_\infty$ a C^*-algebra: **wedge algebra**

GNS $\Rightarrow$ $\Omega \in \mathcal{H}_\omega$ cyclic vector for $\mathcal{A}_\infty$.

+ KMS $\Rightarrow$ Ω is **standard** for $\mathcal{A}_\infty$.

Proposition

Under 'loose assumptions' Ω is **standard** for $\mathcal{A}$.

Q: *What is the modular theory of $\mathcal{A}$ w.r.t. Ω?*

Modular Theory: what?

Standard vector $\Omega \in \mathcal{H}$ for VNA $\mathcal{A} \subset B(\mathcal{H})$:

$$\text{(cyclic): } \overline{\mathcal{A}\Omega} = \mathcal{H}, \quad \text{(separating): } \forall A \in \mathcal{A} : A\Omega = 0 \Rightarrow A = 0$$

Construct modular objects:

$$S : \mathcal{A}\Omega \to \mathcal{A}\Omega, \quad A\Omega \mapsto A^*\Omega$$

$$S = J\Delta^{\frac{1}{2}} \quad \text{and} \quad \Delta^{it}\mathcal{A}\Delta^{-it} = \mathcal{A}$$

Actually more abstract: $H \subset \mathcal{H}$ a *closed* real subspace is **standard subspace** if

$$\text{(cyclic): } \overline{H + iH} = \mathcal{H}, \quad \text{(separating): } H \cap iH = \{0\}$$

Construct modular objects:

$$S : H + iH \to H + iH, \quad h_1 + ih_2 \mapsto h_1 - ih_2$$

$$S = J\Delta^{\frac{1}{2}} \quad \text{and} \quad \Delta^{it}H = H$$

Modular Theory: why?

Many reasons!

- Thermal states: Gibbs state $\leftrightarrow$ KMS state $\leftrightarrow$ Modular theory
- Modular nuclearity: local degrees of freedom & split property
- Modular localization: constructing a QFT from representation theory.
- Structure of algebra: classification and relation to commutant

In general, modular theory is hard to calculate. However:

Theorem (Bisognano-Wichmann)

$\mathcal{A}(W)$ and Ω of Wightman QFT $\quad \Rightarrow \quad \Delta^{it}$ is given by **Lorentz boost**.

What is already known?

The data that we have:

- ω a ... state on $\mathcal{A}_\infty$
- $\alpha : \mathbb{R}^2 \to \mathrm{Aut}(\mathcal{A}_\infty)$
- $\omega \circ \alpha(\vec{x}) = \omega$ for $\vec{x} \in \mathbb{R}^2$
- $\alpha(\vec{x})(\mathcal{A}) \subset \mathcal{A}$ for $\vec{x} \in W$

After GNS it becomes:

- $\Omega \in \mathcal{H}$ cyclic for $\mathcal{A}_\infty$
- $u : \mathbb{R}^2 \to \mathcal{U}(\mathcal{H})$ preserving $\mathcal{A}_\infty$
- $u(\vec{x})\Omega = \Omega$
- $u(\vec{x})\mathcal{A}u(-\vec{x}) \subset \mathcal{A}$ for $\vec{x} \in W$.

	$\mathcal{A}_\infty$	$\mathcal{A}$
Vacuum	Ω not standard	$\Delta^{is}U(\vec{x})\Delta^{-is} = U(\Lambda_{-2\pi s}\vec{x})$
Thermal	$\Delta^{it} = u((-\beta t, 0))$	??

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As standard subspace:

- $H \subset H_\infty \subset \mathcal{H}$ std. subsp. inclusion
- $u : \mathbb{R}^2 \to \mathcal{U}(\mathcal{H})$ preserving H_∞
- $u(\vec{x})H \subset H$ for $\vec{x} \in W$

	$\mathcal{A}_\infty$	$\mathcal{A}$
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Standard Pair

One can prove 2D vacuum **using 1D vacuum case**:

Theorem (Borchers/Florig)

$H \subset \mathcal{H}$ standard subspace, $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ positively generated, $U(\mathbb{R}_{\geq 0})H \subset H$. Then

$$\Delta_H^{is} U(t) \Delta_H^{-is} = U(e^{-2\pi s} t).$$

A pair (H, U) is called a **standard pair**. Note: $\ln \Delta_H$ and $\ln \partial U$ satisfy CCR relations. Consider the inclusion $U(1)H \subset H$. We have

$$\Delta_H^{it} U(1) H = U(e^{-2\pi t}) \Delta_H^{it} H = U(1) U(e^{-2\pi t} - 1) H \subset U(1) H$$

for $t \leq 0$.

Line Picture

So: $H \subset \mathcal{H}$ standard subspace, $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ positively generated one-parameter group, $U(\mathbb{R}_{\geq 0})H \subset H$ can be summarized as:

- H corresponds to right half line;
- U corresponds to right translation;
- Δ_H^{it} corresponds to scaling.

Here we see geometrically: $\Delta_H^{it} U(1) H \subset H$ for $t \leq 0$.

Halfsided Modular Inclusions

Definition

Halfsided Modular Inclusion: Inclusion $H_1 \subset H_2 \subset \mathcal{H}$ such that

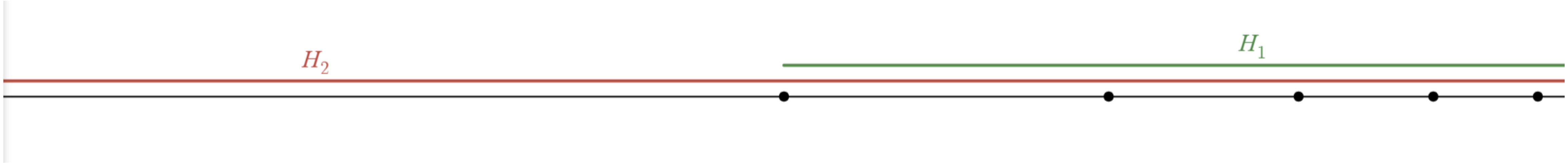
$$\Delta_{H_2}^{-it} H_1 \subset H_1 \quad \text{for } t \geq 0.$$

Theorem (Wiesbrock/Araki-Zsido)

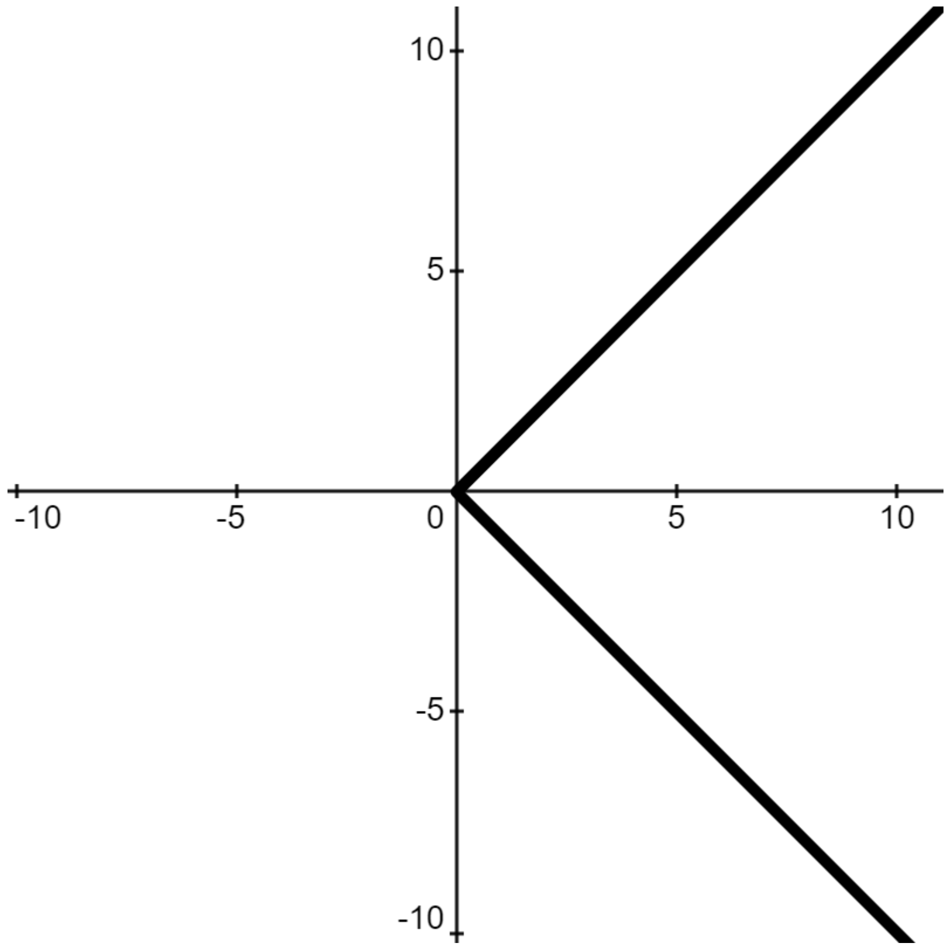
Let $H_1 \subset H_2 \subset \mathcal{H}$ be a Halfsided Modular Inclusion. Then

$$\Delta_{H_2}^{-it} \Delta_{H_1}^{it} = U(1 - e^{2\pi t})$$

defines a positively generated one-parameter group $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$.



Standard Pairs and HSMI's



So there is a correspondence between **standard pairs** and **Half-sided Modular Inclusion**:

$$\begin{aligned} (H, U) &\quad\mapsto\quad U(1)H \subset H \\ (H_2, \Delta_{H_2}^{-if(t)}\Delta_{H_1}^{if(t)}) &\quad\hookleftarrow\quad H_1 \subset H_2 \end{aligned}$$

Q: Generalization to 2 Dimensions?

Chiral example

(Modification of discussion in Borchers and Yngvasson (1999)) Consider:

- $H \subset H_\infty \subset \mathcal{H}$ *positive* HSMI;
- $K \subset K_\infty \subset \mathcal{K}$ *negative* HSMI.

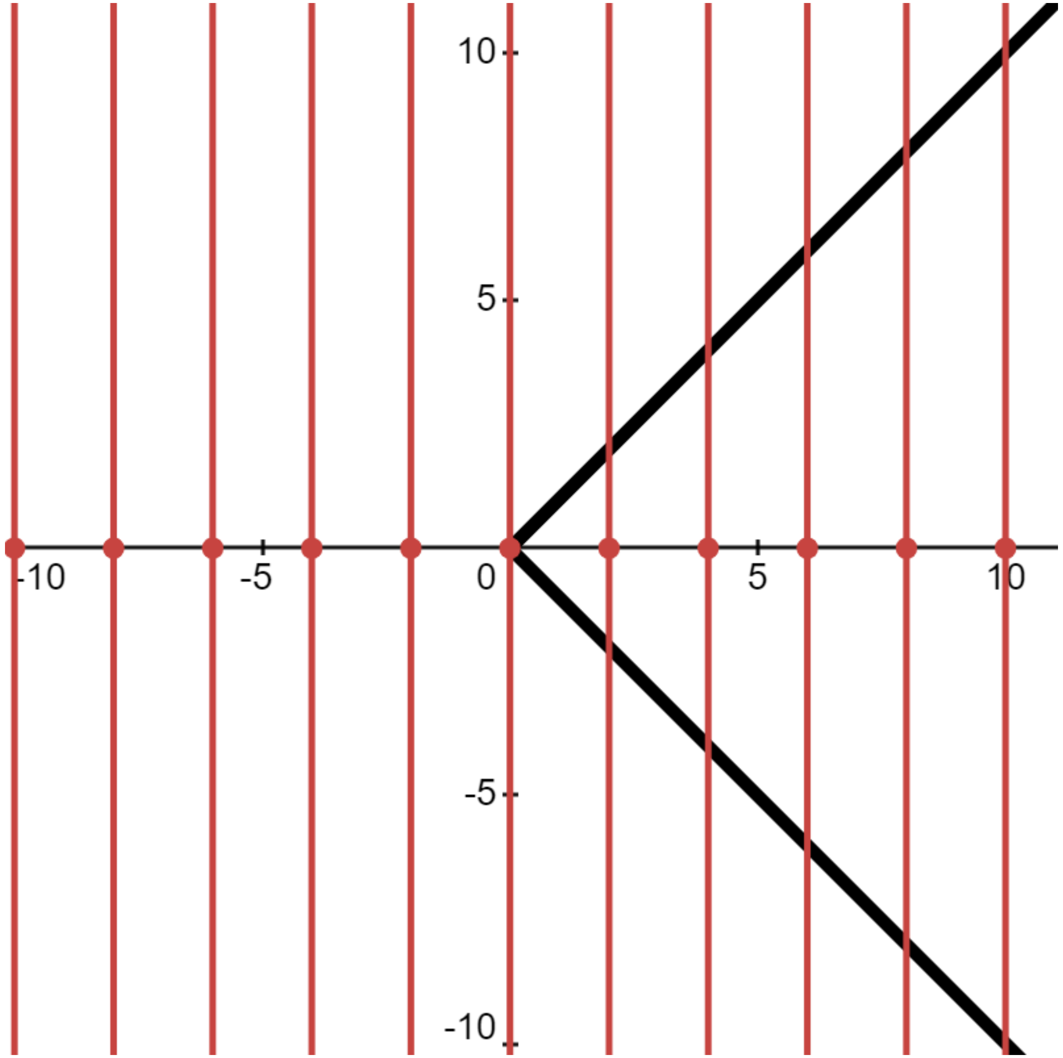
From this we construct the following data:

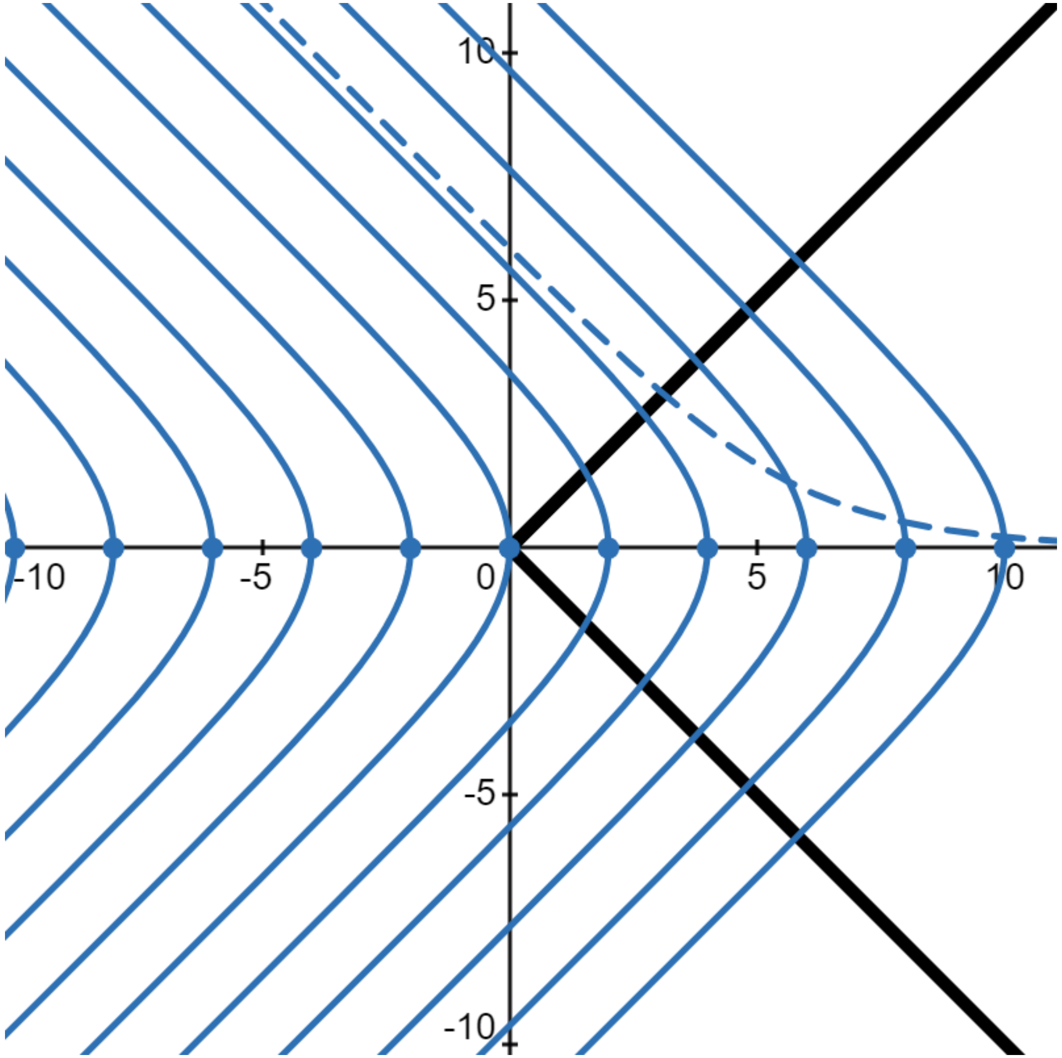
- The inclusion $H \otimes_{\mathbb{R}} K \subset H_\infty \otimes_{\mathbb{R}} K_\infty \subset \mathcal{H} \otimes_{\mathbb{C}} \mathcal{K}$;
- The representation $u(\vec{x}) = \Delta_{H_\infty}^{-i(x_0-x_1)/2\beta} \otimes \Delta_{K_\infty}^{-i(x_0+x_1)/2\beta}$

But HSMI-Standard Pair correspondence: **there is a positive energy representation around!**

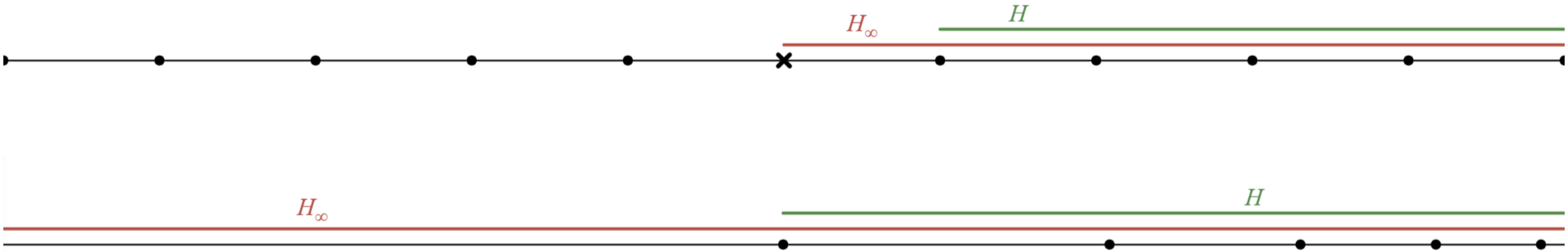
$$U(\vec{x}) = U_H\left(\frac{x_0 - x_1}{2\beta}\right) \otimes U_K\left(\frac{x_0 + x_1}{2\beta}\right)$$

The new $\mathbb{R}^2$ representation

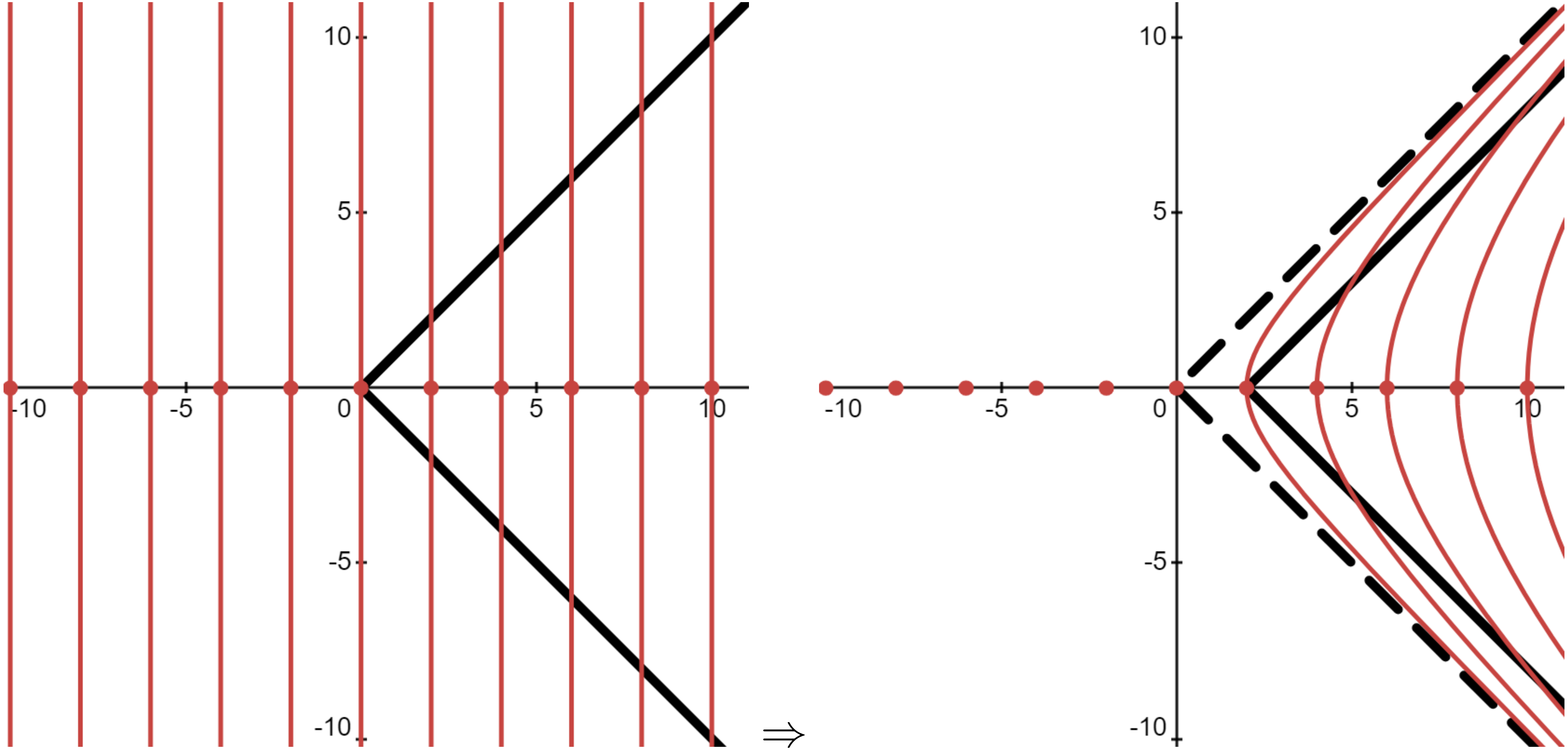




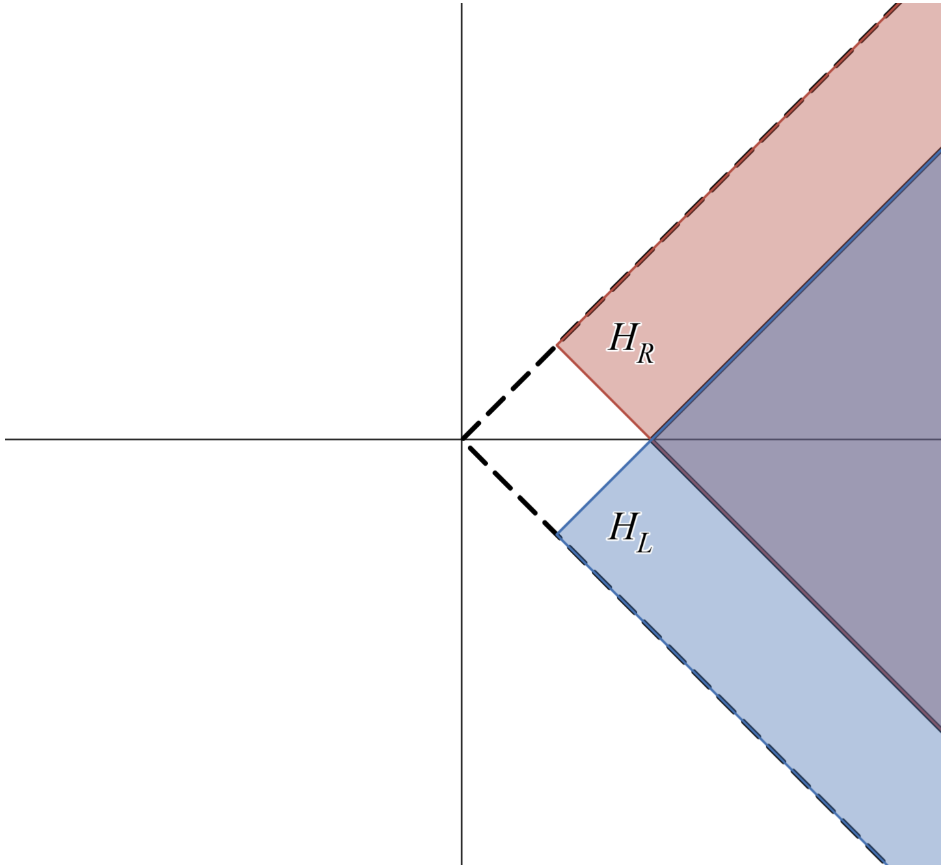
A change of perspective



A change of perspective



Lightray-shifted spaces



How do we find such a U in general?

- $H \subset H_\infty \subset \mathcal{H}$
- $u : \mathbb{R}^2 \to \mathcal{U}(\mathcal{H})$ with $u((t,0)) = \Delta_H^{-it}$
- $u(\vec{x})H_\infty = H_\infty$ for all $\vec{x} \in \mathbb{R}^2$
- $u(\vec{x})H \subset H$ for all $\vec{x} \in W$

There are hidden HSMI's around! We construct

$$H_R := \bigvee_{t \geq 0} u((t,-t))H \subset H_\infty$$

$$H_L := \bigvee_{t \geq 0} u((-t,-t))H \subset H_\infty$$

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Wedge-Modular Inclusions

Definition

Let $H \subset H_\infty \subset \mathcal{H}$ be an inclusion of standard subspaces, $u : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ a strongly continuous one-parameter group such that

- $u(x)H_\infty = H_\infty$ (so we extend u with $\Delta_{H_\infty}^{it}$)
- $u(\vec{x})H \subset H$ for $\vec{x} \in W$
- $H_L \cap H_R = H$.

"Theorem"

The U_L and U_R from the HSMI's $H_L \subset H$ and $H_R \subset H$ commute.

Wedge-Modular Inclusions

"Theorem"

Let $H \subset H_\infty \subset \mathcal{H}$ with $u : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ a Wedge-Modular Inclusion. Then there exists a positive energy representation $U : \mathbb{R}^2 \to \mathcal{U}(\mathcal{H})$ defined by

$$U(2\sinh(2\pi t), 0) = \Delta_{H_\infty}^{-it} \Delta_H^{i2t} \Delta_{H_\infty}^{-it}$$
$$U(0, 2(\cosh(2\pi t) - 1)) = \Delta_{H_\infty}^{it} \Delta_H^{-it} \Delta_{H_\infty}^{-it} \Delta_H^{it}$$

We then have

$$\Delta_{H_\infty}^{it} U(\vec{x}) \Delta_{H_\infty}^{-it} = \Delta_H^{it} U(\vec{x}) \Delta_H^{-it} = U(\Lambda_{-2\pi t}\vec{x})$$
$$u(s)U(\vec{x})u(-s) = U(e^{-2\pi s}\vec{x})$$

A nontrivial example

For the proof, we shift perspective to H_R and H_L, in particular U_R and U_L.
Example: Take (U_0, H_0) *irreducible* standard pair, i.e.

- $\mathcal{H} = L^2(\mathbb{R}, d\lambda)$
- $\ln \varDelta_0 = M[\lambda]$
- $\ln \partial U_0 = i\frac{d}{d\lambda}$

Then we construct

- $H = H_0' \otimes H_0 \subset \mathcal{H} \otimes \mathcal{H}$
- $U_L(t) = U_0(t) \otimes 1$
- $U_R(t) = \exp(it\chi_{[-c,c]}(\ln \varDelta_0) \otimes \partial U_0) = \chi_{[-c,c]}(\ln \varDelta_0) \otimes U_0(t) + \chi_{[-c,c]^c}(\ln \varDelta_0) \otimes 1$

Because $U_0(1)H \cap \chi_{[-c,c]}(\ln \varDelta_0)H = \{0\}$, one has

$$H_L \cap H_R = U_L(-1)H \cap U_R(1) = U_0(-1)H' \otimes U_0(1)H$$

Conclusion

- Wedge Modular Inclusions are a **generalization of Halfsided Modular Inclusions** that model a wedge in a thermal equilibrium.
- It seems that, similar to Halfsided Modular Inclusions, the global **modular objects** and those of the wedge are related by a **positive energy representation** (but one that can be extended to include scaling)
- By focusing on the ‘lightray spaces’, one can construct **interesting inclusions** of standard subspaces.

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