Algebraic Topology

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To prepare this lecture I used the following textbooks that I also recommend as reading:

- G. E. Bredon,**Topology and geometry**, Springer Science & Business Media.
- A. Dold, **Lectures on Algebraic topology**, Springer Science & Business Media,
- Allen Hatcher, **Algebraic Topology**, Cambridge University Press,
- W. S. Massey, **Algebraic Topology: An Introduction**, Springer Graduate Texts,
- W. S. Massey, **A Basic Course in Algebraic Topology**, Springer Graduate Texts,
- J. P. May, **A concise course in Algebraic Topology**, University of Chicago Press,
- R. Stöcker, H. Zieschang, **Algebraische Topologie**, Teubner,
- T. tom Diek, **Algebraic Topology**, EMS Textbooks in Mathematics.

I also used Birgit Richter's and Rune Haugseng's Lecture Notes on Algebraic Topology.

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Contents

1 Background

1.1 Abelian groups

In this section we assemble the necessary background on abelian groups. For an abelian group A we denote by 0 the neutral element and by $-a$ for the inverse of an element $a \in A$. For any element $a \in A$ and $n \in \mathbb{Z}$ we use the notation

$$n \cdot a = \begin{cases} \underbrace{a + \ldots + a}_{n\times} & n > 0 \\ 0 & n = 0 \\ \underbrace{(-a) + \ldots + (-a)}_{|n|\times} & n < 0. \end{cases}$$

We first recall the concepts of a generating set and a basis of an abelian group A. A generating set is defined analogously to a a generating set for a vector space, but in terms of finite linear combinations with coefficients in $\mathbb{Z}$. The same holds for a basis, which is a defined as a linearly independent generating set. However, unlike vector spaces, abelian groups do not need to have bases. Abelian groups with bases have special properties and are called *free abelian groups.*

Definition 1.1.1: Let A be an abelian group.

1. The subgroup $\langle M\rangle$ generated by a subset $M \subset A$ is $\langle \emptyset \rangle = 0$ and for $M \neq \emptyset$
$$\langle M\rangle = \{n_1 \cdot m_1 + \ldots + n_k \cdot m_k \mid k \in \mathbb{N}_0, m_1, ..., m_k \in M, n_1, ..., n_k \in \mathbb{Z}\}.$$
2. A **generating set** for A is a subset $M \subset A$ with $\langle M\rangle = A$.
3. A generating set M for A is called a **basis** for A, if for all $m_1, ..., m_k \in M$ and $n_1, ..., n_k \in \mathbb{Z}$
$$n_1 \cdot m_1 + \ldots + n_k \cdot m_k = 0 \quad \Rightarrow \quad n_1 = \ldots = n_k = 0.$$

Definition 1.1.2: An abelian group A is called

- **finitely generated**, if it is generated by a finite subset $M \subset A$,
- **cyclic**, if there is an element $a \in A$ such that $\{a\}$ generates A,
- **free**, if it has a basis.

Example 1.1.3:

1. The group $\mathbb{Z}$ is a free group with basis $\{1\}$. The subgroup generated by a subset $\{n_1, \ldots, n_k\} \subset \mathbb{Z}$ is $\gcd(n_1, \ldots, n_k) \cdot \mathbb{Z}$.
2. The groups $\mathbb{Z}/n\mathbb{Z}$ are not free, as $n \cdot \bar{1} = \bar{1} + \ldots + \bar{1} = 0$ with $n \neq 0$. The set $\{\bar{1}\}$ is a generating set, but not a basis.
3. Every subgroup of a free abelian group is free.
4. The group $(\mathbb{Q}, +)$ is not finitely generated. For any finite set $M = \{q_1, \ldots, q_k\}$ of rational numbers $z_j = p_j/q_j \in \mathbb{Q}$ any $\mathbb{Z}$-linear combination can be expressed as a fraction with denominator $q_1 \cdots q_k$ and hence M cannot generate $\mathbb{Q}$.

5. The group $(\mathbb{Q}, +)$ is not free. Given fractions $q_1 = z_1/n_1$ and $q_2 = z_2/n_2$, with $q_1 \neq q_2$, $z_1, z_2 \in \mathbb{Z} \setminus \{0\}$ and $n_1, n_2 \in \mathbb{N}$ we have $(z_2 n_1) \cdot q_1 - (z_1 n_2) \cdot q_2 = 0$, with $z_2 n_1, z_1 n_2 \neq 0$. So a basis could contain at most one non-trivial fraction, in contradiction to 4.

Finitely generated abelian groups, in particular, finite abelian groups can be classified. They are given as finite products of the group $\mathbb{Z}$ and the groups $\mathbb{Z}/n\mathbb{Z}$, where n is a power of a prime number. If an abelian group is finite, only the latter are present.

Theorem 1.1.4 (**classification of finitely generated abelian groups**):
For any finitely generated abelian group A, there are unique $n, k \in \mathbb{N}_0$ and unique prime powers $p_i^{n_i}$ with p_i prime and $n_i \in \mathbb{N}$ such that

$$A \cong \mathbb{Z}^n \times \mathbb{Z}/p_1^{n_1}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k^{n_k}\mathbb{Z}.$$

Corollary 1.1.5 (**classification of finite abelian groups**):
For every finite abelian group there is a unique $k \in \mathbb{N}_0$ and unique prime powers $p_i^{n_i}$ such that

$$A \cong \mathbb{Z}/p_1^{n_1}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k^{n_k}\mathbb{Z}.$$

Elements of finite order in an abelian group A are an obstacle to A being a free group. Whenever there is an element $a \in A$ of finite order, we have $n \cdot a = 0$ for some $n \in \mathbb{N}$. Such an element cannot be contained in a basis. Such elements form a subgroup of A, its *torsion subgroup*.

Definition 1.1.6: Let A be an abelian group.

1. The **torsion subgroup** of A is the subgroup of elements of finite order

$$\mathrm{Tor}(A) = \{a \in A \mid \exists n \in \mathbb{N} \text{ with } na = 0\}.$$

2. For $n \in \mathbb{N}$ the **n-torsion subgroup** of A is the subgroup of elements of order n

$$\mathrm{Tor}_n(A) = \{a \in A \mid na = 0\}.$$

3. The group A is called **torsion free**, if its torsion subgroup is trivial: $\mathrm{Tor}(A) = \{0\}$.

Example 1.1.7:

1. Every free abelian group A is torsion free.

   If $M \subset A$ is a basis of A, any element of A is of the form $a = n_1 m_1 + \ldots + n_k m_k$ with $m_i \in M$ and $n_i \in \mathbb{Z}$. Then $0 = na = (nn_1)m_1 + \ldots + (nn_k)m_k$ with $n \in \mathbb{N}$ implies $nn_1 = \ldots = nn_k = 0$ and $n_1 = \ldots = n_k = 0$. So 0 is the only torsion element.

2. A finitely generated abelian group is free if and only of it is torsion free.

   By Theorem 1.1.4 a finitely generated abelian group is isomorphic to a group $A \cong \mathbb{Z}^n \times \mathbb{Z}/p_1^{n_1}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k^{n_k}\mathbb{Z}$ with the torsion subgroup $\mathrm{Tor}(A) = \mathbb{Z}/p_1^{n_1}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k^{n_k}\mathbb{Z}$. It is torsion free, if and only if $A \cong \mathbb{Z}^n$, which is equivalent to A free.

3. The abelian group $(\mathbb{Q}, +)$ is torsion free, but not free.

   The identity $nq = 0$ for $n \in \mathbb{N}$ and $q \in \mathbb{Q}$ implies $n = 0$, so 0 is the only torsion element. By Example 1.1.3. 5. the group $(\mathbb{Q}, +)$ is not free.

There is an alternative description of a free group that does not require the concept of a basis. It is based on the direct sum of abelian groups. The dual concept is the product of abelian groups. Both constructions are characterised by a universal property, and they coincide for finite families of abelian groups.

Definition 1.1.8: For a family $(A_i)_{i\in I}$ of abelian groups their **direct sum** and the **direct product** are the abelian groups

$$\begin{aligned} \oplus_{i\in I} A_i &= \{(a_i)_{i\in I} \mid a_i \in A_i \,\forall i \in I, a_i = 0 \text{ for almost all } i \in I\} \\ \Pi_{i\in I} A_i &= \{(a_i)_{i\in I} \mid a_i \in A_i \,\forall i \in I\}. \end{aligned}$$

with the addition $(a_i)_{i\in I} + (b_i)_{i\in I} = (a_i + b_i)_{i\in I}$. The inclusion and projection maps are the maps $\iota_j : A_j \to \oplus_{i\in I} A_i$, $a \mapsto (\delta_{ij} a)_{i\in I}$ and $\pi_j : \Pi_{i\in I} A_i \to A$, $(a_i)_{i\in I} \mapsto a_j$.

Remark 1.1.9: For any *finite* family of abelian groups, their direct sum and their product coincide. In this case, one often writes $A_1 \times \ldots \times A_n$ instead of $\Pi_{i=1}^n A_i = \oplus_{i=1}^n A_i$.

Proposition 1.1.10: Let $(A_i)_{i\in I}$ be a family of abelian groups.

1. The direct sum of abelian groups has the following **universal property**:

   The inclusion maps $\iota_j : A_j \to \oplus_{i\in I} A_i$, $a \mapsto (\delta_{ij} a)_{i\in I}$ are group homomorphisms. For every family $(f_i)_{i\in I}$ of group homomorphisms $f_i : A_i \to B$ into an abelian group B, there is a unique group homomorphism $f : \oplus_{i\in I} A_i \to B$ with $f \circ \iota_j = f_j : A_j \to B$ for all $j \in I$.

$$\begin{tikzcd} \oplus_{i\in I} A_i \arrow[rr, dashed, "\exists! f"] & & B \\ & A_j \arrow[ul, "\iota_j"] \arrow[ur, "f_j"'] & \end{tikzcd}$$

2. The product of abelian groups has the following **universal property**:

   The projection maps $\pi_j : \Pi_{i\in I} A_i \to A_j$, $(a_i)_{i\in I} \mapsto a_j$ are group homomorphisms. For every family $(f_i)_{i\in I}$ of group homomorphisms $f_i : B \to A_i$ from an abelian group B, there is a unique group homomorphism $f : B \to \Pi_{i\in I} A_i$ with $\pi_j \circ f = f_j : B \to A_j$ for all $j \in I$.

$$\begin{tikzcd} \Pi_{i\in I} A_i \arrow[dr, "\pi_j"'] & & B \arrow[ll, dashed, "\exists! f"'] \arrow[dl, "f_j"] \\ & A_j & \end{tikzcd}$$

The concept of a direct sum allows one give an alternative definition of a free abelian group. Given a basis of an abelian group, one can assign to each basis element a copy of the group $\mathbb{Z}$, which corresponds to taking multiples of this basis element. Finite $\mathbb{Z}$-linear combinations of the basis elements can then be realised by taking the direct sum of the associated copies of $\mathbb{Z}$.

This allows one to associate to any set M a free group with basis M. Non-free groups may then be constructed as factor groups of free groups, by specifying a generating set of the subgroup that is quotiented out. This leads to the concept of a presentation of an abelian group with generators and relations.

Definition 1.1.11: Let A be an abelian group.

1. The **free abelian group** generated by a set M is the direct sum $\langle M\rangle_{\mathbb{Z}} = \oplus_{m\in M}\mathbb{Z}$.
2. If $A \cong \langle M\rangle_{\mathbb{Z}}/\langle U\rangle$ for a subset $U \subset \langle M\rangle_{\mathbb{Z}}$ one writes $A = \langle M \mid U\rangle$ and speaks of a **presentation** of A with generator set M and set of relations U.

Remark 1.1.12:

1. If A is a free abelian group with basis M, then A is isomorphic to $\langle M\rangle_{\mathbb{Z}} = \oplus_M\mathbb{Z}$ with the group isomorphism $\phi : A \to \langle M\rangle_{\mathbb{Z}}$, $\Sigma_{m\in M} n_m m \mapsto (n_m)_{m\in M}$.
2. The free abelian group generated by M has the following universal property:

   For every map $f : M \to B$ into an abelian group B, there is a unique group homomorphism $f' : \langle M\rangle_{\mathbb{Z}} \to B$ with $f' \circ \iota_m(1) = f(m)$ for all $m \in M$:

$$f' : \langle M\rangle_{\mathbb{Z}} \to B,\ (n_m)_{m\in M} \mapsto \Sigma_{m\in M} n_m f(m).$$

3. Presentations $\langle M \mid U\rangle$ of abelian groups have the following universal property:

   For every map $f : M \to B$ into an abelian group B such that $U \subset \ker f'$, there is a unique group homomorphism $f'' : \langle M \mid U\rangle \to B$ with $f'' \circ \pi = f'$. This follows from the universal property of the free group and the factor group.

If an abelian group A is presented with a finite set of generators $M = \{x_1, \ldots, x_n\}$ and a finite set $U = \{r_1, \ldots, r_k\}$ one often omits the brackets and denotes the presentation by

$$A = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_k\rangle \qquad \text{or} \qquad A = \langle x_1, \ldots, x_n \mid r_1 = 0, \ldots, r_k = 0\rangle.$$

Example 1.1.13:

1. Every abelian group A has a presentation $A = \langle A\rangle_{\mathbb{Z}}/\ker(\pi)$ with

$$\pi : \langle A\rangle_{\mathbb{Z}} \to A,\ (n_a)_{a\in A} \mapsto \Sigma_{a\in A} n_a\, a.$$

2. The group $\mathbb{Z}/n\mathbb{Z}$ has the presentation $\langle x \mid nx = 0\rangle_{\mathbb{Z}}$.
3. The abelian group $A = \mathbb{Z}^n \times \mathbb{Z}/p_1^{n_1}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k^{n_k}\mathbb{Z}$ has the presentation

$$A = \langle x_1, \ldots, x_n, y_1, \ldots, y_k \mid p_1^{n_1} y_1 = 0, \ldots, p_k^{n_k} y_k = 0\rangle.$$

1.2 Categories, functors, natural transformations

In this section we summarise the required background on categories, functors and natural transformations. The concept of a category encodes many examples of mathematical structures and structure preserving maps between them, but it goes beyond them and replaces structure preserving maps by the more abstract notion of a morphism. The crucial features are that morphisms have a fixed source and target, can be composed and can be identity morphisms. This generalises the notions of domain and codomain of structure preserving maps, of their composition and of the structure preserving identity maps.

Definition 1.2.1: A **category** $\mathcal{C}$ consists of:

- a class $\mathrm{Ob}\,\mathcal{C}$ of **objects**,
- for each pair of objects $X, Y \in \mathrm{Ob}\,\mathcal{C}$ a class $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of **morphisms**,
- for each triple of objects X, Y, Z a **composition map**

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(X,Z),$$

such that the following axioms are satisfied:

(C1) The classes $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of morphisms are pairwise disjoint,

(C2) The composition is associative: $f\circ(g\circ h) = (f\circ g)\circ h$ for all morphisms $h \in \mathrm{Hom}_{\mathcal{C}}(W,X)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, $f \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$,

(C3) For every object X there is a morphism $1_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$, the **identity morphism** on X, with $1_X \circ f = f$ and $g \circ 1_X = g$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(W,X)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$.

Instead of $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, we also write $f : X \to Y$. The object X is called the **source** of f, and the object Y the **target** of f. A morphism $f : X \to X$ is called an **endomorphism**.

A morphism $f : X \to Y$ is called an **isomorphism**, if there is a morphism $g : Y \to X$ with $g \circ f = 1_X$ and $f \circ g = 1_Y$. In this case, we call the objects X and Y **isomorphic**.

Often, one requires that the *morphisms* between fixed objects form not only a class, but a set. This is the case in essentially all familiar categories from algebra and topology. Nevertheless, it is sometimes necessary to relax this condition. In contrast, requiring that the *objects* of a category form a set is very restrictive and excludes many familiar and important categories.

Definition 1.2.2: A category $\mathcal{C}$ is called

- **locally small**, if $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a set for all objects $X, Y \in \mathrm{Ob}\mathcal{C}$,
- **small**, if it is locally small and $\mathrm{Ob}\mathcal{C}$ is a set.

The following examples of categories are all locally small, but none of them is small.

Example 1.2.3:

1. The category Set: objects are sets, morphisms are maps. The composition is the composition of maps, and identity morphisms are identity maps. Isomorphisms are bijective maps.

2. The category Top of topological spaces: objects are topological spaces, morphisms are continuous maps, isomorphisms are homeomorphisms.

3. The category Top* of **pointed topological spaces**: Objects are pairs (X,x) of a topological space X and a point $x \in X$, morphisms $f : (X,x) \to (Y,y)$ are continuous maps $f : X \to Y$ with $f(x) = y$.

4. The category Top(2) of **pairs of topological spaces**: Objects are pairs (X,A) of a topological space X and a subspace $A \subset X$, morphisms $f : (X,A) \to (Y,B)$ are continuous maps $f : X \to Y$ with $f(A) \subset B$. Isomorphisms are homeomorphisms $f : X \to Y$ with $f(A) = B$.

5. Many examples of categories we will use in the following are categories of algebraic structures. This includes the following:
    - the category $\mathrm{Vect}_{\mathbb{F}}$ of vector spaces over a field $\mathbb{F}$:
      objects: vector spaces over $\mathbb{F}$, morphisms: $\mathbb{F}$-linear maps,
    - the category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ of finite dimensional vector spaces over a field $\mathbb{F}$:
      objects: finite-dimensional vector spaces over $\mathbb{F}$, morphisms: $\mathbb{F}$-linear maps,
    - the category Grp of groups:
      objects: groups, morphisms: group homomorphisms,
    - the category Ab of abelian groups:
      objects: abelian groups, morphisms: group homomorphisms,
    - the category Ring of unital rings:
      objects: unital rings, morphisms: unital ring homomorphisms,
    - the category Field of fields:
      objects: fields, morphisms: field homomorphisms,
    - the category $\mathrm{Alg}_{\mathbb{F}}$ of algebras over a field $\mathbb{F}$:
      objects: algebras over $\mathbb{F}$, morphisms: algebra homomorphisms,
    - the categories R-Mod and Mod-R of left and right modules over a ring R:
      objects: R-left or right modules, morphisms: R-left or right module homomorphisms.
    - the category R-Mod-S of (R,S)-bimodules:
      objects: (R,S)-bimodules, morphisms: (R,S)-bimodule homomorphisms.

In all of the categories in Example 1.2.3 the morphisms are *maps.* A category for which this is the case is called a **concrete category**. A category that is not concrete is the category of sets and relations in Exercise 2. Further examples of non-concrete categories arise from some of the basic categorical concepts and constructions in the next example.

Example 1.2.4:

1. A small category $\mathcal{C}$ in which all morphisms are isomorphisms is called a **groupoid**.
2. A category with a single object is a **monoid** and a groupoid with a single object a **group**.

   Group elements are identified with endomorphisms of the object, and the composition of morphisms is the group multiplication.

   More generally, for any object X in a groupoid $\mathcal{C}$, the set $\mathrm{End}_{\mathcal{C}}(X) = \mathrm{Hom}_{\mathcal{C}}(X,X)$ with the composition $\circ : \mathrm{End}_{\mathcal{C}}(X) \times \mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{C}}(X)$ is a group.
3. For every category $\mathcal{C}$, there is an **opposite category** $\mathcal{C}^{op}$, which has the same objects as $\mathcal{C}$, whose morphisms are given by $\mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$ and in which the order of the composition is reversed.
4. The **cartesian product** of categories $\mathcal{C}, \mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$ with pairs (C,D) of objects in $\mathcal{C}$ and $\mathcal{D}$ as objects, with $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((C,D),(C',D')) = \mathrm{Hom}_{\mathcal{C}}(C,C') \times \mathrm{Hom}_{\mathcal{D}}(D,D')$ and the composition of morphisms $(h,k) \circ (f,g) = (h \circ f, k \circ g)$.

5. A **subcategory** of a category $\mathcal{C}$ is a category $\mathcal{D}$, such that $\mathrm{Ob}(\mathcal{D}) \subset \mathrm{Ob}(\mathcal{C})$ is a subclass, $\mathrm{Hom}_{\mathcal{D}}(D, D') \subset \mathrm{Hom}_{\mathcal{C}}(D, D')$ for all objects D, D' in $\mathcal{D}$ and the composition of morphisms of $\mathcal{D}$ coincides with their composition in $\mathcal{C}$. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is called a **full subcategory** if $\mathrm{Hom}_{\mathcal{D}}(D, D') = \mathrm{Hom}_{\mathcal{C}}(D, D')$ for all objects D, D' in $\mathcal{D}$.

6. **Quotient categories:** Let $\mathcal{C}$ be a category with an equivalence relation $\sim_{X,Y}$ on each morphism set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ that is compatible with the composition of morphisms:
   $f \sim_{X,Y} g$ and $h \sim_{Y,Z} k$ implies $h \circ f \sim_{X,Z} k \circ g$.

   Then one obtains a category $\mathcal{C}'$, the **quotient category** of $\mathcal{C}$, with the same objects as $\mathcal{C}$ and equivalence classes of morphisms in $\mathcal{C}$ as morphisms.

   The composition of morphisms in $\mathcal{C}'$ is given by $[h] \circ [f] = [h \circ f]$, and the identity morphisms by $[1_X]$. Isomorphisms in $\mathcal{C}'$ are equivalence classes of morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for which there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ with $f \circ g \sim_{Y,Y} 1_Y$ and $g \circ f \sim_{X,X} 1_X$.

The construction in the last example plays an important role in classification problems, in particular in topology. Classifying the objects of a category $\mathcal{C}$ usually means classifying them up to isomorphism - giving a list of objects in $\mathcal{C}$ such that every object in $\mathcal{C}$ is isomorphic to exactly one object in this list.

This is possible in some contexts - for instance for the category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ of finite dimensional vector spaces over $\mathbb{F}$. In this case the list contains the vector spaces $\mathbb{F}^n$ with $n \in \mathbb{N}_0$. However, it is often too difficult to solve this problem in full generality. In this case, it is sometimes simpler to consider instead a quotient category $\mathcal{C}'$ and to attempt a partial classification.

If two objects are isomorphic in $\mathcal{C}$, they are by definition isomorphic in $\mathcal{C}'$, as any isomorphism $f : X \to Y$ with inverse $g : Y \to X$ yields $[g] \circ [f] = [g \circ f] = [1_X]$ and $[f] \circ [g] = [f \circ g] = [1_Y]$. However, the converse does not hold - the category $\mathcal{C}'$ yields a weaker classification than $\mathcal{C}$.

To relate different categories, one must not only relate their objects but also their morphisms, in a way that is compatible with source and target objects, the composition of morphisms and the identity morphisms. This leads to the concept of a *functor*.

Definition 1.2.5: Let $\mathcal{C}, \mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of:

- an assignment of an object $F(C)$ in $\mathcal{D}$ to every object C in $\mathcal{C}$,
- for each pair of objects C, C' in $\mathcal{C}$, a map

$$\mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C')), \quad f \mapsto F(f),$$

that is compatible with the composition of morphisms and with the identity morphisms

$$\begin{aligned} &F(g \circ f) = F(g) \circ F(f) && \forall f \in \mathrm{Hom}_{\mathcal{C}}(C, C'), g \in \mathrm{Hom}_{\mathcal{C}}(C', C'') \\ &F(1_C) = 1_{F(C)} && \forall C \in \mathrm{Ob}\, \mathcal{C}. \end{aligned}$$

- A functor $F : \mathcal{C} \to \mathcal{C}$ is called an **endofunctor**.
- A functor $F : \mathcal{C}^{op} \to \mathcal{D}$ is sometimes called a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$.
- The **composite** of two functors $F : \mathcal{B} \to \mathcal{C}$, $G : \mathcal{C} \to \mathcal{D}$ is the functor $GF : \mathcal{B} \to \mathcal{D}$ given by the assignment $B \mapsto GF(B)$ for all objects B in $\mathcal{B}$ and the maps

$$\mathrm{Hom}_{\mathcal{B}}(B, B') \to \mathrm{Hom}_{\mathcal{D}}(GF(B), GF(B')), \quad f \mapsto G(F(f)).$$

Example 1.2.6:

1. For any category $\mathcal{C}$, the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ that assigns each object and morphism in $\mathcal{C}$ to itself is an endofunctor of $\mathcal{C}$.

2. The **forgetful functor** $\mathrm{Vect}_{\mathbb{F}} \to \mathrm{Ab}$ assigns to each vector space the underlying abelian group and to each linear map the associated group homomorphism. There are analogous forgetful functors $\mathrm{Vect}_{\mathbb{F}} \to \mathrm{Set}$, $\mathrm{Ring} \to \mathrm{Set}$, $\mathrm{Grp} \to \mathrm{Set}$, $\mathrm{Top} \to \mathrm{Set}$ that assign to each vector space, ring, group, topological space the underlying set and to each morphism the underlying map.

3. Vector space duals define a functor $* : \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Vect}_{\mathbb{F}}^{op}$ that assigns to
   - a vector space V its dual V^*,
   - a linear map $f : V \to W$ its adjoint $f^* : W^* \to V^*$, $\alpha \mapsto \alpha \circ f$.

4. A group G defines a category BG with a single object, the **delooping** of G, with elements of G as morphisms, and with the multiplication of G as the composition.
   - Functors $F : BG \to \mathrm{Set}$ correspond to G-sets $X = F(\bullet)$ with the group action $\rhd : G \times X \to X$, $g \rhd x = F(g)(x)$.
   - Functors $F : BG \to \mathrm{Vect}_{\mathbb{F}}$ correspond to representations of G over $\mathbb{F}$, with the representation space $V = F(\bullet)$ and $\rho = F(g) : G \to \mathrm{Aut}_{\mathbb{F}}V$.

5. **Hom-functors**: Let $\mathcal{C}$ be a category and C an object in $\mathcal{C}$.
   - The functor $\mathrm{Hom}(C, -) : \mathcal{C} \to \mathrm{Set}$ assigns to
   - an object C' the set $\mathrm{Hom}_{\mathcal{C}}(C, C')$,
   - a morphism $f : C' \to C''$ the map $\mathrm{Hom}(C, f) : \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{C}}(C, C'')$, $g \mapsto f \circ g$.
   - The functor $\mathrm{Hom}(-, C) : \mathcal{C}^{op} \to \mathrm{Set}$ assigns to
   - an object C' the set $\mathrm{Hom}_{\mathcal{C}}(C', C)$,
   - a morphism $f : C' \to C''$ the map $\mathrm{Hom}(f, C) : \mathrm{Hom}_{\mathcal{C}}(C'', C) \to \mathrm{Hom}_{\mathcal{C}}(C', C)$, $g \mapsto g \circ f$.

6. The **path component functor** $\pi_0 : \mathrm{Top} \to \mathrm{Set}$ assigns to
   - a topological space X the set $\pi_0(X)$ of its path components,
   - a continuous map $f : X \to Y$ the map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$, $P(x) \mapsto P(f(x))$.

7. The **fundamental group** defines a functor $\pi_1 : \mathrm{Top}^* \to \mathrm{Grp}$ that assigns to
   - a pointed topological space (x, X) its fundamental group $\pi_1(x, X)$,
   - a morphism $f : (x, X) \to (y, Y)$ of pointed topological spaces the group homomorphism $\pi_1(f) : \pi_1(x, X) \to \pi_1(y, Y)$, $[\gamma] \mapsto [f \circ \gamma]$.

8. **Abelisation**: The abelisation functor $F : \mathrm{Grp} \to \mathrm{Ab}$ assigns to
   - a group G the abelian group $F(G) = G/[G, G]$, where $[G, G]$ is the normal subgroup generated by the set of all elements $ghg^{-1}h^{-1}$ for $g, h \in G$,
   - a group homomorphism $f : G \to H$ the induced group homomorphism $F(f) : G/[G, G] \to H/[H, H]$, $g + [G, G] \mapsto f(g) + [H, H]$.

9. The **discrete topology** functor $D : \mathrm{Set} \to \mathrm{Top}$ and the **indiscrete topology** functor $I : \mathrm{Set} \to \mathrm{Top}$ assign to
   - a set X the topological space X with the discrete (indiscrete) topology,
   - a map $f : X \to Y$ the associated continuous map $f : X \to Y$.

There is another structure that relates functors. As a functor $F : \mathcal{C} \to \mathcal{D}$ involves maps between the sets $\mathrm{Hom}_{\mathcal{C}}(C, C')$ and $\mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$, a structure that relates functors $F, G : \mathcal{C} \to \mathcal{D}$ must in particular relate the sets $\mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$ and $\mathrm{Hom}_{\mathcal{D}}(G(C), G(C'))$. The simplest way to do this is to assign to each object C in $\mathcal{C}$ a morphism $\eta_C : F(C) \to G(C)$ in $\mathcal{D}$. One then requires compatibility with the images $F(f)$ and $G(f)$ for all morphisms $f : C \to C'$ in $\mathcal{C}$.

Definition 1.2.7: A **natural transformation** $\eta : F \Rightarrow G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is an assignment of a morphism $\eta_C : F(C) \to G(C)$ in $\mathcal{D}$ to every object C in $\mathcal{C}$ such that the following diagram commutes for all morphisms $f : C \to C'$ in $\mathcal{C}$

$$\begin{tikzcd} F(C) \arrow[r, "\eta_C"] \arrow[d, "F(f)"] & G(C) \arrow[d, "G(f)"] \\ F(C') \arrow[r, "\eta_{C'}"] & G(C'). \end{tikzcd}$$

A **natural isomorphism** is a natural transformation $\eta : F \Rightarrow G$, for which all morphisms $\eta_X : F(X) \to G(X)$ are isomorphisms. Two functors that are related by a natural isomorphism are called **naturally isomorphic**.

Example 1.2.8:

1. For any functor $F : \mathcal{C} \to \mathcal{D}$ the identity natural transformation $\mathrm{id}_F : F \Rightarrow F$ with component morphisms $(\mathrm{id}_F)_X = 1_{F(X)} : F(X) \to F(X)$ is a natural isomorphism.

2. Consider the functors $\mathrm{id} : \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Vect}_{\mathbb{F}}$ and $** : \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Vect}_{\mathbb{F}}$. Then there is a canonical natural transformation $\mathrm{can} : \mathrm{id} \Rightarrow **$, whose component morphisms $\eta_V : V \to V^{**}$ assign to a vector $v \in V$ the unique vector $v^{**} \in V^{**}$ with $v^{**}(\alpha) = \alpha(v)$ for all $\alpha \in V^*$. It is a natural isomorphism if and only if V is finite-dimensional.

3. Consider the category CRing of commutative unital rings and unital ring homomorphisms and the category Grp of groups and group homomorphisms.

   Let $F : \mathrm{CRing} \to \mathrm{Grp}$ the functor that assigns to
   - a commutative unital ring k the group $\mathrm{GL}_n(k)$ of invertible $n \times n$-matrices in k,
   - a unital ring homomorphism $f : k \to l$ the group homomorphism

   $$\mathrm{GL}_n(f) : \mathrm{GL}_n(k) \to \mathrm{GL}_n(l), \quad M = (m_{ij})_{i,j=1,\ldots,n} \mapsto f(M) = (f(m_{ij}))_{i,j=1,..,n}.$$

   Let $G : \mathrm{CRing} \to \mathrm{Grp}$ be the functor that assigns to
   - a commutative unital ring k the group $G(k) = k^\times$ of units in k,
   - a unital ring homomorphism $f : k \to l$ the induced group homomorphism

   $$G(f) = f|_{k^\times} : k^\times \to l^\times.$$

   The determinant defines a natural transformation $\det : F \to G$ with component morphisms $\det_k : \mathrm{GL}_n(k) \Rightarrow k^\times$, because the following diagram commutes for every unital ring homomorphism $f : k \to l$

   $$\begin{tikzcd} \mathrm{GL}_n(k) \arrow[r, "\det_k"] \arrow[d, "\mathrm{GL}_n(f)"'] & k^\times \arrow[d, "f|_{k^\times}"] \\ \mathrm{GL}_n(l) \arrow[r, "\det_l"] & l^\times. \end{tikzcd}$$

4. Let G be a group and BG its delooping. Then functors $F : BG \to \text{Set}$ are G-sets by Example 1.2.6, 4. Natural transformations between them are G-equivariant maps.

   A natural transformation $\eta : F \Rightarrow F'$ has a single component $\eta_\bullet : F(\bullet) \to F'(\bullet)$. The naturality condition states that $\eta_\bullet(g \rhd x) = g \rhd' \eta_\bullet(x)$ for all $g \in G$, $x \in X$.

   Similarly, by Example 1.2.6, 4. functors $F : BG \to \text{Vect}_{\mathbb{F}}$ are representations of G over $\mathbb{F}$, and natural transformations between them are homomorphisms of representations.

Remark 1.2.9:

1. For any small category $\mathcal{C}$ and category $\mathcal{D}$, the functors $F : \mathcal{C} \to \mathcal{D}$ and natural transformations between them form a category, denoted $\text{Fun}(\mathcal{C}, \mathcal{D})$ or $\mathcal{D}^{\mathcal{C}}$, the **functor category**.

   The composite of natural transformations $\eta : F \Rightarrow G$ and $\kappa : G \Rightarrow H$ is the natural transformation $\kappa \circ \eta : F \Rightarrow H$ with component morphisms $(\kappa \circ \eta)_X = \kappa_X \circ \eta_X : F(X) \to H(X)$ and the identity morphisms are the identity natural transformations $1_F = \text{id}_F : F \Rightarrow F$.

   If $\mathcal{C}$ is a category that is not small, the functor category $\mathcal{D}^{\mathcal{C}}$ is defined analogously, but it is no longer locally small.

2. Natural transformations can be composed with functors.

   If $F, F' : \mathcal{C} \to \mathcal{D}$ are functors and $\eta : F \Rightarrow F'$ a natural transformation, then for any functor $G : \mathcal{B} \to \mathcal{C}$ one obtains a natural transformation $\eta G : FG \Rightarrow F'G$ with component morphisms $(\eta G)_B = \eta_{G(B)} : FG(B) \to F'G(B)$. Similarly, any functor $E : \mathcal{D} \to \mathcal{E}$ defines a natural transformation $E\eta : EF \Rightarrow EF'$ with $(E\eta)_C = E(\eta_C) : EF(C) \to EF'(C)$.

The notions of natural transformations and natural isomorphisms are particularly important as they allow one to generalise the notion of an inverse map and of a bijection to functors. An **inverse** of a functor $F : \mathcal{C} \to \mathcal{D}$ is by definition a functor $G : \mathcal{D} \to \mathcal{C}$ with $GF = \text{id}_{\mathcal{C}}$ and $FG = \text{id}_{\mathcal{D}}$, and an **isomorphism** of categories is a functor $F : \mathcal{C} \to \mathcal{D}$ with an inverse.

However, it turns out that there are very few examples of functors with an inverse. A more useful generalisation is obtained by weakening this requirement. Instead of requiring $FG = \text{id}_{\mathcal{D}}$ and $GF = \text{id}_{\mathcal{C}}$, one requires only that these functors are *naturally isomorphic* to the identity functors. This leads to the concept of an equivalence of categories.

Definition 1.2.10: A functor $F : \mathcal{C} \to \mathcal{D}$ is called an **equivalence of categories** if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\kappa : GF \Rightarrow \text{id}_{\mathcal{C}}$ and $\eta : FG \Rightarrow \text{id}_D$. In this case, the categories $\mathcal{C}$ and $\mathcal{D}$ are called **equivalent**.

Sometimes it is easier to use a more direct characterisation of an equivalences of categories in terms of its behaviour on objects and morphisms. This is the categorical equivalent of the statement that a map between sets is an isomorphism if and only if it is injective and surjective. The proof of the following lemma makes use of the axiom of choice and can be found for instance in [K], Chapter XI, Prop XI.1.5.

Lemma 1.2.11: A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if it is:

1. **essentially surjective**:
   for every object D in $\mathcal{D}$ there is an object C of $\mathcal{C}$ such that D is isomorphic to $F(C)$.
2. **fully faithful**:
   all maps $\text{Hom}_{\mathcal{C}}(C, C') \to \text{Hom}_{\mathcal{D}}(F(C), F(C'))$, $f \mapsto F(f)$ are bijections.

Example 1.2.12:

1. The category $\mathrm{Vect}_{\mathbb{F}}^{fin}$ of finite-dimensional vector spaces over $\mathbb{F}$ is equivalent to the category $\mathcal{C}$, whose objects are non-negative integers $n \in \mathbb{N}_0$, whose morphisms $f : n \to m$ are $m \times n$-matrices with entries in $\mathbb{F}$ and with the matrix multiplication as composition.
2. The category Set^{fin} of finite sets is equivalent to the category Ord^{fin}, whose objects are finite **ordinal numbers** $[n] = \{0, 1, ..., n-1\}$ for all $n \in \mathbb{N}_0$ and whose morphisms $f : [m] \to [n]$ are maps $f : \{0, 1, ..., m-1\} \to \{0, 1, ..., n-1\}$ with the composition of maps as the composition of morphisms.

1.3 Products and Coproducts

Many concepts and constructions from algebra or topology can be generalised straightforwardly to categories. This works, whenever it is possible to characterise them in terms of *universal properties* involving only the *morphisms* in the category. In particular, there are concepts of a categorical product and coproduct that generalise cartesian products and disjoint unions of sets and products and sums of topological spaces.

Definition 1.3.1: Let $\mathcal{C}$ be a category and $(C_i)_{i\in I}$ a family of objects in $\mathcal{C}$.

1. A **product** of the family $(C_i)_{i\in I}$ is an object $\Pi_{i\in I} C_i$ in $\mathcal{C}$ together with a family of morphisms $\pi_i : \Pi_{j\in I} C_j \to C_i$, such that for all families of morphisms $f_i : W \to C_i$ there is a unique morphism $f : W \to \Pi_{i\in I} C_i$ such that the diagram

$$\begin{tikzcd} W \arrow[r, dashed, "\exists! f"] \arrow[dr, "f_i"'] & \Pi_{j\in I} C_j \arrow[d, "\pi_i"] \\ & C_i \end{tikzcd} \qquad (1)$$

commutes for all $i \in I$. This is called the **universal property** of the product.

2. A **coproduct** of the family $(C_i)_{i\in I}$ is an object $\amalg_{i\in I} C_i$ in $\mathcal{C}$ with a family $(\iota_i)_{i\in I}$ of morphisms $\iota_i : C_i \to \amalg_{j\in I} C_j$, such that for every family $(f_i)_{i\in I}$ of morphisms $f_i : C_i \to Y$ there is a unique morphism $f : \amalg_{i\in I} C_i \to Y$ such that the diagram

$$\begin{tikzcd} Y & \amalg_{j\in I} C_j \arrow[l, dashed, "\exists! f"'] \\ & C_i \arrow[ul, "f_i"] \arrow[u, "\iota_i"'] \end{tikzcd} \qquad (2)$$

commutes for all $i \in I$. This is called the **universal property** of the coproduct.

Remark 1.3.2: Products or coproducts do not necessarily exist for a given family of objects $(C_i)_{i\in I}$ in a category $\mathcal{C}$, but if they exist, they are **unique up to unique isomorphism**:

If $(\Pi_{i\in I} C_i, (\pi_i)_{i\in I})$ and $(\Pi'_{i\in I} C_i, (\pi'_i)_{i\in I})$ are two products for a family of objects $(C_i)_{i\in I}$ in $\mathcal{C}$, then there is a unique morphism $\pi' : \Pi'_{i\in I} C_i \to \Pi_{i\in I} C_i$ with $\pi_i \circ \pi' = \pi'_i$ for all $i \in I$, and this morphism is an isomorphism.

By the universal property of the product $\Pi_{i\in I}C_i$ applied to the family $f_i := \pi'_i : \Pi'_{i\in I}C_i \to C_i$, there is a unique morphism $\pi' : \Pi'_{i\in I}C_i \to \Pi_{i\in I}C_i$ such that $\pi_i \circ \pi' = \pi'_i$ for all $i \in I$. Similarly, the universal property of $\Pi'_{i\in I}C_i$ implies that for the family of morphisms $\pi_i : \Pi_{i\in I}C_i \to C_i$ there is a unique morphism $\pi : \Pi_{i\in I}C_i \to \Pi'_{i\in I}C_i$ with $\pi'_i \circ \pi = \pi_i$ for all $i \in I$. It follows that $\pi' \circ \pi : \Pi_{i\in I}C_i \to \Pi_{i\in I}C_i$ is a morphism with $\pi_i \circ \pi \circ \pi' = \pi'_i \circ \pi = \pi_i$ for all $i \in I$. As the identity morphism on $\Pi_{i\in I}C_i$ is another morphism with this property, the uniqueness implies $\pi' \circ \pi = 1_{\Pi_{i\in I}C_i}$. By the same argument one obtains $\pi \circ \pi' = 1_{\Pi'_{i\in I}C_i}$ and hence π' is an isomorphism with inverse π.

$$
\begin{tikzcd}
\Pi'_{i\in I}C_i \arrow[r, "\pi'"] \arrow[rr, bend left, "1_{\Pi'_{i\in I}C_i}"] \arrow[dr, "\pi'_i"'] & \Pi_{i\in I}C_i \arrow[r, "\pi"] \arrow[d, "\pi_i"] & \Pi'_{i\in I}C_i \arrow[dl, "\pi'_i"] \\
 & C_i. &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Pi_{i\in I}C_i \arrow[r, "\pi"] \arrow[rr, bend left, "1_{\Pi_{i\in I}C_i}"] \arrow[dr, "\pi_i"'] & \Pi'_{i\in I}C_i \arrow[r, "\pi'"] \arrow[d, "\pi'_i"] & \Pi_{i\in I}C_i \arrow[dl, "\pi_i"] \\
 & C_i. &
\end{tikzcd}
$$

Example 1.3.3:

1. The cartesian product of sets is a product in Set, and the disjoint union of sets is a coproduct in Set. The product of topological spaces is a product in Top and the topological sum is a coproduct in Top. They exist for all families of objects.

2. The direct sum of vector spaces is a coproduct and the direct product of vector spaces a product in $\mathrm{Vect}_\mathbb{F}$.
$$\oplus_{i\in I}V_i = \{(v_i)_{i\in I} \mid v_i \in V_i \,\forall i \in I, v_i = 0 \text{ for almost all } i \in I\}$$
$$\Pi_{i\in I}V_i = \{(v_i)_{i\in I} \mid v_i \in V_i \,\forall i \in I\},$$
with the addition $(v_i)_{i\in I}+(v'_i)_{i\in I} = (v_i+v'_i)_{i\in I}$ and scalar multiplication $\lambda(v_i)_{i\in I} = (\lambda v_i)_{i\in I}$.

3. The direct sum of abelian groups is a coproduct in Ab and the product of abelian groups a product in Ab. They are given by
$$\oplus_{i\in I}A_i = \{(a_i)_{i\in I} \mid a_i \in A_i \,\forall i \in I, a_i = 0 \text{ for almost all } i \in I\}$$
$$\Pi_{i\in I}A_i = \{(a_i)_{i\in I} \mid a_i \in A_i \,\forall i \in I\},$$
with the group multiplication $(a_i)_{i\in I} + (a'_i)_{i\in I} = (a_i + a'_i)_{i\in I}$.

4. The direct product of groups is a product in Grp and the free product of groups is a coproduct in Grp. They exist for all families of groups.

In particular, we can consider categorical products and coproducts over empty index sets I. By definition, a categorical product for an empty family of objects is an object $T = \Pi_\emptyset$ such that for every object C in $\mathcal{C}$ there is a unique morphism $t_C : C \to T$. Similarly, a coproduct over an empty index set I is an object $I := \amalg_\emptyset$ in $\mathcal{C}$ such that for every object C in $\mathcal{C}$, there is a unique morphism $i_C : I \to C$. Such objects are called *terminal* and *initial* objects in $\mathcal{C}$. Initial and terminal objects need not exist in every category $\mathcal{C}$, but if they exist they are unique up to unique isomorphism by the universal property of the products and coproducts.

An object that is both, terminal and initial, is called a *zero object* or *null object*. If it exists, it is unique up to unique isomorphism. It also gives rise to a distinguished morphism, the *zero morphism* $0 = i_{C'} \circ t_C : C \to C'$, between any two objects C, C' in $\mathcal{C}$.

Definition 1.3.4: Let $\mathcal{C}$ be a category. An object X in a category $\mathcal{C}$ is called:

1. A **final** or **terminal object** in $\mathcal{C}$ is an object T in $\mathcal{C}$ such that for every object C in $\mathcal{C}$ there is a unique morphism $t_C : C \to T$.
2. A **cofinal** or **initial object** in $\mathcal{C}$ is an object I in $\mathcal{C}$ such that for every object C in $\mathcal{C}$ there is a unique morphism $i_C : I \to C$,
3. A **null object** or **zero object** in $\mathcal{C}$ is an object 0 in $\mathcal{C}$ that is both final and initial: for every object C in $\mathcal{C}$ there are a unique morphisms $t_C : C \to 0$ and $i_C : 0 \to C$.
4. If $\mathcal{C}$ has a zero object, then the morphism $0 = i_{C'} \circ t_C : C \to 0 \to C'$ is called the **trivial morphism** or **zero morphism** from C to C'.

Example 1.3.5:

1. The empty set is an initial object in Set and the empty topological space an initial object in Top. Any set with one element is a final object in Set and any one point space a final object in Top. The categories Set and Top do not have null objects.

2. The null vector space $\{0\}$ is a null object in the category $\mathrm{Vect}_{\mathbb{F}}$.

3. The trivial group $G = \{e\}$ is a null object in Grp and in Ab.

4. The ring $\mathbb{Z}$ is an initial object in the category Ring, since for every unital ring R, there is exactly one ring homomorphism $f : \mathbb{Z} \to R$, namely the one determined by $f(0) = 0_R$ and $f(1) = 1_R$. The zero ring $R = \{0\}$ with $0 = 1$ is a final object in Ring, but not an initial one. The category Ring has no zero object.

5. The category Field does not have initial or final objects. As any ring homomorphism $f : \mathbb{F} \to \mathbb{K}$ between fields is injective, an initial object in Field would be a subfield of all other fields, and every field would be a subfield of a final field. Either of them would imply that each field has the same characteristic as an initial or final field, a contradiction.

2 Chain complexes

In this section, we introduce the essential algebraic tools of algebraic topology, chain complexes, chain maps and chain homotopies. These concepts also play a fundamental role in algebra and group theory. Their fundamental advantage is that they are $\mathbb{Z}$-linear - they involve abelian groups and group homomorphisms between them - and sufficiently complex to store enough information about the topological spaces. Unlike homotopy groups, they are rather easy to compute due to their $\mathbb{Z}$-linearity.

2.1 Chain complexes, chain maps and chain homotopies

In this section, we consider chain complexes in the category Ab of abelian groups. Chain complexes can be formulated in more generality in the categories of modules over unital rings or even more generally in abelian categories.

Definition 2.1.1:

1. A **chain complex** $X_\bullet$ consists of
   - a family $(X_n)_{n\in\mathbb{Z}}$ of abelian groups X_n,
   - a family $(d_n)_{n\in\mathbb{Z}}$ of group homomorphisms $d_n : X_n \to X_{n-1}$, **boundary operators**, such that $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{Z}$.

   It is denoted $\ldots \xrightarrow{d_{n+2}} X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \ldots$.
   - Elements of the group X_n are called n-**chains**.
   - The subgroup of n-**cycles** is $Z_n(X_\bullet) = \ker d_n \subset X_n$.
   - The subgroup of n-**boundaries** is $B_n(X_\bullet) = \mathrm{im}\, d_{n+1} \subset Z_n(X_\bullet)$.

2. A **chain map** $f_\bullet : X_\bullet \to X'_\bullet$ is a family $(f_n)_{n\in\mathbb{Z}}$ of group homomorphisms $f_n : X_n \to X'_n$ such that $d'_n \circ f_n = f_{n-1} \circ d_n$ for all $n \in \mathbb{Z}$.

$$\begin{tikzcd}
\ldots \arrow[r, "d_{n+2}"] & X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] & X_n \arrow[r, "d_n"] \arrow[d, "f_n"'] & X_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"'] & \ldots \\
\ldots \arrow[r, "d'_{n+2}"] & X'_{n+1} \arrow[r, "d'_{n+1}"] & X'_n \arrow[r, "d'_n"] & X'_{n-1} \arrow[r, "d'_{n-1}"] & \ldots
\end{tikzcd}$$

Notation 2.1.2: It is standard to omit subsequences of trivial groups and morphisms between them from chain complexes:

- $0 \to X_m \xrightarrow{d_m} X_{m-1} \xrightarrow{d_{m-1}} \ldots$ stands for a chain complex with $X_k = 0$ for all $k > m$. Such a chain complex is called **bounded above**. It is called **negative** if $m = 0$.
- $\ldots \xrightarrow{d_{m+2}} X_{m+1} \xrightarrow{d_{m+1}} X_m \to 0$ stands for a chain complex with $X_k = 0$ for all $k < m$. Such a chain complex is called **bounded below**. It is called **positive** if $m = 0$.
- If $X_k = 0$ for all $k < m$ and $k > l > m$, the chain complex is called **bounded** and denoted

$$0 \to X_l \xrightarrow{d_l} X_{l-1} \xrightarrow{d_{l-1}} \ldots \xrightarrow{d_{m+2}} X_{m+1} \xrightarrow{d_{m+1}} X_m \to 0.$$

Remark 2.1.3:

1. Chain complexes and chain maps between them form a category $\mathrm{Ch}_{\mathrm{Ab}}$ with the composition $f_\bullet \circ g_\bullet = (f_n \circ g_n)_{n\in\mathbb{Z}}$ and the identity morphism $\mathrm{id}_{X_\bullet} = (\mathrm{id}_{X_n})_{n\in\mathbb{Z}}$.

2. The category $\mathrm{Ch}_{\mathrm{Ab}}$ has all products and coproducts. For a family $(X^i_\bullet)_{i\in I}$ of chain complexes, their **direct sum** or **coproduct** and their **product** are given by

$$(\oplus_{i\in I} X^i_\bullet)_n = \oplus_{i\in I} X^i_n \qquad\qquad (\Pi_{i\in I} X^i_\bullet)_n = \Pi_{i\in I} X^i_n$$

$$\begin{tikzcd}
\oplus_{i\in I} X^i_n \arrow[r, "d_n"] & \oplus_{i\in I} X^i_{n-1} \\
X^j_n \arrow[u, "\iota^j_n"] \arrow[r, "d^j_n"'] & X^j_{n-1} \arrow[u, "\iota^j_{n-1}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\Pi_{i\in I} X^i_n \arrow[r, "d_n"] \arrow[d, "\pi^j_n"'] & \Pi_{i\in I} X^i_{n-1} \arrow[d, "\pi^j_{n-1}"] \\
X^j_n \arrow[r, "d^j_n"'] & X^j_{n-1}.
\end{tikzcd}$$

There is an analogous concept of a *cochain complex* with *coboundary operators* that raise the degree instead of lowering it. This is usually emphasised by writing upper instead of lower indices. Although chain complexes and cochain complexes can be transformed into each other by renumbering the indices, cochain complexes are more natural in many applications, such as functions and n-forms on manifolds.

Definition 2.1.4:

1. A **cochain complex** $X^\bullet$ consists of
   - a family $(X^n)_{n\in\mathbb{Z}}$ of abelian groups X^n,
   - a family $(d^n)_{n\in\mathbb{Z}}$ of group homomorphisms $d^n : X^n \to X^{n+1}$, the **coboundary operators**, such that $d^{n+1} \circ d^n = 0$ for all $n \in \mathbb{Z}$.

   It is denoted $\ldots \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \ldots$.
   - Elements of the group X^n are called n-**cochains**.
   - The subgroup of n-**cocycles** is $Z^n(X^\bullet) = \ker d^n \subset X^n$.
   - The subgroup of n-**coboundaries** is $B^n(X^\bullet) = \mathrm{im}\, d^{n-1} \subset Z^n(X^\bullet)$.

2. A **cochain map** $f^\bullet : X^\bullet \to X'^\bullet$ is a family $(f^n)_{n\in\mathbb{Z}}$ of R-linear maps $f^n : X^n \to X'^n$ such that $d'^n \circ f^n = f^{n+1} \circ d^n$ for all $n \in \mathbb{Z}$.

$$\begin{tikzcd}
\ldots \arrow[r, "d^{n-2}"] & X^{n-1} \arrow[r, "d^{n-1}"] \arrow[d, "f^{n-1}"'] & X^n \arrow[r, "d^n"] \arrow[d, "f^n"'] & X^{n+1} \arrow[r, "d^{n+1}"] \arrow[d, "f^{n+1}"'] & \ldots \\
\ldots \arrow[r, "d'^{n-2}"'] & X'^{n-1} \arrow[r, "d'^{n-1}"'] & X'^n \arrow[r, "d'^n"'] & X'^{n+1} \arrow[r, "d'^{n+1}"'] & \ldots
\end{tikzcd}$$

The category of cochain complexes and cochain maps is denoted $\mathrm{Ch}^{\mathrm{Ab}}$.

In the following we mostly restrict attention to chain complexes, since a cochain complex $X^\bullet$ can be transformed into a chain complex $X'_\bullet$ by setting $X'_n = X^{-n}$ and $d'_n = d^{-n} : X'_n \to X'_{n-1}$ for all $n \in \mathbb{Z}$. Nevertheless, sometimes it is necessary to consider both structures.

The essential feature of a chain complex $X_\bullet$ is the condition $d_{n-1} \circ d_n = 0$ for its boundary operators. It ensures that the abelian group $B_n(X_\bullet)$ of n-boundaries is a subgroup of the abelian group $Z_n(X_\bullet)$ of n-cycles. An analogous condition holds for n-coboundaries and n-cocycles

$$B_n(X_\bullet) = \mathrm{im}\, d_{n+1} \subset \ker d_n = Z_n(X_\bullet) \qquad\qquad B^n(X^\bullet) = \mathrm{im}\, d^{n-1} \subset \ker d^n = Z^n(X^\bullet).$$

This allows one to consider the associated factor groups, the *homologies* of a chain complex or *cohomologies* of a cochain complex.

Definition 2.1.5:

1. The nth **homology** of a chain complex $X_\bullet$ is the factor group
$$H_n(X_\bullet) = Z_n(X_\bullet)/B_n(X_\bullet).$$
2. The nth **cohomology** of a cochain complex $X^\bullet$ is the factor group
$$H^n(X^\bullet) = Z^n(X^\bullet)/B^n(X^\bullet).$$

A chain complex $X_\bullet$ or cochain complex $X^\bullet$ is called **exact in** X_n or X^n, if $H_n(X_\bullet) = 0$ or $H^n(X^\bullet) = 0$, and **exact**, if it is exact in all X_n or X^n for $n \in \mathbb{Z}$.

Remark 2.1.6: A chain complex $X_\bullet$ is exact in X_n, if and only if $\ker d_{n-1} = \mathrm{im}\, d_n$.

- a chain complex $0 \to X_0 \xrightarrow{d_0} X_{-1} \to \ldots$ is exact in X_0, if and only if d_0 is injective,
- a chain complex $\cdots \to X_1 \xrightarrow{d_1} X_0 \to 0$ is exact in X_0, if and only if d_1 is surjective,
- a chain complex $0 \to X_1 \xrightarrow{d_1} X_0 \to 0$ is exact, if and only if d_1 is an isomorphism.

Example 2.1.7:

1. For $m \in \mathbb{Z}$ the chain complex $0 \xrightarrow{d_2} X_1 = \mathbb{Z} \xrightarrow{d_1: z \mapsto mz} X_0 = \mathbb{Z} \xrightarrow{d_0} 0$ has homologies $H_0(X_\bullet) = \ker(d_0)/\mathrm{im}\,(d_1) = \mathbb{Z}/m\mathbb{Z}$ and $H_n(X_\bullet) = 0$ for $n \neq 0$.
2. The chain complex $X_\bullet = \ldots \xrightarrow{\bar z \mapsto 2\bar z} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\bar z \mapsto 2\bar z} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\bar z \mapsto 2\bar z} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\bar z \mapsto 2\bar z} \ldots$ has homologies $H_n(X_\bullet) = \ker(\bar z \mapsto 2\bar z)/\mathrm{im}\,(\bar z \mapsto 2\bar z) = \{\bar 0, \bar 2\}/\{\bar 0, \bar 2\} = 0$ for all $n \in \mathbb{Z}$.
3. The chain complex $X_\bullet = \ldots \xrightarrow{0} X_n \xrightarrow{0} X_{n-1} \xrightarrow{0} X_{n-2} \xrightarrow{0} X_{n-3} \xrightarrow{0} \ldots$ has homologies $H_n(X_\bullet) = \ker(0)/\mathrm{im}\,(0) = X_n$ for all $n \in \mathbb{Z}$.

To see why the homologies of a chain complex are important and useful quantities, we investigate their algebraic properties. By definition, the nth homology of a chain complex assigns to a chain complex an abelian group. To be useful in practice, this assignment should define a functor from the category $\mathrm{Ch}_{\mathrm{Ab}}$ of chain complexes and chain maps between them to the category Ab of abelian groups. This ensures that isomorphic chain complexes have isomorphic homologies.

Proposition 2.1.8: The nth homologies and cohomologies define functors
$$H_n : \mathrm{Ch}_{\mathrm{Ab}} \to \mathrm{Ab} \qquad\qquad H^n : \mathrm{Ch}^{\mathrm{Ab}} \to \mathrm{Ab}$$

Proof:
We prove the claim for the homologies.

1. We define H_n on the chain maps:
Any chain map $f_\bullet : X_\bullet \to X'_\bullet$ satisfies $d_n \circ f_n(x) = f_{n-1} \circ d_n(x) = 0$ for all $x \in Z_n(X_\bullet)$ and hence $f_n(Z_n(X_\bullet)) \subset Z_n(X'_\bullet)$. Likewise, for any n-boundary $x = d_{n+1}(y) \in B_n(X_\bullet)$, we have

$f_n(x) = f_n \circ d_{n+1}(y) = d_{n+1} \circ f_{n+1}(y) = d_{n+1}(f_{n+1}(y))$ and hence $f_n(B_n(X_\bullet)) \subset B_n(X'_\bullet)$. By the universal property of the factor group, this defines a group homomorphism

$$H_n(f_\bullet) : H_n(X_\bullet) \to H_n(X'_\bullet),\ [x] \mapsto [f_n(x)].$$

2. We show that H_n respects the composition of morphisms:
For chain maps $f_\bullet : X_\bullet \to X'_\bullet$ and $g_\bullet : X'_\bullet \to X''_\bullet$ and all n-cycles $x \in Z_n(X_\bullet)$ we have

$$H_n(g_\bullet \circ f_\bullet)([x]) = [g_n \circ f_n(x)] = H_n(g_\bullet)([f_n(x)]) = H_n(g_\bullet) \circ H_n(f_\bullet)([x]).$$

3. We show that H_n respects identity morphisms: for all chain complexes $X_\bullet$ and $x \in Z_n(X_\bullet)$

$$H_n(\mathrm{id}_{X_\bullet})([x]) = [\mathrm{id}_{X_n}(x)] = [x] = \mathrm{id}_{H_n(X_\bullet)}([x]).$$

□

The essential feature of chain complexes and chain maps that makes them so useful in topology is that there is another layer of structure beyond chain maps, namely chain homotopies between chain maps. Chain homotopies can be viewed as the counterpart of homotopies between continuous maps in topology. Recall that a homotopy between continuous maps $f, g : X \to X'$ with the same topological spaces as sources and targets is a continuous map $h : [0,1] \times X \to X'$ such that $h(0,x) = f(x)$ and $h(1,x) = g(x)$ for all $x \in X$. Equivalently, one can impose the condition that the following diagram of topological spaces and continuous maps commutes for the inclusions $\iota^j : X \to [0,1] \times X$, $x \mapsto (j,x)$

$$\begin{tikzcd}
X \arrow[r, "\iota^0"] \arrow[dr, "f"'] & {[0,1] \times X} \arrow[d, "h"] & X \arrow[l, "\iota^1"'] \arrow[dl, "g"] \\
& X'. &
\end{tikzcd}$$

Definition 2.1.9:

1. A **chain homotopy** $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ from a chain map $f_\bullet : X_\bullet \to X'_\bullet$ to a chain map $f'_\bullet : X_\bullet \to X'_\bullet$ is a family $(h_n)_{n\in\mathbb{Z}}$ of group homomorphisms $h_n : X_n \to X'_{n+1}$ with

$$f'_n - f_n = h_{n-1} \circ d_n + d'_{n+1} \circ h_n \qquad \forall n \in \mathbb{Z}. \tag{3}$$

If there is a chain homotopy $h_\bullet : f_\bullet \Rightarrow f'_\bullet$, then $f_\bullet$ und $f'_\bullet$ are called **chain homotopic**, and one writes $f_\bullet \sim f'_\bullet$.

2. A chain map $f_\bullet : X_\bullet \to X'_\bullet$ is called a **chain homotopy equivalence** if there is a chain map $g : X'_\bullet \to X_\bullet$ with $g_\bullet \circ f_\bullet \sim 1_{X_\bullet}$ and $f_\bullet \circ g_\bullet \sim 1_{X'_\bullet}$. In this case the chain complexes $X_\bullet$ und $X'_\bullet$ are called **chain homotopy equivalent**, and one writes $X_\bullet \simeq X'_\bullet$.

Remark 2.1.10:

1. For given chain complexes $X_\bullet$, $X'_\bullet$ the chain maps $f_\bullet : X_\bullet \to X'_\bullet$ and chain homotopies between them form a groupoid.

   The composite of two chain homotopies $h : f_\bullet \Rightarrow f'_\bullet$ and $h'_\bullet : f'_\bullet \Rightarrow f''_\bullet$ is the chain homotopy $h'_\bullet \circ h_\bullet = (h_n + h'_n)_{n\in\mathbb{Z}} : f_\bullet \Rightarrow f''_\bullet$. The identity morphisms are trivial chain homotopies $1_{f_\bullet} = (0)_{n\in\mathbb{Z}}$, and the inverse of $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ is $h_\bullet^{-1} = (-h_n)_{n\in\mathbb{Z}} : f'_\bullet \Rightarrow f_\bullet$.

2. For all chain complexes $X_\bullet, X'_\bullet$ being chain homotopic is an equivalence relation on the set $\mathrm{Hom}_{\mathrm{Ch}_{\mathrm{Ab}}}(X_\bullet, X'_\bullet)$ of chain maps from $X_\bullet$ to $X'_\bullet$. It is compatible with the composition of morphisms:

   For all chain maps $f_\bullet, f'_\bullet : X_\bullet \to X'_\bullet$ and $g_\bullet, g'_\bullet : X'_\bullet \to X''_\bullet$ and chain homotopies $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ and $h'_\bullet : g_\bullet \Rightarrow g'_\bullet$, the family of morphisms $k_\bullet = (g_{n+1} \circ h_n + h'_n \circ f'_n)_{n\in\mathbb{Z}}$ is a chain homotopy $k_\bullet : g_\bullet \circ f_\bullet \Rightarrow g'_\bullet \circ f'_\bullet$ since

$$\begin{aligned} g'_n \circ f'_n - g_n \circ f_n &= (g'_n - g_n) \circ f'_n + g_n \circ (f'_n - f_n) \\ &= (h'_{n-1} \circ d'_n + d''_{n+1} \circ h'_n) \circ f'_n + g_n \circ (h_{n-1} \circ d_n + d'_{n+1} \circ h_n) \\ &= (g_n \circ h_{n-1} + h'_{n-1} \circ f'_{n-1}) \circ d_n + d''_{n+1} \circ (g_{n+1} \circ h_n + h'_n \circ f'_n) \\ &= k_{n-1} \circ d_n + d''_{n+1} \circ k_n. \end{aligned}$$

3. We obtain a category $\mathcal{K}\mathrm{Ch}_{\mathrm{Ab}}$, the **homotopy category of chain complexes**, whose objects are chain complexes and whose morphisms are chain homotopy classes of chain maps. The isomorphisms in $\mathcal{K}\mathrm{Ch}_{\mathrm{Ab}}$ are chain homotopy classes of chain homotopy equivalences.

Remark 2.1.10 shows that chain homotopies have nice algebraic properties. However, the rather technical Definition 2.1.9 makes it difficult to view chain homotopies as a counterpart of topological homotopies. In particular, a homotopy $h : f \Rightarrow g$ between continuous maps $f, g : X \to X'$ is itself a continuous map $h : [0,1] \times X \to X'$. This is not directly apparent for chain homotopies. However, one can show that for chain maps $f_\bullet, g_\bullet : X_\bullet \to X'_\bullet$ chain homotopies from $f_\bullet$ to $g_\bullet$ are in fact chain maps from a chain complex $X''_\bullet$ that replaces the space $[0,1] \times X$ to $X'_\bullet$.

Remark 2.1.11: Let $X_\bullet$, $X'_\bullet$ be chain complexes and $X''_\bullet$ the chain complex with

$$X''_n = X_n \oplus X_n \oplus X_{n-1}, \qquad d''_n : X''_n \to X''_{n-1}, \;\; (x, x', x'') \mapsto (d_n(x) + x'', d_n(x') - x'', -d_{n-1}(x'')).$$

The inclusions $\iota^0_n : X_n \to X''_n$, $x \mapsto (x, 0, 0)$ and $\iota^1_n : X_n \to X''_n$, $x \mapsto (0, x, 0)$ define chain maps $\iota^0_\bullet, \iota^1_\bullet : X_\bullet \to X''_\bullet$. Chain homotopies $h_\bullet : f_\bullet \Rightarrow g_\bullet$ between chain maps $f_\bullet, g_\bullet : X_\bullet \to X'_\bullet$ are in bijection with chain maps $k_\bullet : X''_\bullet \to X'_\bullet$ such that $k_\bullet \circ \iota^0_\bullet = f_\bullet$ and $k_\bullet \circ \iota^1_\bullet = g_\bullet$.

$$\begin{tikzcd} X_\bullet \arrow[r, "\iota^0_\bullet"] \arrow[dr, "f_\bullet"'] & X''_\bullet \arrow[d, "k_\bullet"] & X_\bullet \arrow[l, "\iota^1_\bullet"'] \arrow[dl, "g_\bullet"] \\ & X'_\bullet & \end{tikzcd}$$

There are two reasons that homotopies are so useful in topology. The first is that they are often easy to visualise and behave rather intuitively. The second is that essential quantities that characterise a topological space such as the number of path components and its homotopy groups are isomorphic, whenever the spaces are homotopy equivalent. This is a consequence of the fact that homotopic maps between topological spaces induce the same group homomorphisms between their homotopy groups.

An analogous statement holds for chain maps between chain complexes and chain homotopies between them. Chain homotopic chain maps induce the same group homomorphisms between the homologies. As a consequence, the homologies of a chain complex depend only on its chain homotopy equivalence class.

Proposition 2.1.12:

1. Chain homotopic chain maps in Ab induce the same maps on the homologies: if $f_\bullet \sim g_\bullet$ then $H_n(f_\bullet) = H_n(g_\bullet)$ for all $n \in \mathbb{Z}$.
2. The nth homology induces a functor $H_n : \mathcal{K}\mathrm{Ch}_{\mathrm{Ab}} \to \mathrm{Ab}$ for all $n \in \mathbb{Z}$.
3. Chain homotopy equivalences induce isomorphisms on the homologies: if $X_\bullet \simeq X'_\bullet$, then $H_n(X_\bullet) \cong H_n(X'_\bullet)$ for all $n \in \mathbb{Z}$.

Proof:
1. To prove the first claim, let $f_\bullet, g_\bullet : X_\bullet \to X'_\bullet$ be chain maps and $h_\bullet : f_\bullet \Rightarrow g_\bullet$ a chain homotopy. Then we have for all $x \in Z_n(X_\bullet)$

$$H_n(f_\bullet)([x]) - H_n(g_\bullet)([x]) = [f_n(x) - g_n(x)] \overset{(3)}{=} [h_{n-1} \circ d_n(x) + d'_{n+1} \circ h_n(x)] = 0.$$

2. This follows directly from 1.

3. If $f_\bullet : X_\bullet \to X'_\bullet$ and $g_\bullet : X'_\bullet \to X_\bullet$ are chain maps with $f_\bullet \circ g_\bullet \sim 1_{X'_\bullet}$ and $g_\bullet \circ f_\bullet \sim 1_{X_\bullet}$, then

$$\begin{aligned} H_n(f_\bullet) \circ H_n(g_\bullet) &= H_n(f_\bullet \circ g_\bullet) = H_n(\mathrm{id}_{X'_\bullet}) = \mathrm{id}_{H_n(X'_\bullet)} \\ H_n(g_\bullet) \circ H_n(f_\bullet) &= H_n(g_\bullet \circ f_\bullet) = H_n(\mathrm{id}_{X_\bullet}) = \mathrm{id}_{H_n(X_\bullet)}. \end{aligned}$$

This shows that $H_n(f_\bullet)$ is an isomorphism with inverse $H_n(f_\bullet)^{-1} = H_n(g_\bullet)$. □

2.2 The long exact homology sequence

In this section we introduce *short exact sequences* of chain complexes. Short exact sequences of chain complexes can be viewed as triples of a chain complex $X_\bullet$, a subcomplex $W_\bullet \subset X_\bullet$ and the associated quotient complex $X_\bullet/W_\bullet$. They are the counterparts of subgroups and factor groups for abelian groups. We will show that the homologies of the chain complexes in short exact sequence form an exact chain complex, the *long exact sequence of homologies.*

Definition 2.2.1:

1. A **long exact sequence** in Ab is an exact chain complex.
2. A **long exact sequence** in $\mathrm{Ch}_{\mathrm{Ab}}$ is a sequence of chain complexes and chain maps

$$\ldots \xrightarrow{f^{k+2}_\bullet} A^{k+1}_\bullet \xrightarrow{f^{k+1}_\bullet} A^k_\bullet \xrightarrow{f^k_\bullet} A^{k-1}_\bullet \xrightarrow{f^{k-1}_\bullet} \ldots$$

   such that for all $n \in \mathbb{Z}$ their components form a long exact sequence in Ab

$$\ldots \xrightarrow{f^{k+2}_n} A^{k+1}_n \xrightarrow{f^{k+1}_n} A^k_n \xrightarrow{f^k_n} A^{k-1}_n \xrightarrow{f^{k-1}_n} \ldots$$

3. A **short exact sequence** in Ab is an exact chain complex $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$.
4. A **short exact sequence** in $\mathrm{Ch}_{\mathrm{Ab}}$ is a sequence $0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{\pi_\bullet} C_\bullet \to 0$ of chain complexes in Ab such that $0 \to A_n \xrightarrow{\iota_n} B_n \xrightarrow{\pi_n} C_n \to 0$ is exact for all $n \in \mathbb{Z}$.

Short exact sequences are called short exact sequences, because they are the shortest exact sequences that carry information that cannot be stated in a much simpler way. A chain complex of the form $0 \to X \to 0$ is exact if and only if X is the trivial group, and a chain complex of the form $0 \to X \xrightarrow{f} Y \to 0$ is exact if and only if f is a group isomorphism. The information contained in short exact sequences is less trivial. The following example shows that they are related to factor groups.

Remark 2.2.2:

1. The exactness of a short sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ in Ab is equivalent to: (i) ι injective, (ii) π surjective, (iii) $\ker \pi = \operatorname{im} \iota$.
2. An exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ in Ab corresponds to a triple (A, B, C) such that $A \subset B$ is a subgroup and $C = B/A$ the associated factor group.

   If $A \subset B$ is a subgroup and $C = B/A$ the associated factor group, then the injective inclusion $\iota : A \to B$ and the canonical surjection $\pi : B \to B/A$ satisfy $\operatorname{im} \iota = \ker \pi$ and thus form a short exact sequence in Ab.

   Conversely, in a short exact sequence injectivity of ι implies $A \cong \operatorname{im} \iota \subset B$. Conditions (ii) and (iii) imply $C \cong \operatorname{im} \pi \cong B/\ker \pi \cong B/\operatorname{im} \iota \cong B/A$.
3. Exactness of a sequence $0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{\pi_\bullet} C_\bullet \to 0$ is equivalent to (i) ι_n injective, (ii) π_n surjective, (iii) $\ker \pi_n = \operatorname{im} \iota_n$ for all $n \in \mathbb{Z}$.

We can visualise a long exact sequence in $\mathrm{Ch}_{\mathrm{Ab}}$ as a commuting diagram

$$\begin{tikzcd}
 & \ldots \arrow[d, "d^{k+1}_{n+2}"] & \ldots \arrow[d, "d^{k}_{n+2}"] & \ldots \arrow[d, "d^{k-1}_{n+2}"] & \\
\ldots \arrow[r, "f^{k+2}_{n+1}"] & A^{k+1}_{n+1} \arrow[r, "f^{k+1}_{n+1}"] \arrow[d, "d^{k+1}_{n+1}"] & A^{k}_{n+1} \arrow[r, "f^{k}_{n+1}"] \arrow[d, "d^{k}_{n+1}"] & A^{k-1}_{n+1} \arrow[r, "f^{k-1}_{n+1}"] \arrow[d, "d^{k-1}_{n+1}"] & \ldots \\
\ldots \arrow[r, "f^{k+2}_{n}"] & A^{k+1}_{n} \arrow[r, "f^{k+1}_{n}"] \arrow[d, "d^{k+1}_{n}"] & A^{k}_{n} \arrow[r, "f^{k}_{n}"] \arrow[d, "d^{k}_{n}"] & A^{k-1}_{n} \arrow[r, "f^{k-1}_{n}"] \arrow[d, "d^{k-1}_{n}"] & \ldots \\
\ldots \arrow[r, "f^{k+2}_{n-1}"] & A^{k+1}_{n-1} \arrow[r, "f^{k+1}_{n-1}"] \arrow[d, "d^{k+1}_{n-1}"] & A^{k}_{n-1} \arrow[r, "f^{k}_{n-1}"] \arrow[d, "d^{k}_{n-1}"] & A^{k-1}_{n-1} \arrow[r, "f^{k-1}_{n-1}"] \arrow[d, "d^{k-1}_{n-1}"] & \ldots \\
 & \ldots & \ldots & \ldots &
\end{tikzcd}$$

in which all columns are chain complexes and all rows exact chain complexes. The squares in this diagram commute, because the maps $f^k_n : A^k_n \to A^{k-1}_n$ form chain maps $f^k_\bullet : A^k_\bullet \to A^{k-1}_\bullet$. A short exact sequence of chain complexes is given by an analogous diagram, in which the dots at the left and right of the diagram are replaced by zeros.

We will now show that a short exact sequence of chain complexes generalises the concepts of a subgroup and a factor group from the category Ab to $\mathrm{Ch}_{\mathrm{Ab}}$. We start by introducing a subcomplex of a chain complex.

Definition 2.2.3: A **subcomplex** $A_\bullet \subset X_\bullet$ of a chain complex $X_\bullet$ is a family $(A_n)_{n\in\mathbb{Z}}$ of subgroups $A_n \subset X_n$ such that $d_n(A_n) \subset A_{n-1}$ for all $n \in \mathbb{Z}$.

The condition on a subcomplex ensures that the boundary operators $d^X_n : X_n \to X_{n-1}$ restrict to group homomorphisms $d^A_n : A_n \to A_{n-1}$ with $d^A_{n-1} \circ d^A_n = 0$ for all $n \in \mathbb{Z}$. This gives the family of subgroups $A_n \subset X_n$ the structure of a chain complex $A_\bullet$. By definition, its boundary operators satisfy $d^X_n \circ i_n = i_{n-1} \circ d^A_n$ for the inclusions $i_n : A_n \to X_n$. In other words, the inclusions define a chain map $i_\bullet : A_\bullet \to X_\bullet$ with injective components.

It is then plausible to define an associated quotient complex by considering the factor groups X_n/A_n and the canonical surjections $\pi_n : X_n \to X_n/A_n$ that satisfy $\pi_n \circ i_n = 0$ for all $n \in \mathbb{Z}$.

The boundary operator of the quotient complex $X_\bullet/A_\bullet$ should be induced by the universal property of the quotient and thus be given by $d_n^{X/A}: X_n/A_n \to X_{n-1}/A_{n-1}$, $[x] \mapsto [d_n(x)]$.

We also expect that any chain map $f_\bullet: X_\bullet \to Y_\bullet$ that satisfies $f_n(A_n) \subset B_n$ for all $n \in \mathbb{Z}$ and subcomplexes $A_\bullet \subset X_\bullet$ and $B_\bullet \subset Y_\bullet$ should induce chain maps $f^A_\bullet: A_\bullet \to B_\bullet$ and $f'_\bullet: X_\bullet/A_\bullet \to Y_\bullet/B_\bullet$. An analogous statement should hold for chain homotopies.

Proposition 2.2.4: Let $X_\bullet, Y_\bullet$ be chain complexes.

1. A subcomplex $A_\bullet \subset X_\bullet$ defines a **quotient complex** $X_\bullet/A_\bullet$ and a short exact sequence of chain complexes $0 \to A_\bullet \xrightarrow{i_\bullet} X_\bullet \xrightarrow{\pi_\bullet} X_\bullet/A_\bullet \to 0$ given by the commuting diagrams
$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "d_n^A"] & X_n \arrow[r, "\pi_n"] \arrow[d, "d_n^X"] & X_n/A_n \arrow[r] \arrow[d, "d_n^{X/A}"] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r, "i_{n-1}"'] & X_{n-1} \arrow[r, "\pi_{n-1}"'] & X_{n-1}/A_{n-1} \arrow[r] & 0. \end{tikzcd} \tag{4}$$

2. If $A_\bullet \subset X_\bullet$ and $B_\bullet \subset Y_\bullet$ are subcomplexes and $f_\bullet: X_\bullet \to Y_\bullet$ is a chain map with $f_n(A_n) \subset B_n$ for all $n \in \mathbb{Z}$, then we obtain a commuting diagram in $\mathrm{Ch}_{\mathrm{Ab}}$ with exact rows
$$\begin{tikzcd} 0 \arrow[r] & A_\bullet \arrow[r, "i_\bullet"] \arrow[d, "f^A_\bullet"] & X_\bullet \arrow[r, "\pi_\bullet"] \arrow[d, "f_\bullet"] & X_\bullet/A_\bullet \arrow[r] \arrow[d, "f'_\bullet"] & 0 \\ 0 \arrow[r] & B_\bullet \arrow[r, "i'_\bullet"'] & Y_\bullet \arrow[r, "\pi'_\bullet"'] & Y_\bullet/B_\bullet \arrow[r] & 0. \end{tikzcd} \tag{5}$$

3. Let $A_\bullet \subset X_\bullet$ and $B_\bullet \subset Y_\bullet$ be subcomplexes and $f_\bullet, g_\bullet: X_\bullet \to Y_\bullet$ chain maps with $f_n(A_n) \subset B_n$ and $g_n(A_n) \subset B_n$ for all $n \in \mathbb{Z}$. Then any chain homotopy $h_\bullet: f_\bullet \Rightarrow g_\bullet$ with $h_n(A_n) \subset B_{n+1}$ for all $n \in \mathbb{Z}$ induces chain homotopies $h^A_\bullet: f^A_\bullet \Rightarrow g^A_\bullet$ and $h'_\bullet: f'_\bullet \Rightarrow g'_\bullet$.

Proof:
1. As $A_n \subset X_n$ is a subgroup and X_n/A_n the associated factor group, both rows of diagram (4) are short exact sequences in Ab. The condition that $A_\bullet \subset X_\bullet$ is a subcomplex implies $d_n^X \circ i_n = i_{n-1} \circ d_n^A$ and hence the left square in (4) commutes.

We define $d_n^{X/A}: X_n/A_n \to X_{n-1}/A_{n-1}$ in (4) and show that it satisfies $d_{n-1}^{X/A} \circ d_n^{X/A} = 0$ for all $n \in \mathbb{Z}$. As the left square in (4) commutes and both rows are exact, we have

$$\pi_{n-1} \circ d_n^X \circ i_n = \pi_{n-1} \circ i_{n-1} \circ d_n^A = 0 \circ d_n^A = 0 \quad \Rightarrow \quad \ker \pi_n = \mathrm{im}\, i_n \subset \ker(\pi_{n-1} \circ d_n^X).$$

The universal property of the quotient defines a homomorphism $d_n^{X/A}: X_n/A_n \to X_{n-1}/A_{n-1}$ with $d_n^{X/A} \circ \pi_n = \pi_{n-1} \circ d_n^X$. This states that the right square in (4) commutes and implies

$$d_{n-1}^{X/A} \circ d_n^{X/A} \circ \pi_n = d_{n-1}^{X/A} \circ \pi_{n-1} \circ d_n^X = \pi_{n-2} \circ d_{n-1}^X \circ d_n^X = \pi_{n-2} \circ 0 = 0.$$

With the surjectivity of π_n we obtain $d_{n-1}^{X/A} \circ d_n^{X/A} = 0$ for all $n \in \mathbb{Z}$. Thus, we have a quotient complex $X_\bullet/A_\bullet$. The inclusions i_n and canonical surjections π_n define chain maps $i_\bullet: A_\bullet \to X_\bullet$ and $\pi_\bullet: X_\bullet \to X_\bullet/A_\bullet$, because diagram (4) commutes.

2. If $f_\bullet: X_\bullet \to Y_\bullet$ is a chain map with $f_n(A_n) \subset B_n$ for all $n \in \mathbb{Z}$, its components restrict to group homomorphisms $f_n^A: A_n \to B_n$ with $i'_n \circ f_n^A = f_n \circ i_n$ for the inclusions $i_n: A_n \to X_n$

and $i'_n : B_n \to Y_n$. The left square in the following diagram with exact rows commutes

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "f_n^A"] & X_n \arrow[r, "\pi_n"] \arrow[d, "f_n"] & X_n/A_n \arrow[r] \arrow[d, "f'_n"] & 0 \\
0 \arrow[r] & B_n \arrow[r, "i'_n"'] & Y_n \arrow[r, "\pi'_n"'] & Y_n/B_n \arrow[r] & 0.
\end{tikzcd}$$

We define the group homomorphism $f' : X_n/A_n \to Y_n/B_n$ via the universal property of the quotient. By exactness of the rows and because the left square commutes, we have

$$\pi'_n \circ f_n \circ i_n = \pi'_n \circ i'_n \circ f_n^A = 0 \circ f_n^A = 0 \quad \Rightarrow \quad \ker \pi_n = \operatorname{im} i_n \subset \ker (\pi'_n \circ f_n).$$

The universal property of the quotient gives a unique homomorphism $f'_n : X_n/A_n \to Y_n/B_n$ with $f'_n \circ \pi_n = \pi'_n \circ f_n$. We show that these maps define a chain map $f'_\bullet : X_\bullet/A_\bullet \to Y_\bullet/B_\bullet$:

$$d_n^{Y/B} \circ f'_n \circ \pi_n = d_n^{Y/B} \circ \pi'_n \circ f_n = \pi'_{n-1} \circ d_n^Y \circ f_n = \pi'_{n-1} \circ f_{n-1} \circ d_n^X = f'_{n-1} \circ \pi_{n-1} \circ d_n^X = f'_{n-1} \circ d_n^{X/A} \circ \pi_n.$$

By surjectivity of π_n this implies $d_n^{Y/B} \circ f'_n = f'_{n-1} \circ d_n^{X/A}$ for all $n \in \mathbb{Z}$.

3. As $h_n(A_n) \subset B_{n+1}$, the components $h_n : X_n \to Y_{n+1}$ of the chain homotopy $h_\bullet$ define group homomorphisms $h_n^A : A_n \to B_{n+1}$ with $i'_{n+1} \circ h_n^A = h_n \circ i_n$ for all $n \in \mathbb{Z}$. This implies

$$\begin{aligned}
i'_n \circ (d_{n+1}^B \circ h_n^A + h_{n-1}^A \circ d_n^A) &= d_{n+1}^Y \circ i'_{n+1} \circ h_n^A + h_{n-1} \circ i_{n-1} \circ d_n^A \\
&= (d_{n+1}^Y \circ h_n + h_{n-1} \circ d_n^X) \circ i_n = (g_n - f_n) \circ i_n = i'_n \circ (g_n^A - f_n^A),
\end{aligned}$$

and by injectivity of i'_n it follows that this defines a chain homotopy $h_\bullet^A : f_\bullet^A \Rightarrow g_\bullet^A$.

We also have the identities

$$\pi'_{n+1} \circ h_n \circ i_n = \pi'_{n+1} \circ i'_{n+1} \circ h_n^A = 0 \circ h_n^A = 0 \quad \Rightarrow \quad \ker \pi_n = \operatorname{im} i_n \subset \ker (\pi'_{n+1} \circ h_n).$$

The universal property of the quotient yields a unique homomorphism $h'_n : X_n/A_n \to Y_{n+1}/B_{n+1}$ with $h'_n \circ \pi_n = \pi'_{n+1} \circ h_n$. We show that the maps h_n define a chain homotopy from f' to g':

$$\begin{aligned}
&(d_{n+1}^{Y/B} \circ h'_n + h'_{n-1} \circ d_n^{X/A}) \circ \pi_n = d_{n+1}^{Y/B} \circ \pi'_{n+1} \circ h_n + h'_{n-1} \circ \pi_{n-1} \circ d_n^X \\
&= \pi'_n \circ (d_{n+1}^Y \circ h_n + h_{n-1} \circ d_n^X) = \pi'_n \circ (g_n - f_n) = (g'_n - f'_n) \circ \pi_n.
\end{aligned}$$

Surjectivity of π_n implies that the morphisms h'_n define a chain homotopy $h'_\bullet : f'_\bullet \Rightarrow g'_\bullet$. □

We now investigate the relation between the homologies of chain complexes in a short exact sequence $0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{\pi_\bullet} C_\bullet \to 0$. A first naive guess would be that that these homologies also form short exact sequences. This would amount to the identities $H_n(C_\bullet) = H_n(B_\bullet)/H_n(A_\bullet)$ for all $n \in \mathbb{Z}$. The following example shows that this first guess is wrong and that the relation between the homologies must be more complicated than simply a quotient.

Example 2.2.5: We consider the short exact sequence $0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{\pi_\bullet} C_\bullet \to 0$ of chain complexes given by the following commuting diagram with exact rows

$$\begin{tikzcd}
& A_\bullet & & B_\bullet & & C_\bullet & \\
& 0 \arrow[d, "d_2^A"] & & 0 \arrow[d, "d_2^B"] & & 0 \arrow[d, "d_2^C"] & \\
0 \arrow[r] & A_1 = 0 \arrow[rr, "\iota_1"] \arrow[d, "d_1^A"] & & B_1 = \mathbb{Z} \arrow[rr, "\pi_1 = \mathrm{id}"] \arrow[d, "d_1^B : z \mapsto (z,z)"] & & C_1 = \mathbb{Z} \arrow[r] \arrow[d, "d_1^C : z \mapsto 0"] & 0 \\
0 \arrow[r] & A_0 = \mathbb{Z} \arrow[rr, "\iota_0 : z \mapsto (z,z)"] \arrow[d, "d_0^A"] & & B_0 = \mathbb{Z} \oplus \mathbb{Z} \arrow[rr, "\pi_0 : (z_1,z_2) \mapsto z_2 - z_1"] \arrow[d, "d_0^B"] & & C_0 = \mathbb{Z} \arrow[r] \arrow[d, "d_0^C"] & 0 \\
& 0 & & 0 & & 0 &
\end{tikzcd}$$

Then we have

$$\begin{array}{lll}
H_n(A_\bullet) = 0, & H_n(B_\bullet) = 0, & H_n(C_\bullet) = 0 \quad n \geq 2 \text{ or } n < 0 \\
H_1(A_\bullet) = \ker d_1^A = 0, & H_1(B_\bullet) = \ker d_1^B = 0, & H_1(C_\bullet) = \ker d_1^C = \mathbb{Z} \\
H_0(A_\bullet) = \ker d_0^A = \mathbb{Z}, & H_0(B_\bullet) = \ker d_0^B / \mathrm{im}\, d_1^B \cong \mathbb{Z}, & H_0(C_\bullet) = \ker d_0^C = \mathbb{Z}.
\end{array}$$

Clearly, $H_1(C_\bullet) = \mathbb{Z} \ncong 0 = H_1(B_\bullet)/H_1(A_\bullet)$.

This example shows that the relation between the homologies of chain complexes in a short exact sequence is more complicated than apparent at first sight. We will now clarify this relation. Our first ingredient is the connecting homomorphism. It relates the nth homology of the last chain complex in a short exact sequence to the $(n-1)$st homology of the first chain complex. Despite its awkward definition, it is conceptual and natural with respect to chain maps.

Proposition 2.2.6 (**connecting homomorphism**)**:**

Let $0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{\pi_\bullet} C_\bullet \to 0$ a short exact sequence of chain complexes. Then there are group homomorphisms

$$\partial_n : H_n(C_\bullet) \to H_{n-1}(A_\bullet), \quad [c] \mapsto [a] \qquad \exists b \in B_n : \pi_n(b) = c \text{ and } d_n(b) = \iota_{n-1}(a)$$

for all $n \in \mathbb{Z}$, the **connecting homomorphisms** such that for all commuting diagrams

$$\begin{tikzcd}
0 \arrow[r] & A_\bullet \arrow[r, "\iota_\bullet"] \arrow[d, "\alpha_\bullet"] & B_\bullet \arrow[r, "\pi_\bullet"] \arrow[d, "\beta_\bullet"] & C_\bullet \arrow[r] \arrow[d, "\gamma_\bullet"] & 0 \\
0 \arrow[r] & A'_\bullet \arrow[r, "\iota'_\bullet"] & B'_\bullet \arrow[r, "\pi'_\bullet"] & C'_\bullet \arrow[r] & 0
\end{tikzcd} \tag{6}$$

of chain complexes with exact rows, we have $H_{n-1}(\alpha_\bullet) \circ \partial_n = \partial'_n \circ H_n(\gamma_\bullet)$

$$\begin{tikzcd}
H_n(C_\bullet) \arrow[r, "\partial_n"] \arrow[d, "H_n(\gamma_\bullet)"'] & H_{n-1}(A_\bullet) \arrow[d, "H_{n-1}(\alpha_\bullet)"] \\
H_n(C'_\bullet) \arrow[r, "\partial'_n"'] & H_{n-1}(A'_\bullet).
\end{tikzcd}$$

Proof:
Let $c \in Z_n(C_\bullet)$. Then by surjectivity of π_n there is an $b \in B_n$ with $\pi_n(b) = c$. Because $\pi_\bullet$ is a chain map and $c \in Z_n(C_\bullet)$, this implies $\pi_{n-1} \circ d_n(b) = d_n \circ \pi_n(b) = d_n(c) = 0$. This shows that $d_n(b) \in \ker(\pi_{n-1}) = \mathrm{im}(\iota_{n-1})$, and there is an $a \in A_{n-1}$ with $\iota_{n-1}(a) = d_n(b)$. By injectivity of ι_{n-1}, this element $a \in A_{n-1}$ is unique. We have $\iota_{n-1} \circ d_{n-1}(a) = d_{n-1} \circ \iota_n(a) = d_{n-1} \circ d_n(b) = 0$. By injectivity of ι_{n-1}, this implies $d_{n-1}(a) = 0$ and hence $a \in Z_{n-1}(A_\bullet)$.

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "\iota_n"] \arrow[d, "d_n"] & b \in B_n \arrow[r, "\pi_n"] \arrow[d, "d_n"] & c \in C_n \arrow[r] \arrow[d, "d_n"] & 0 \\
0 \arrow[r] & a \in A_{n-1} \arrow[r, "\iota_{n-1}"'] & d_n(b) \in B_{n-1} \arrow[r, "\pi_{n-1}"'] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

We aim to define

$$\partial_n[c] = [a], \text{ where } c = \pi_n(b),\ \iota_{n-1}(a) = d_n(b) \text{ for some } b \in B_n. \tag{7}$$

1. We show that this is well-defined: (i) it does not depend on the choice of b and (ii) it depends only on the homology class of c.

(i) Let $b' \in B_n$ another element with $\pi_n(b') = c$ and $a' \in A_{n-1}$ with $\iota_{n-1}(a') = d_n(b')$. Then $b' - b \in \ker(\pi_n) = \mathrm{im}(\iota_n)$, so there is an $a'' \in A_n$ with $b' - b = \iota_n(a'')$. This implies

$$\iota_{n-1}(a' - a) = d_n(b' - b) = d_n \circ \iota_n(a'') = \iota_{n-1} \circ d_n(a'') \quad \Rightarrow \quad a' - a = d_n(a'') \quad \Rightarrow \quad [a] = [a'].$$

(ii) Let $c' = c + d_{n+1}(c'')$. By surjectivity of π_{n+1} there is a $b'' \in B_{n+1}$ with $\pi_{n+1}(b'') = c''$. Because $\pi_\bullet$ is a chain map, this implies $\pi_n \circ d_{n+1}(b'') = d_{n+1} \circ \pi_{n+1}(b'') = d_{n+1}(c'')$. We set $b' = b + d_{n+1}(b'')$ and obtain $\pi_n(b') = \pi_n(b + d_{n+1}(b'')) = c + d_{n+1}(c'') = c'$ as well as $d_n(b') = d_n(b)$. Thus $\iota_{n-1}(a) = d_n(b) = d_n(b')$.

2. We show that $\partial_n : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ is a group homomorphism:

Let $c, c' \in Z_n(C_\bullet)$ with $\pi_n(b) = c$, $\pi_n(b') = c'$ and $\iota_{n-1}(a) = d_n(b)$, $\iota_{n-1}(a') = d_n(b')$. Then we have $\partial_n[c] = a$ and $\partial_n[c'] = [a']$. As ι_n and π_n are group homomorphisms, we have

$$\pi_n(b + b') = \pi_n(b) + \pi_n(b') = c + c' \qquad \iota_{n-1}(a + a') = \iota_{n-1}(a) + \iota_{n-1}(a') = b + b'.$$

This implies $\partial_n[c + c'] = [a + a'] = [a] + [a'] = \partial_n[c] + \partial_n[c']$.

3. We prove the naturality of ∂_n:

Let $c \in Z_n(C_\bullet)$, $b \in B_n$ with $\pi_n(b) = c$ and $d_n(b) = \iota_{n-1}(a)$. Then we have

$$H_{n-1}(\alpha_\bullet) \circ \partial_n[c] = H_{n-1}(\alpha_\bullet)[a] = [\alpha_{n-1}(a)].$$

Because diagram (6) commutes, we have $\pi'_n \circ \beta_n(b) = \gamma_n \circ \pi_n(b) = \gamma_n(c)$ and because $\beta_\bullet$ is a chain map $d'_n \circ \beta_n(b) = \beta_{n-1} \circ d_n(b) = \beta_{n-1} \circ \iota_{n-1}(a) = \iota'_{n-1} \circ \alpha_{n-1}(a)$. This implies

$$\partial'_n \circ H_{n-1}(\gamma_\bullet)[c] = \partial'_n[\gamma_n(c)] = [\alpha_{n-1}(a)] = H_{n-1}(\alpha_\bullet) \circ \partial_n[c]. \qquad \square$$

The connecting homomorphisms allow us to organise the homologies of a short exact sequence $0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{\pi_\bullet} C_\bullet \to 0$ into a long sequence, whose terms are related by the group homomorphisms $H_n(\iota_\bullet) : H_n(A_\bullet) \to H_n(B_\bullet)$ and $H_n(\pi_\bullet) : H_n(B_\bullet) \to H_n(C_\bullet)$ and by the connecting homomorphisms $\partial_n : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$. It turns out that this sequence is exact. The naturality of the connecting homomorphisms and the fact that the homologies are functors $H_n : \mathrm{Ch}_{\mathrm{Ab}} \to \mathrm{Ab}$ ensure that this exact sequence is compatible with triples of chain maps.

Proposition 2.2.7 (**long exact homology sequence**):

1. Any short exact sequence $0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{\pi_\bullet} C_\bullet \to 0$ of chain complexes defines a long exact homology sequence

$$\ldots \xrightarrow{\partial_{n+1}} H_n(A_\bullet) \xrightarrow{H_n(\iota_\bullet)} H_n(B_\bullet) \xrightarrow{H_n(\pi_\bullet)} H_n(C_\bullet) \xrightarrow{\partial_n} H_{n-1}(A_\bullet) \xrightarrow{H_{n-1}(\iota_\bullet)} H_{n-1}(B_\bullet) \xrightarrow{H_n(\pi_\bullet)} \ldots$$

2. Any commuting diagram of chain complexes with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A_\bullet \arrow[r, "\iota_\bullet"] \arrow[d, "\alpha_\bullet"] & B_\bullet \arrow[r, "\pi_\bullet"] \arrow[d, "\beta_\bullet"] & C_\bullet \arrow[r] \arrow[d, "\gamma_\bullet"] & 0 \\
0 \arrow[r] & A'_\bullet \arrow[r, "\iota'_\bullet"'] & B'_\bullet \arrow[r, "\pi'_\bullet"'] & C'_\bullet \arrow[r] & 0
\end{tikzcd}$$

defines a commuting diagram

$$\begin{tikzcd}
\ldots \arrow[r, "H_{n+1}(\pi_\bullet)"] & H_{n+1}(C_\bullet) \arrow[r, "\partial_{n+1}"] \arrow[d, "H_{n+1}(\gamma_\bullet)"] & H_n(A_\bullet) \arrow[r, "H_n(\iota_\bullet)"] \arrow[d, "H_n(\alpha_\bullet)"] & H_n(B_\bullet) \arrow[r, "H_n(\pi_\bullet)"] \arrow[d, "H_n(\beta_\bullet)"] & H_n(C_\bullet) \arrow[r, "\partial_n"] \arrow[d, "H_n(\gamma_\bullet)"] & H_{n-1}(A_\bullet) \arrow[r, "H_{n-1}(\iota_\bullet)"] \arrow[d, "H_{n-1}(\alpha_\bullet)"] & \ldots \\
\ldots \arrow[r, "H_{n+1}(\pi'_\bullet)"'] & H_{n+1}(C'_\bullet) \arrow[r, "\partial'_{n+1}"'] & H_n(A'_\bullet) \arrow[r, "H_n(\iota'_\bullet)"'] & H_n(B'_\bullet) \arrow[r, "H_n(\pi'_\bullet)"'] & H_n(C'_\bullet) \arrow[r, "\partial'_n"'] & H_{n-1}(A'_\bullet) \arrow[r, "H_{n-1}(\iota'_\bullet)"'] & \ldots
\end{tikzcd}$$

Proof:
1.(a) We show exactness in $H_n(B_\bullet)$:
For all $a \in Z_n(A_\bullet)$ we have $H_n(\pi_\bullet) \circ H_n(\iota_\bullet)[a] = H_n(\pi_\bullet)[\iota_n(a)] = [\pi_n \circ \iota_n(a)] = 0$ by exactness of the sequence. This implies $H_n(\pi_\bullet) \circ H_n(\iota_\bullet)[a] = [\pi_n \circ \iota_n(a)] = [0]$ and $\mathrm{im}\, H_n(\iota_\bullet) \subset \ker H_n(\pi_\bullet)$.

Let now $b \in Z_n(B_\bullet)$ with $H_n(\pi_\bullet)[b] = [\pi_n(b)] = 0$. Then there is a $c' \in C_{n+1}$ such that $\pi_n(b) = d_{n+1}(c')$ and by surjectivity of π_{n+1} a $b' \in B_{n+1}$ with $\pi_{n+1}(b') = c'$. We then have $\pi_n(b - d_{n+1}(b')) = \pi_n(b) - \pi_n \circ d_{n+1}(b') = d_{n+1}(c') - d_{n+1} \circ \pi_{n+1}(b') = d_{n+1}(c') - d_{n+1}(c') = 0$. This gives $b - d_{n+1}(b') \in \ker \pi_n$, and by exactness of the sequence there is an $a \in A_n$ with $\iota_n(a) = b - d_{n+1}(b')$. This implies $\iota_{n-1} \circ d_n(a) = d_n \circ \iota_n(a) = d_n(b) - d_n \circ d_{n+1}(b') = 0$, because $\iota_\bullet$ is a chain map, and $d_n(a) = 0$ by injectivity of ι_{n-1}. Hence, $a \in Z_n(A_\bullet)$ with $H_n(\iota_\bullet)[a] = [\iota_n(a)] = [b - d_{n+1}(b')] = [b]$, and this shows $\ker H_n(\pi_\bullet) \subset \mathrm{im}\, H_n(\iota_\bullet)$.

1.(b) We show exactness in $H_{n-1}(A_\bullet)$:
Let $a \in Z_{n-1}(A_\bullet)$ and $[c] \in H_n(C_\bullet)$ with $\partial_n[c] = [a]$. Then there is a $b \in B_n$ with $\pi_n(b) = c$ and $\iota_{n-1}(a) = d_n(b)$. This implies $H_{n-1}(\iota_\bullet)[a] = [\iota_{n-1}(a)] = [d_n(b)] = 0$. Hence, we have $\mathrm{im}\, \partial_n \subset \ker H_{n-1}(\iota_\bullet)$.

Conversely, let $[a] \in \ker H_{n-1}(\iota_\bullet)$. Then there is a $b \in B_n$ with $\iota_{n-1}(a) = d_n(b)$. With the definition of the connection homomorphism, this implies $\partial_n[\pi_n(b)] = [a]$ and $[a] \in \partial_n H_n(C_\bullet)$. Thus we have $\ker H_{n-1}(\iota_\bullet) \subset \mathrm{im}\, \partial_n$.

1.(c) We show exactness in $H_n(C_\bullet)$:
Because $\pi_\bullet$ is a chain map, we have $d_n \circ \pi_n(b) = \pi_{n-1} \circ d_n(b) = 0 = \iota_{n-1}(0)$ for all $b \in Z_n(B_\bullet)$. This implies $\partial_n \circ H_n(\pi_\bullet)[b] = \partial_n[\pi_n(b)] = [0]$ and $\mathrm{im}\, H_n(\pi_\bullet) \subset \ker \partial_n$.

Conversely, let $c \in Z_n(C_\bullet)$ with $\partial_n[c] = 0$. Then by definition of the connecting homomorphism there is a $b \in B_n$ and an $a \in A_n$ with $\pi_n(b) = c$ and $d_n(b) = \iota_{n-1} \circ d_n(a) = d_n \circ \iota_n(a)$. This implies $d_n(b - \iota_n(a)) = 0$ and hence $b - \iota_n(a) \in Z_n(B_\bullet)$. By exactness of the sequence we also have $\pi_n(b - \iota_n(a)) = \pi_n(b) - \pi_n \circ \iota_n(a) = \pi_n(b) = c$. This shows that $[c] \in \mathrm{im}\, H_n(\pi_\bullet)$, because $H_n(\pi_\bullet)[b - \iota_n(a)] = [\pi_n(b - \iota_n(a))] = [c]$, and that $\ker \partial_n \subset \mathrm{im}\, H_n(\pi_\bullet)$.

2. The diagram of chain complexes and chain maps defines two long exact sequences of homologies and group homomorphisms on the vertical arrows between them. All squares in the diagram of homologies that do not involve the connecting homomorphisms commute by functoriality of the homologies and because the associated squares of chain complexes commute. The squares involving the connecting homomorphisms commute by Proposition 2.2.6. □

Example 2.2.8: Consider the short exact sequence $0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{\pi_\bullet} C_\bullet \to 0$ of chain complexes from Example 2.2.5 given by the following commuting diagram with exact rows

$$\begin{tikzcd}
 & A_\bullet & B_\bullet & C_\bullet & \\
 & 0 \arrow[d, "d_2^A"] & 0 \arrow[d, "d_2^B"] & 0 \arrow[d, "d_2^C"] & \\
0 \arrow[r] & A_1 = 0 \arrow[r, "\iota_1"] \arrow[d, "d_1^A"] & B_1 = \mathbb{Z} \arrow[r, "\pi_1 = \mathrm{id}"] \arrow[d, "d_1^B : z \mapsto (z,z)"] & C_1 = \mathbb{Z} \arrow[r] \arrow[d, "d_1^C : z \mapsto 0"] & 0 \\
0 \arrow[r] & A_0 = \mathbb{Z} \arrow[r, "\iota_0 : z \mapsto (z,z)"] \arrow[d, "d_0^A"] & B_0 = \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\pi_0 : (z_1, z_2) \mapsto z_2 - z_1"] \arrow[d, "d_0^B"] & C_0 = \mathbb{Z} \arrow[r] \arrow[d, "d_0^C"] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

From Example 2.2.5 it follows that there can be only one non-trivial connecting homomorphism, namely $\partial_1 : H_1(C_\bullet) \to H_0(A_\bullet)$. We have $Z_1(C_\bullet) = \ker d_1^C = \mathbb{Z}$ and for any $c \in Z_1(C_\bullet) = \mathbb{Z}$ we have $c = \pi_1(b)$ and $\iota_0(a) = d_n(b)$ for some $b \in B_1$ and $a \in A_0$ if and only if $b = c$ and $\iota_0(a) = (a, a) = (c, c) = d_1^B(b)$. This implies $a = c$ and

$$\partial_1 = \mathrm{id} : \mathbb{Z} \to \mathbb{Z}.$$

Using the results from Example 2.2.5 we find the long exact homology sequence

$$\begin{tikzcd}
0 \arrow[r] & H_1(A_\bullet) \arrow[r, "H_1(\iota_\bullet)"] \arrow[d, equal] & H_1(B_\bullet) \arrow[r, "H_1(\pi_\bullet)"] \arrow[d, equal] & H_1(C_\bullet) \arrow[r, "\partial_1"] \arrow[d, equal] & H_0(A_\bullet) \arrow[r, "H_0(\iota_\bullet)"] \arrow[d, equal] & H_0(B_\bullet) \arrow[r, "H_0(\pi_\bullet)"] \arrow[d, equal] & H_0(C_\bullet) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r, "0"'] & \mathbb{Z} \arrow[r, "\mathrm{id}"'] & \mathbb{Z} \arrow[r, "0"'] & \mathbb{Z} \arrow[r, "\mathrm{id}"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where the last two labelled arrows in the bottom row are already determined uniquely by ∂_1 and the requirement that the sequence is exact. They can also be computed directly from the commuting diagram that defines the chain complexes.

3 Singular homology

In this section, we associate to every topological space a chain complex, its *singular chain complex*, and to each continuous map a chain map, the *singular map*. We will also show that homotopies between continuous maps give rise to chain homotopies and hence do not affect the homologies of these chain complexes. This leads to functors $H_n : \mathrm{Top} \to \mathrm{Ab}$ that are constant on homotopy classes of continuous maps and hence descend to functors $H_n : \mathcal{K}\mathrm{Top} \to \mathrm{Ab}$ from the homotopy category of topological spaces. We will then investigate its behaviour with respect to subspaces and quotients.

In the following we focus on chain complexes in the category $\mathbb{Z}\text{-Mod} = \mathrm{Ab}$ of abelian groups and group homomorphisms between them. Chain complexes with coefficients in other abelian groups are considered in Section 5.

3.1 The singular chain complex

The idea of singular homology is to probe a topological space X by considering continuous maps $f : \Delta^n \to X$ from certain standard spaces, the *standard simplexes* Δ^n. They are examples of affine simplexes, convex hulls of discrete sets of point in $\mathbb{R}^n$.

Recall that an affine subspace A of an $\mathbb{F}$-vector space V with associated subspace $V_A \subset V$ is a subset $A \subset V$ such that $a' - a \in V_A$ for all $a, a' \in A$. This implies $\Sigma_{i=0}^n \lambda_i a_i \in A$ for all $a_i \in A$ and $\lambda_i \in \mathbb{F}$ with $\lambda_0 + \ldots + \lambda_n = 1$. Such an expression is called an affine linear combination.

An affine map $f : A \to B$ between non-empty affine subspaces $A, B \subset V$ is a map that induces a linear map $f : V_A \to V_B$. This implies that f respects affine linear combinations: $f(\Sigma_{i=0}^n \lambda_i a_i) = \Sigma_{i=0}^n \lambda_i f(a_i)$ for all $a_i \in A$ and $\lambda_i \in \mathbb{F}$ with $\lambda_0 + \ldots + \lambda_n = 1$.

Definition 3.1.1:

1. An **affine m-simplex** $\Delta \subset \mathbb{R}^n$ is the convex hull of $m+1$ points $v_0, ..., v_m \in \mathbb{R}^n$

$$\Delta = \mathrm{conv}\{v_0, ..., v_m\} = \{\Sigma_{i=0}^m t_i v_i \mid 0 \le t_i \le 1, \Sigma_{i=0}^m t_i = 1\}.$$

   It is called **degenerate**, if the vectors $v_1 - v_0, \ldots, v_m - v_0$ are linearly dependent.

2. The k-simplexes conv $\{v_{i_0}, \ldots, v_{i_k}\}$ for subsets $\{v_{i_0}, \ldots, v_{i_k}\} \subset \{v_0, \ldots, v_n\}$ are called the **k-faces** of Δ, the 0-faces are called **vertices** and the 1-faces **edges**.

3. An **ordered m-simplex** is an affine m-simplex with an ordering of its vertices. We write $[v_0, ..., v_m]$ for $\Delta = \mathrm{conv}\{v_0, ..., v_m\}$ with ordering $v_0 < v_1 < ... < v_m$.

The advantage of working with affine simplexes is that all of their k-faces are again affine simplexes. They are obtained by omitting an arbitrary subset of vertices from the simplex. Thus, affine simplexes are simple building blocks for topological spaces that exist in any dimension and can be reduced to specifying points. If we are just interested in affine simplexes up to invertible affine maps, we can restrict attention to a set of non-degenerate standard simplexes that are given by the origin and a standard basis of $\mathbb{R}^n$. The ordering of the standard basis equips them with an ordering.

Definition 3.1.2: Let $(e_1, ..., e_n)$ be the standard basis of $\mathbb{R}^n$ and $e_0 := 0 \in \mathbb{R}^n$.

1. The **standard n-simplex** is the ordered n-simplex

$$\Delta^n = [e_0, ..., e_n] = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid 0 \le t_i \le 1, \Sigma_{i=1}^n t_i \le 1\}.$$

2. For $n \in \mathbb{N}$ and $i \in \{0, ..., n\}$ the ith **face map** $f_i^n : \Delta^{n-1} \to \Delta^n$ is the affine map with

$$f_i^n(e_j) = \begin{cases} e_j & j < i \\ e_{j+1} & j \ge i. \end{cases} \tag{8}$$

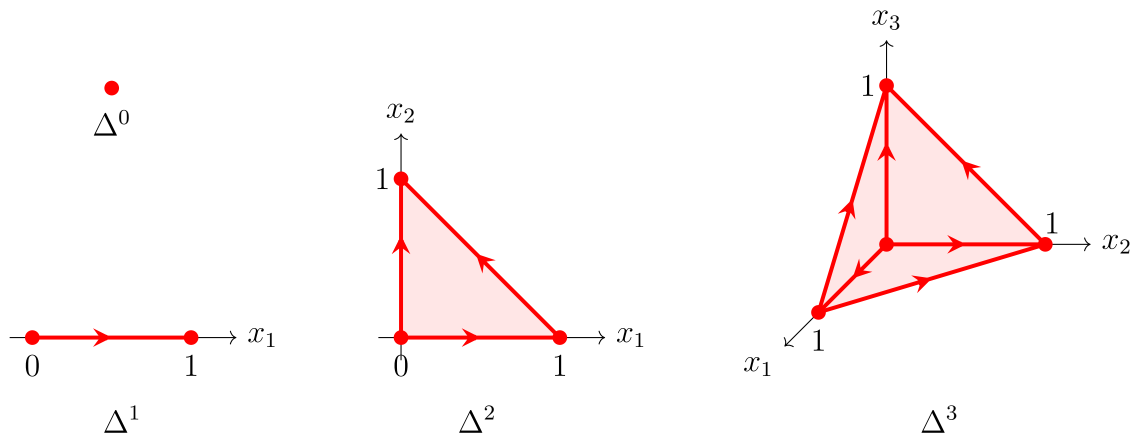
More precisely, we should say that the face maps are the restrictions to Δ^{n-1} and corestrictions to Δ^n of the affine linear map $f_i^n : \mathbb{R}^{n-1} \to \mathbb{R}^n$ given by (8). As this is too cumbersome, we will also use the term affine map for their restrictions to non-degenerate affine simplexes and corestrictions to their images.

Note in particular that an affine map $f : \mathbb{R}^n \to \mathbb{R}^m$ sends the standard n-simplex Δ^n to a possibly degenerate ordered n-simplex $\sigma = [v_0, \ldots, v_n] \subset \mathbb{R}^m$ with $v_k = f(e_k)$. As the affine map f is determined by the images of the vertices in Δ^n, we write $f : \Delta^n \to \sigma$, $e_i \mapsto v_i$ or $f = [v_0, \ldots, v_n]$ and do not distinguish affine maps $f : \mathbb{R}^n \to \mathbb{R}^m$ and affine simplexes in $\mathbb{R}^m$.

In particular, this identifies the *face map* $f_i^n : \Delta^{n-1} \to \Delta^n$ with the affine $(n-1)$-simplex $f_i^n = [e_0, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n] \subset \mathbb{R}^n$, which is the face opposite e_i in Δ^n. This motivates the name face map. In coordinates it is given by

$$f_i^n : \Delta^{n-1} \to \Delta^n, \ (t_1, \ldots, t_{n-1}) \mapsto (t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_n).$$

The ordering of an affine m-simplex is pictured by drawing an arrow on each edge that points from its vertex of lower order to its vertex of higher order. Note that the face maps respect the ordering of vertices in the standard n-simplexes. They omit vertices, but do not change their order. Hence, the ordering of the vertices in the $(n-1)$-face $f_i^n(\Delta^{n-1}) \subset \Delta^n$ induced by the ordering of Δ^{n-1} coincides with the one induced by the ordering of Δ^n.



The standard n-simplexes for $n = 0, 1, 2, 3$.

To describe faces of dimension $< n-1$ in standard n-simplexes we can consider composites of face maps. We then need to keep track of the relations between such composites. One can show that all relations between different composites of face maps are consequences of the relations in the following lemma.

Lemma 3.1.3: The face maps satisfy the relations

$$f_i^n \circ f_j^{n-1} = f_j^n \circ f_{i-1}^{n-1} \qquad 0 \le j < i \le n. \tag{9}$$

Proof. We have for $0 \le j < i \le n$ and $0 \le k < n-1$

$$f_i^n \circ f_j^{n-1}(e_k) = \begin{cases} f_i^n(e_k) & 0 \le k < j \\ f_i^n(e_{k+1}) & j \le k < n-1 \end{cases} = \begin{cases} e_k & 0 \le k < j \\ e_{k+1} & j \le k < i \\ e_{k+2} & i \le k < n-1 \end{cases}$$

$$f_j^n \circ f_{i-1}^{n-1}(e_k) = \begin{cases} f_n^j(e_k) & 0 \le k < i-1 \\ f_j^n(e_{k+1}) & i \le k < n-1 \end{cases} = \begin{cases} e_k & 0 \le k < j \\ e_{k+1} & j \le k < i \\ e_{k+2} & i \le k < n-1. \end{cases}$$

□

In a general topological space X there is no notion of an affine n-simplex. However, we can still consider continuous maps $\sigma : \Delta^n \to X$. Such continuous maps are called *singular* n-simplexes, because we impose no injectivity conditions. All continuous maps are admissible, even the most singular ones that send the entire standard n-simplex to a point.

As we wish to work in a linear setting, we not only consider singular simplexes, but also finite formal linear combinations of singular simplexes with integer coefficients. Such linear combinations are called *singular n-chains*. They are realised as elements of the free abelian group generated by the set of singular simplexes in X.

Definition 3.1.4: Let X be a topological space.

1. A **singular n-simplex** in X is a continuous map $\sigma : \Delta^n \to X$.
2. Elements of the free abelian group $S_n(X) = \langle \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X) \rangle_{\mathbb{Z}}$ generated by the set of singular n-simplexes in X are called **singular n-chains** in X.

As the singular n-simplexes form a basis of the abelian group $S_n(X)$, any group homomorphism $\phi : S_n(X) \to A$ into an abelian group A is determined by the images of the singular n-simplexes. We therefore write $f : S_n(X) \to A$, $\sigma \mapsto f(\sigma)$ without further comments in the following.

We now associate to each topological space a chain complex $S_\bullet(X)$ consisting of the abelian groups $S_n(X)$ of singular n-chains in X. A boundary operator must assign to each singular n-chain a singular $(n-1)$-chain in X. Given the available structures, it seems plausible to send singular n-simplexes $\sigma : \Delta^n \to X$ to their composites with face maps $\sigma \circ f_i^n : \Delta^{n-1} \to X$. To obtain a boundary operator, we then take the alternating sum of these composites for all face maps $f_i^n : \Delta^{n-1} \to \Delta^n$.

In order to be useful, such an assignment of chain complexes to topological spaces must also assign a chain map between them to each continuous map $f : X \to Y$, in such a way that this is compatible with the composition of continuous maps and with the identity maps. The most obvious way to define such a map is to map each singular n-simplex $\sigma : \Delta^n \to X$ to the composite $f \circ \sigma : \Delta^n \to Y$.

Proposition 3.1.5: There is a functor $S_\bullet : \text{Top} \to \text{Ch}_{\text{Ab}}$, the **singular chain complex functor**, that assigns

- to a topological space X the chain complex $S_\bullet(X)$ with abelian groups $S_n(X)$ and
$$d_n : S_n(X) \to S_{n-1}(X),\ \sigma \mapsto \Sigma_{i=0}^n (-1)^i \sigma \circ f_i^n \text{ for } n > 0.$$
- to a continuous map $f : X \to X'$ the chain map $S_\bullet(f) : S_\bullet(X) \to S_\bullet(X')$ given by
$$S_n(f) : S_n(X) \to S_n(X'),\ \sigma \mapsto f \circ \sigma.$$

Proof:
1. We show that $S_\bullet(X)$ is a chain complex: for all singular n-simplexes $\sigma : \Delta^n \to X$ we have

$$\begin{aligned}
d_{n-1} \circ d_n(\sigma) &= \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} \sigma \circ f_i^n \circ f_j^{n-1} \\
&= \sum_{i=0}^{n} \sum_{j=0}^{i-1} (-1)^{i+j} \sigma \circ f_i^n \circ f_j^{n-1} + \sum_{i=0}^{n} \sum_{j=i}^{n-1} (-1)^{i+j} \sigma \circ f_i^n \circ f_j^{n-1} \\
&\overset{(9)}{=} \sum_{i=0}^{n} \sum_{j=0}^{i-1} (-1)^{i+j} \sigma \circ f_j^n \circ f_{i-1}^{n-1} + \sum_{i=0}^{n} \sum_{j=i}^{n-1} (-1)^{i+j} \sigma \circ f_i^n \circ f_j^{n-1} \\
&= \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} (-1)^{i+j+1} \sigma \circ f_j^n \circ f_i^{n-1} + \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} (-1)^{i+j} \sigma \circ f_i^n \circ f_j^{n-1} = 0.
\end{aligned}$$

2. We show that $S_\bullet(f)$ is a chain map: for all singular n-simplexes $\sigma : \Delta^n \to X$ we have

$$d_n \circ S_n(f)(\sigma) = d_n(f \circ \sigma) = \sum_{i=0}^{n} (-1)^i f \circ \sigma \circ f_i^n = S_{n-1}(f) \circ d_n(\sigma).$$

3. We show that $S_\bullet$ respects the composition of morphisms: for all continuous maps $f : X \to X'$ and $g : X' \to X''$ and all singular n-simplexes $\sigma : \Delta^n \to X$ we have

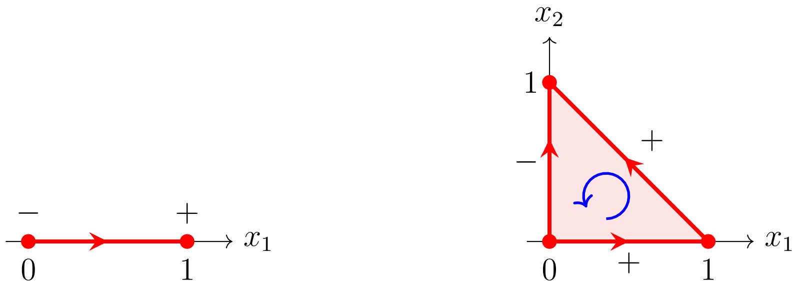
$$S_n(g) \circ S_n(f)(\sigma) = g \circ f \circ \sigma = S_n(g \circ f)(\sigma).$$

4. We show that $S_\bullet$ respects identity morphisms: for all singular n-simplexes $\sigma : \Delta^n \to X$

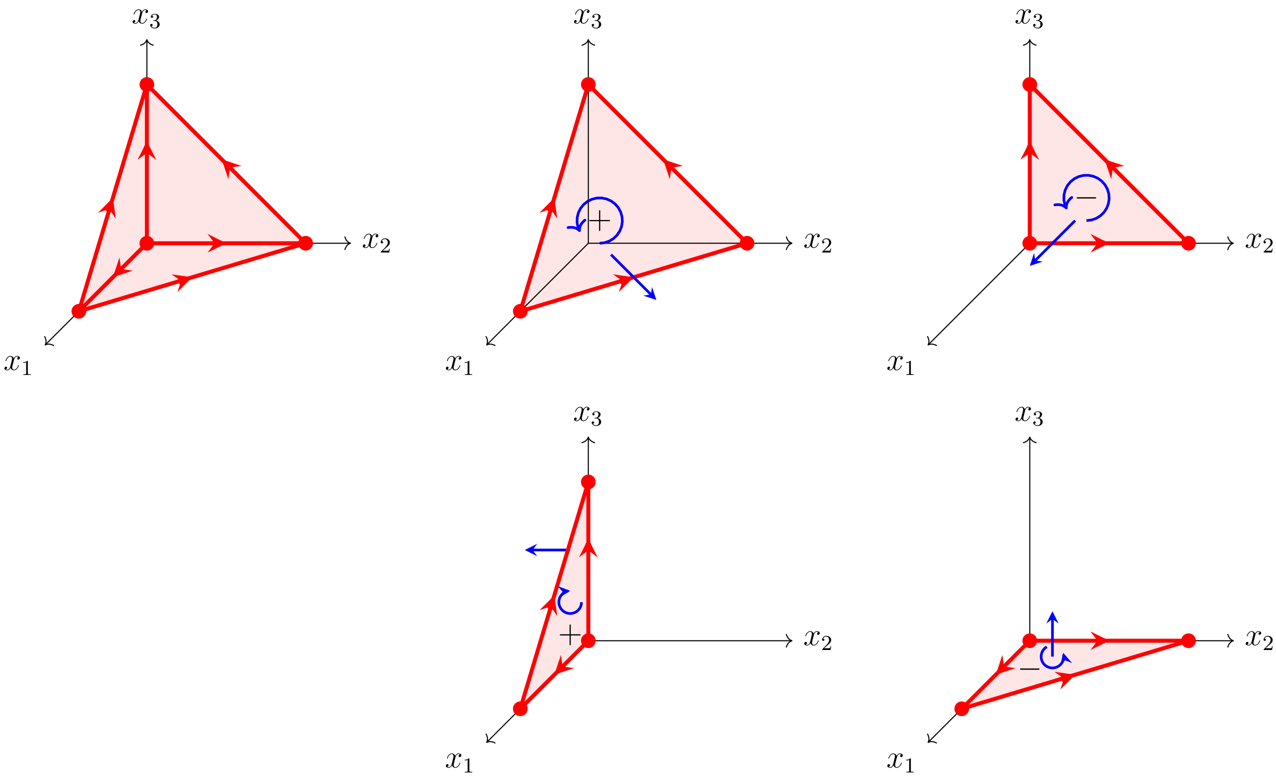
$$S_n(\text{id}_X)(\sigma) = \text{id}_X \circ \sigma = \sigma.$$

□

The signs of the singular boundary operators have a geometrical interpretation and can be visualised for $n = 1, 2, 3$. For a 1-simplex $\sigma : \Delta^1 \to X$ the sign in front of the term $\sigma \circ f_i^1$ is $+1$ if the arrow on the ordered 1-simplex $\Delta^1 = [e_0, e_1]$ points towards e_i and -1 if it points away from e_i. For a 2-simplex $\sigma : \Delta^2 \to X$ the sign of the term $\sigma \circ f_i^2$ is given by the orientation of Δ^2. If we orient $\Delta^2 = [e_0, e_1, e_2]$ according to the ordering of the vertices from the vertex of lowest to the vertex of highest order, as indicated by the blue arrow, then the sign is $+1$ if the arrow on the 1-simplex $f_i^2(\Delta^1)$ is oriented parallel to this and -1 if it is oriented against it.



For a 3-simplex $\sigma : \Delta^3 \to X$ the sign in front of the term $\sigma \circ f_i^3$ is given by the *right hand rule*. If one equips each 2-face of Δ^3 with the orientation defined above and the fingers of the right hand follow this orientation, then the sign is $+1$ if the thumb of the right hand points out of Δ^3 and -1 if it points inside Δ^3.



The *boundary operator* is called boundary operator, because it assigns to a singular n-simplex $\sigma : \Delta^n \to X$ the alternating sum of the singular $(n-1)$-simplexes $\sigma \circ f_i^n : \Delta^{n-1} \to X$ that are the restrictions of σ to the $(n-1)$-faces of Δ^n. Together, these $(n-1)$-faces form the boundary $\partial\Delta^n$ of $\Delta^n \subset \mathbb{R}^n$.

The signs in front of the terms $\sigma \circ f_i^n$ ensure that applying the boundary operator twice gives zero. This has a geometrical interpretation. Each $(n-2)$-face f of Δ^n is contained in the boundary of exactly two $(n-1)$-faces. In one of them f is oriented parallel to the orientation of the $(n-1)$-face, in the other against it. Hence, the two contributions have opposite signs and cancel. This encodes the fact that the boundary of the boundary of Δ^n is empty: one has $\partial\Delta^n = \cup_{i=0}^n f_i^n(\Delta^{n-1})$ and $\partial(\partial\Delta^n) = 0$.

Composing the functor $S_\bullet : \mathrm{Top} \to \mathrm{Ch}_{\mathrm{Ab}}$ from Proposition 3.1.5 with the homology functor $H_n : \mathrm{Ch}_{\mathrm{Ab}} \to \mathrm{Ab}$ from Proposition 2.1.8 yields for all $n \in \mathbb{N}_0$ functors $H_nS_\bullet : \mathrm{Top} \to \mathrm{Ab}$, the *singular homology functors*. They assign to each topological space an abelian group, the singular homology group.

Definition 3.1.6:
The nth **singular homology functor** is the functor $H_nS_\bullet : \mathrm{Top} \to \mathrm{Ab}$ that assigns to

- a topological space X the abelian group $H_n(X) := H_nS_\bullet(X)$,
- a continuous map $f : X \to Y$ the homomorphism $H_n(f) := H_nS_\bullet(f) : H_n(X) \to H_n(Y)$.

The abelian group $H_n(X)$ is called the nth **singular homology group** of X.

The homology groups of a topological space X have several advantages over its singular chain complex $S_\bullet(X)$. The first is that they are much more manageable and smaller than the groups $S_n(X)$ of n-chains. The latter have a huge basis consisting of all singular n-simplexes in X, which is not even countable for most relevant examples of topological spaces. In contrast, the homology groups are finitely generated for many relevant examples. This will become apparent in Example 3.1.10 and Theorem 3.1.11 below. We start with some simple and structural examples.

Example 3.1.7: For the empty topological space $\emptyset$ there are no n-simplexes for any $n \in \mathbb{N}_0$. Thus we have $S_n(\emptyset) = 0$ for all $n \in \mathbb{N}_0$ and $H_n(\emptyset) = 0$ for all $n \in \mathbb{N}_0$.

Example 3.1.8: The one point space $\bullet$ has a single n-simplex $\sigma^n : \Delta^n \to \bullet$ for each $n \in \mathbb{N}_0$ and hence $S_n(\bullet) = \langle \sigma \rangle_{\mathbb{Z}} \cong \mathbb{Z}$ for all $n \in \mathbb{N}_0$. Its boundary operator is given by

$$d_n(\sigma^n) = \sum_{i=0}^{n} (-1)^i \sigma^n \circ f_i^n = \sum_{i=0}^{n} (-1)^i \sigma^{n-1} = \begin{cases} \sigma^{n-1} & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

Its singular chain complex is $S_\bullet(\bullet) = \ldots \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$. Its homologies are

$$H_n(X) = \begin{cases} \mathbb{Z}/0 = \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

A more structural result that can be obtained directly from the definition of the singular homologies are the homologies of sums of topological spaces. The proof of this claim is left as an exercise (Exercise 18).

Proposition 3.1.9: Let $X = \amalg_{i \in I} X_i$ be a topological sum. Then the inclusions $\iota^i : X_i \to X$ induce isomorphisms

$$I_n : \oplus_{i \in I} H_n(X_i) \xrightarrow{\sim} H_n(X) \qquad n \in \mathbb{N}_0.$$

We will now determine the zeroth and first homology groups for more general topological spaces X. Both of them have a geometric interpretation and encode relevant topological information.

Example 3.1.10: Let X be a topological space. Then

$$H_0(X) = \frac{\langle X \rangle_{\mathbb{Z}}}{\langle \sigma(1) - \sigma(0) \mid \sigma : [0,1] \to X \text{ continuous} \rangle_{\mathbb{Z}}} \cong \bigoplus_{\pi_0(X)} \mathbb{Z}$$

where $\pi_0(X)$ is the set of path components of X.

Proof:
As $\Delta^0 = \{0\}$ all maps $\sigma : \Delta^0 \to X$ are continuous, and they are in bijection with points of X. Continuous maps $\sigma : \Delta^1 \to X$ are simply paths in X. This gives

$$\begin{aligned} &S_0(X) = \langle X \rangle_{\mathbb{Z}} && d_0 : S_0(X) \to \{0\},\ x \mapsto 0 \\ &S_1(X) = \langle \sigma : [0,1] \to X \text{ continuous} \rangle_{\mathbb{Z}} && d_1 : S_1(X) \to S_0(X),\ \sigma \mapsto \sigma(1) - \sigma(0). \end{aligned}$$

We obtain $Z_0(X) = S_0(X)$ and $B_0(X) = \langle \sigma(1) - \sigma(0) \mid \sigma : [0,1] \to X \text{ continuous} \rangle_{\mathbb{Z}}$. Hence, two 0-cycles $x, y \in X$ are related by a 0-boundary if and only if there is a continuous map $\sigma : [0,1] \to X$ with $\sigma(0) = x$ and $\sigma(1) = y$, i. e. a path from x to y. Hence $x, y \in X$ are identified if and only if they are in the same path component of X.

By selecting a point x_P in each path component $P \in \pi_0(X)$, we can rewrite any $\mathbb{Z}$-linear combination $v = \Sigma_{i=1}^n z_i x_i$ of points $x_i \in X$ uniquely as $v = \Sigma_{i=1}^n z_i x_{P_i} + \sum_{i=1}^n z_i (x_i - x_{P_i})$, where x_{P_i} represents the path component of x_i. This defines an isomorphism $H_0(X) \cong \oplus_{\pi_0(X)} \mathbb{Z}$. $\square$

As the 0th homology group $H_0(X)$ of a topological space X is given by its set $\pi_0(X)$ of path components, one expects that the first homology group $H_1(X)$ of a path connected topological space X should be related to its fundamental group $\pi_1(X)$. A 1-chain on X is a $\mathbb{Z}$-linear combination of paths $\sigma : [0,1] \to X$. The identity $d_1(\sigma) = \sigma(1) - \sigma(0)$ implies that a singular 1-simplex $\sigma : [0,1] \to X$ is a 1-cycle if and only if the path $\sigma : [0,1] \to X$ is closed: $\sigma(0) = \sigma(1)$. One expects homotopies between paths with the same endpoints to be related to 2-simplexes.

However, there is an essential difference between the fundamental group $\pi_1(X)$ and the first homology group $H_1(X)$ of a path connected topological space X. The group multiplication of $\pi_1(X)$ is induced by the concatenation of paths and in general *not abelian*, whereas the composition of 1-cycles is given by the addition in the *abelian* group $Z_n(X)$. For a collection of paths based at a point $x \in X$ their product in the fundamental group $\pi_1(X)$ keeps track of the *order* in which the paths are composed. The sum of their homology classes in $H_1(X)$ only takes into account *how often* each path in the collection is traversed and in which direction.

We now show that for a path connected topological space X the first homology group $H_1(X)$ is the *abelisation* of the fundamental group $\pi_1(X)$. For this, recall that the **commutator subgroup** $[G, G]$ of a group G is the normal subgroup of G generated by the **group commutators** $[g, h] = ghg^{-1}h^{-1}$ of all elements $g, h \in G$. The factor group $G/[G, G]$ is abelian by construction and called the **abelisation** $\mathrm{Ab}(G)$ of G . One can show that abelisation is characterised by a universal property and defines a functor $\mathrm{Ab} : \mathrm{Grp} \to \mathrm{Ab}$ from the category Grp of groups to the category Ab of abelian groups (Exercise 3).

Theorem 3.1.11: Let X be a path connected topological space and $x \in X$.

1. The map $\phi : \pi_1(x, X) \to H_1(X)$, $[\gamma]_{\pi_1} \mapsto [\gamma]_{H_1}$ is a group homomorphism.
2. It induces an isomorphism $\phi : \mathrm{Ab}\, \pi_1(x, X) \to H_1(X)$, the **Huréwicz isomorphism**.

Proof:
1. We show that $\phi : \pi_1(x, X) \to H_1(X)$, $[\gamma]_{\pi_1} \mapsto [\gamma]_{H_1}$ is well-defined.

Note first that any path $\gamma : [0, 1] \to X$ with $\gamma(0) = \gamma(1)$ is a singular 1-cycle, as $\Delta^1 = [0, 1]$ and $d_1(\gamma) = \gamma \circ f_0^1 - \gamma \circ f_1^1 = \gamma(1) - \gamma(0) = 0$.

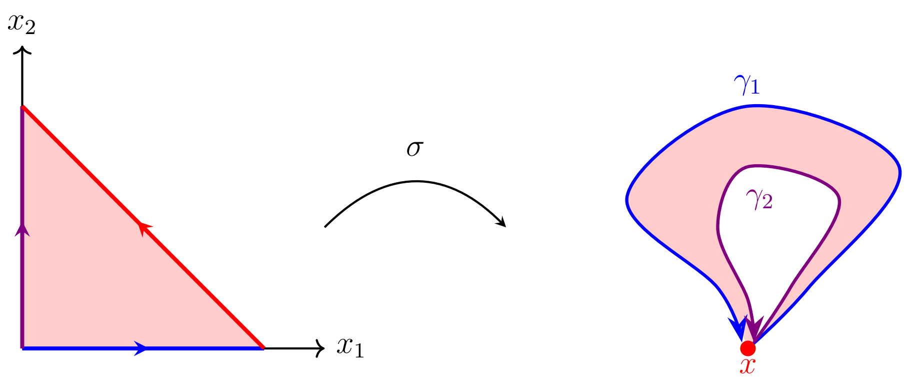
It remains to show that homotopic paths are related by a 1-boundary. Let $\gamma_1, \gamma_2 : [0, 1] \to X$ be paths with $\gamma_i(0) = \gamma_i(1) = x$ and $h : [0, 1] \times [0, 1] \to X$ a homotopy of paths from γ_1 to γ_2. Then we have $h(0, t) = \gamma_1(t)$, $h(1, t) = \gamma_2(t)$ and $h(s, 0) = h(s, 1) = x$ for all $t, s \in [0, 1]$. From the homotopy h we construct a map $\sigma : \Delta^2 \to X$ defined by

$$\sigma(s,t) = h(\tfrac{t}{s+t}, s+t) \text{ for } (s,t) \neq (0,0), \qquad \sigma(0,0) = x.$$

This map is continuous, because $h : [0, 1] \times [0, 1] \to X$ is continuous with $h(s, 0) = x$ for all $s \in [0, 1]$. By applying the boundary operator, we obtain $d_2(\sigma) = \sigma \circ f_0^2 - \sigma \circ f_1^2 + \sigma \circ f_2^2$ with

$$\sigma \circ f_0^2(t) = \sigma(1-t,t) = x, \qquad \sigma \circ f_1^2(t) = \sigma(0,t) = \gamma_2(t), \qquad \sigma \circ f_2^2(t) = \sigma(t,0) = \gamma_1(t).$$

Hence, σ sends the face $[e_1, e_2]$ of Δ^2 to x, the face $[e_0, e_2]$ to $\mathrm{im}\,(\gamma_2)$ and the face $[e_0, e_1]$ to $\mathrm{im}\,(\gamma_1)$. We have $d_2(\sigma) = \gamma_x - \gamma_2 + \gamma_1$ with the constant 1-cycle $\gamma_x : [0, 1] \to X$, $t \mapsto x$.



As γ_x is a boundary $\gamma_x = d_2(\rho_x)$ of the constant 2-simplex $\rho_x : \Delta^2 \to X$, $(s, t) \mapsto x$, we have $0 = [\gamma_x]_{H_1} = [d_2(\sigma)]_{H_1} + [\gamma_2]_{H_1} - [\gamma_1]_{H_1} = [\gamma_2]_{H_1} - [\gamma_1]_{H_1}$. This shows that $[\gamma]_{H_1}$ depends only on the homotopy class of γ and ϕ is well-defined.

2. We show that $\phi : \pi_1(x, X) \to H_1(X)$, $[\gamma]_{\pi_1} \mapsto [\gamma]_{H_1}$ is a group homomorphism.

Let $\gamma_1, \gamma_2 : [0, 1] \to X$ be paths with $\gamma_i(0) = \gamma_i(1) = x$. By composing their concatenation

$$\gamma_2 \star \gamma_1 : [0,1] \to X, \quad t \mapsto \begin{cases} \gamma_1(2t) & t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & t \in [\frac{1}{2}, 1 \end{cases}$$

with the affine map $g : \Delta^2 \to [0, 1]$, $(s, t) \mapsto \frac{1}{2}s + t$, we obtain a 2-simplex

$$\sigma = (\gamma_2 \star \gamma_1) \circ g : \Delta^2 \to X, \quad (s,t) \mapsto (\gamma_2 \star \gamma_1)(\tfrac{s}{2} + t)$$

that satisfies

$$\begin{aligned} & \sigma \circ f_0^2(t) = \sigma(1-t,t) = \gamma_2 \star \gamma_1(\tfrac{1}{2} + \tfrac{t}{2}) = \gamma_2(t) \\ & \sigma \circ f_1^2(t) = \sigma(0,t) = \gamma_2 \star \gamma_1(t) \\ & \sigma \circ f_2^2(t) = \sigma(t,0) = \gamma_2 \star \gamma_1(g(t,0)) = \gamma_2 \star \gamma_1(\tfrac{t}{2}) = \gamma_1(t). \\ \Rightarrow \quad & d_2(\sigma) = \gamma_1 + \gamma_2 - \gamma_2 \star \gamma_1. \end{aligned}$$

This implies

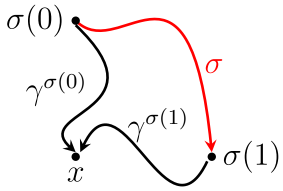
$$\phi([\gamma_1]_{\pi_1}) + \phi([\gamma_2]_{\pi_1}) = [\gamma_1]_{H_1} + [\gamma_2]_{H_1} = [\gamma_2 \star \gamma_1]_{H_1} = \phi([\gamma_2 \star \gamma_1]_{\pi_1}) = \phi([\gamma_2]_{\pi_1} \cdot [\gamma_1]_{\pi_1}).$$

As $H_1(X)$ is abelian, this induces a group homomorphism $\phi : \mathrm{Ab}\,\pi_1(x, X) \to H_1(X)$.

3. We show that $\phi : \mathrm{Ab}\,\pi_1(x, X) \to H_1(X)$ is a group isomorphism by constructing its inverse. We choose for every point $y \in X$ a path $\gamma^y : [0,1] \to X$ with $\gamma^y(0) = y$, $\gamma^y(1) = x$ and define

$$K : S_1(X) \to \mathrm{Ab}\,\pi_1(x, X), \qquad \sigma \mapsto [\gamma^{\sigma(1)} \star \sigma \star \overline{\gamma}^{\sigma(0)}]_{\mathrm{Ab}(\pi_1)}$$

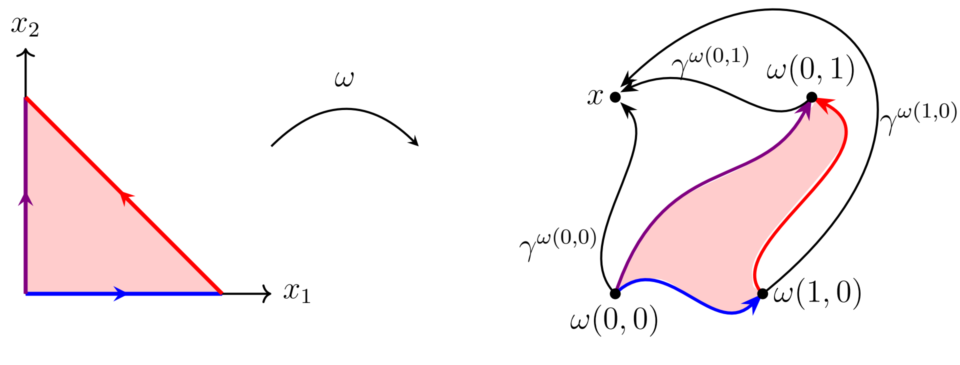
for singular 1-simplexes $\sigma : [0,1] \to X$. As $S_1(X)$ is the free abelian group generated by the singular 1-simplexes, this defines a group homomorphism.



To show that $K : S_1(X) \to \mathrm{Ab}\pi_1(x, X)$ induces a group homomorphism $K : H_1(X) \to \mathrm{Ab}\pi_1(x, X)$, we show that $K(d_2(\omega)) = 0$ for every singular 2-simplex $\omega : \Delta^2 \to X$:

$$\begin{aligned}
K(d_2\omega) &= K(\omega \circ f_0^2 - \omega \circ f_1^2 + \omega \circ f_2^2) = K(\omega \circ f_0^2) - K(\omega \circ f_1^2) + K(\omega \circ f_2^2) \\
&= [\gamma^{\omega(0,1)} \star (\omega \circ f_0^2) \star \overline{\gamma}^{\omega(1,0)}]_{\mathrm{Ab}(\pi_1)} - [\gamma^{\omega(0,1)} \star (\omega \circ f_1^2) \star \overline{\gamma}^{\omega(0,0)}]_{\mathrm{Ab}(\pi_1)} \\
&+ [\gamma^{\omega(1,0)} \star (\omega \circ f_2^2) \star \overline{\gamma}^{\omega(0,0)}]_{\mathrm{Ab}(\pi_1)} \\
&= [\gamma^{\omega(0,0)} \star \overline{(\omega \circ f_1^2)} \star \overline{\gamma}^{\omega(0,1)} \star \gamma^{\omega(0,1)} \star (\omega \circ f_0^2) \star \overline{\gamma}^{\omega(1,0)} \star \gamma^{\omega(1,0)} \star (\omega \circ f_2^2) \star \overline{\gamma}^{\omega(0,0)}]_{\mathrm{Ab}(\pi_1)} \\
&= [\gamma^{\omega(0,0)} \star \overline{(\omega \circ f_1^2)} \star (\omega \circ f_0^2) \star (\omega \circ f_2^2) \star \overline{\gamma}^{\omega(0,0)}]_{\mathrm{Ab}(\pi_1)} = [\gamma]_{\mathrm{Ab}(\pi_1)},
\end{aligned}$$

where $\gamma : [0,1] \to X$ is a loop with base point x that circles the boundary $\partial\omega(\Delta^2) \subset X$ counterclockwise and we suppress the bracketing in the concatenation of paths. As γ is null homotopic, we have $K(d_2\omega) = [\gamma]_{\mathrm{Ab}(\pi_1)} = 0$. This implies $B_1(X) \subset \ker K$, and K induces a group homomorphism $K : H_1(X) \to \mathrm{Ab}\pi_1(x, X)$.



We show that $K : H_1(X) \to \mathrm{Ab}\pi_1(x, X)$ is the inverse of $\phi : \mathrm{Ab}\pi_1(x, X) \to H_1(X)$. For any path $\delta : [0,1] \to X$ with $\delta(0) = \delta(1) = x$ we have

$$K \circ \phi\, [\delta]_{\mathrm{Ab}(\pi_1)} = [\gamma^x \star \delta \star \overline{\gamma}^x]_{\mathrm{Ab}(\pi_1)} = [\gamma^x]_{\mathrm{Ab}(\pi_1)} - [\gamma^x]_{\mathrm{Ab}(\pi_1)} + [\delta]_{\mathrm{Ab}(\pi_1)} = [\delta]_{\mathrm{Ab}(\pi_1)}.$$

A linear combination $x = \Sigma_{i=0}^n z_i \sigma_i \in S_1(X)$ of paths $\sigma_i : [0,1] \to X$ is a 1-cycle if and only if $\Sigma_{i=0}^n z_i(\sigma_i(1) - \sigma_i(0)) = 0$. We then obtain

$$\begin{aligned}
\phi \circ K\, [x]_{H_1} &= \Sigma_{i=0}^n z_i [\gamma^{\sigma_i(1)} \star \sigma_i \star \overline{\gamma}^{\sigma_i(0)}]_{H_1} = \Sigma_{i=0}^n z_i \left([\gamma^{\sigma_i(1)}]_{H_1} - [\gamma^{\sigma_i(0)}]_{H_1} + [\sigma_i]_{H_1} \right) \\
&= \Sigma_{i=0}^n z_i [\sigma_i]_{H_1} = [x]_{H_1}.
\end{aligned}$$

Hence $K = \mathrm{Ab}(\phi)^{-1}$ and $\phi : \mathrm{Ab}\pi_1(x, X) \to H_1(X)$ is a group isomorphism. $\square$

If the fundamental group of a path connected topological space is known, we can use the Huréwicz isomorphism to compute its first homology group. In particular, if the fundamental group is abelian, the abelisation has no effect and the first homology group is $H_1(X) = \pi_1(X)$

Example 3.1.12:

1. Every simply connected and path connected topological space has a trivial fundamental group $\pi_1(X) = \{1\}$ and hence a trivial first homology group $H_1(X) = 0$.
2. The circle S^1 has fundamental group and first homology group $\pi_1(S^1) = H_1(S^1) = \mathbb{Z}$.
3. The torus T has fundamental group and first homology group $\pi_1(T) = H_1(T) = \mathbb{Z} \times \mathbb{Z}$.
4. Projective space $\mathbb{R}\mathrm{P}^2$ has $\pi_1(\mathbb{R}\mathrm{P}^2) = H_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}/2\mathbb{Z}$.
5. The Klein bottle K has fundamental group $\pi_1(K) = \mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$ and $H_1(K) = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
6. The fundamental group of an oriented surface Σ of genus $g \geq 0$ has the presentation

$$\pi_1(\Sigma) = \langle a_1, b_1, \ldots, a_g, b_g \mid [b_g, a_g] \cdots [b_1, a_1] = 1 \rangle.$$

   Its first homology group is $H_1(\Sigma) = \mathbb{Z}^{\times 2g}$.

Remark 3.1.13: There are analogues of this statement for higher homology and homotopy groups, the **Huréwicz theorem**:

1. For any path connected topological space X and point $x \in X$ there are group homomorphisms $\phi_n : \pi_n(X) \to H_n(X)$ for all $n \geq 2$.
2. If X is $(n-1)$-**connected**, that is non-empty and path-connected with $\pi_k(x, X) = \{1\}$ for $1 \leq k \leq n-1$, then ϕ_n is a group isomorphism.

Note that abelisation is not required for $n \geq 2$, as the homotopy groups $\pi_n(X)$ are then abelian.

3.2 Homotopy invariance

In this section we will derive the second advantage that the singular homologies of a topological space have over its singular chain complex. In contrast to the singular chain complex they are homotopy invariants. Concretely, this means that homotopic maps induce the same maps between the singular homologies. As a consequence, homotopy equivalent topological spaces have isomorphic singular homologies. This makes them more easy to compute, because one can replace a given topological space by a homotopy equivalent one for which the homologies are simpler to compute. For instance, the homologies of any contractible topological space can be computed from the 1 point-space.

As the singular homology functor is the composite $H_n S_\bullet : \mathrm{Top} \to \mathrm{Ab}$, it is intuitive how to derive this result. We show that a homotopy $h : [0,1] \times X \to Y$ from $f : X \to Y$ to $g : X \to Y$ induces a chain homotopy $h_\bullet : f_\bullet \Rightarrow g_\bullet$ with components $h_n : S_n(X) \to S_{n+1}(Y)$. As the singular n-simplexes $\sigma : \Delta^n \to X$ form a basis of $S_n(X)$, we need to define them by their action on singular n-simplexes. Given a singular n-simplex $\sigma : \Delta^n \to X$ and a homotopy $h : [0,1] \times X \to Y$, the simplest way to construct a singular $(n+1)$-simplex in Y is by constructing an affine linear map $T:\Delta^{n-1}\to[0,1]\times\Delta^n$ and considering the following composite

$$\Delta^{n+1} \xrightarrow{T} [0,1]\times\Delta^n \xrightarrow{\mathrm{id}\times\sigma} [0,1]\times X \xrightarrow{h} Y. \tag{10}$$

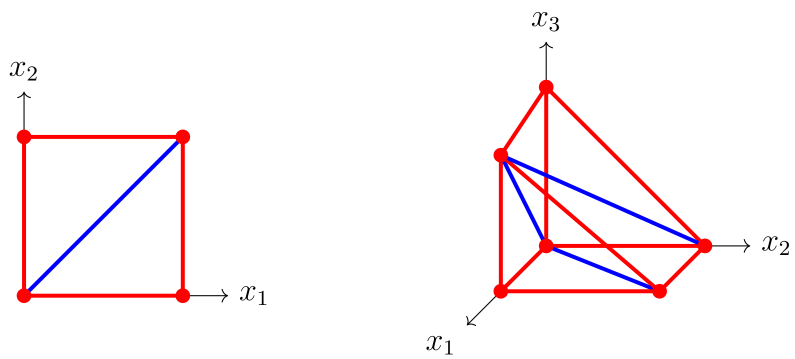
The affine linear map $T:\Delta^{n+1}\to[0,1]\times\Delta^n$ should take the $(n+2)$ vertices of Δ^{n+1} to some of the $2(n+1)$ vertices in the prism $[0,1]\times\Delta^n$, some some on the top face $\{1\}\times\Delta^n$ and some on the bottom face $\{0\}\times\Delta^n$. This is achieved by the prim maps.

Definition 3.2.1: The **prism maps** are the affine linear maps

$$T_n^j:\Delta^{n+1}\to[0,1]\times\Delta^n,\quad T_n^j(e_k)=\begin{cases}(0,e_k) & 0\leq k\leq j\leq n\\ (1,e_{k-1}) & 0\leq j<k\leq n+1.\end{cases}$$

The prism maps decompose the prism $[0,1]\times\Delta^n$ into $(n+1)$ different $(n+1)$-simplexes $T_n^j(\Delta^{n+1})$ for $j=0,\ldots,n$. In coordinates, they read

$$T_n^j:\Delta^{n+1}\to[0,1]\times\Delta^n,\ (t_1,...,t_{n+1})\mapsto(t_1,...,t_{j-1},t_j+t_{j+1},t_{j+2},...,t_{n+1},t_{j+1}+...+t_{n+1}).$$



The prism maps T_n^j for $n=1,2$.

To prove later that our strategy yields indeed a chain homotopy, we need to investigate its interaction with the boundary operator. This requires taking composites of the prism maps with face maps and establishing the relations between such composites.

Lemma 3.2.2: The prism maps satisfy the relations

$$\begin{aligned}
&T_n^j\circ f_i^{n+1}=(\mathrm{id}_{[0,1]}\times f_i^n)\circ T_{n-1}^{j-1} && \forall j>i\\
&T_n^j\circ f_i^{n+1}=(\mathrm{id}_{[0,1]}\times f_{i-1}^n)\circ T_{n-1}^{j} && \forall j<i-1\\
&T_n^i\circ f_i^{n+1}=T_n^{i-1}\circ f_i^{n+1} && \forall i\in\{1,...,n\}\\
&T_n^0\circ f_0^{n+1}=i^1,\quad T_n^n\circ f_{n+1}^{n+1}=i^0,
\end{aligned}\tag{11}$$

where $i^t:\Delta^n\to[0,1]\times\Delta^n$, $x\mapsto(t,x)$ is the inclusion map and $f_j^{n+1}:\Delta^n\to\Delta^{n+1}$ the face map from Definition 3.1.2.

Proof:
This follows by direct computations with the definition of the prism operators and the face maps.
• case 1: For $j > i$ we have

$$T_n^j \circ f_i^{n+1}(e_k) = \begin{cases} T_n^j(e_k) & 0 \le k < i \\ T_n^j(e_{k+1}) & i \le k \le n \end{cases} = \begin{cases} (0, e_k) & 0 \le k < i \\ (0, e_{k+1}) & i \le k < j \\ (1, e_k) & j \le k \le n \end{cases}$$

$$(\mathrm{id}_{[0,1]} \times f_i^n) \circ T_{n-1}^{j-1}(e_k) = \begin{cases} (0, f_i^n(e_k)) & 0 \le k < j \\ (1, f_i^n(e_{k-1})) & j \le k \le n \end{cases} = \begin{cases} (0, e_k) & 0 \le k < i \\ (0, e_{k+1}) & i \le k < j \\ (1, e_k) & j \le k \le n. \end{cases}$$

• case 2: for $j < i - 1$ we have

$$T_n^j \circ f_i^{n+1}(e_k) = \begin{cases} T_n^j(e_k) & 0 \le k < i \\ T_n^j(e_{k+1}) & i \le k \le n \end{cases} = \begin{cases} (0, e_k) & 0 \le k \le j \\ (1, e_{k-1}) & j < k < i \\ (1, e_k) & i \le k \le n \end{cases}$$

$$(\mathrm{id}_{[0,1]} \times f_{i-1}^n) \circ T_{n-1}^{j}(e_k) = \begin{cases} (0, f_{i-1}^n(e_k)) & 0 \le k \le j \\ (1, f_{i-1}^n(e_{k-1})) & j < k \le n \end{cases} = \begin{cases} (0, e_k) & 0 \le k \le j \\ (1, e_{k-1}) & j < k < i \\ (1, e_k) & i \le k \le n. \end{cases}$$

• case 3: for $i \in \{1, \ldots, n\}$ we have

$$T_n^i \circ f_i^{n+1}(e_k) = \begin{cases} T_n^i(e_k) & 0 \le k < i \\ T_n^i(e_{k+1}) & i \le k \le n \end{cases} = \begin{cases} (0, e_k) & 0 \le k < i \\ (0, e_{k-1}) & k = i \\ (1, e_k) & i < k \le n \end{cases}$$

$$T_n^{i-1} \circ f_i^{n+1}(e_k) = \begin{cases} T_n^{i-1}(e_k) & 0 \le k < i \\ T_n^{i-1}(e_{k+1}) & i \le k \le n \end{cases} = \begin{cases} (0, e_k) & 0 \le k < i \\ (0, e_{k-1}) & k = i \\ (1, e_k) & i < k \le n. \end{cases}$$

• case 4: For all $k \in \{0, \ldots, n\}$ we have

$$T_n^0 \circ f_0^{n+1}(e_k) = T_n^0(e_{k+1}) = (1, e_k) \qquad T_n^n \circ f_{n+1}^{n+1}(e_k) = T_n^n(e_k) = (0, e_k).$$

□

With these preliminaries we can now construct chain homotopies from a given homotopy $h : [0,1] \times X \to Y$. The act on a singular n-simplex $\sigma : \Delta^n \to X$ as outlined in (10). In analogy to the definition of the boundary operator, we take an alternating sum over the different prism maps.

Proposition 3.2.3:

1. A homotopy $h : [0,1] \times X \to Y$ from $f : X \to Y$ to $g : X \to Y$ induces a chain homotopy $S_\bullet(h) : S_\bullet(f) \Rightarrow S_\bullet(g)$.
2. If $f \sim g : X \to Y$ are homotopic, then $H_n(f) = H_n(g) : H_n(X) \to H_n(Y)$.

3. Any homotopy equivalence $f: X \to Y$ induces an isomorphism $H_n(f): H_n(X) \xrightarrow{\sim} H_n(Y)$.

Proof:
1. Given a homotopy $h: [0,1]\times X \to Y$ from $f: X \to Y$ to $g: X \to Y$, we define the associated chain homotopy by specifying its components

$$S_n(h): S_n(X) \to S_{n+1}(Y), \quad \sigma \mapsto \sum_{j=0}^{n} (-1)^j\, h \circ (\mathrm{id}_{[0,1]} \times \sigma) \circ T_n^j \tag{12}$$

The relations between the prism operators ensure that this is indeed a chain homotopy:

$$\begin{aligned}
&(d_{n+1} \circ S_n(h) + S_{n-1}(h) \circ d_n)(\sigma)\\
&= \sum_{i=0}^{n+1}\sum_{j=0}^{n} (-1)^{i+j} h \circ (\mathrm{id}_{[0,1]} \times \sigma) \circ T_n^j \circ f_i^{n+1} + \sum_{i=0}^{n}\sum_{j=0}^{n-1} (-1)^{i+j} h \circ (\mathrm{id}_{[0,1]} \times \sigma) \circ (\mathrm{id}_{[0,1]} \times f_i^n) \circ T_{n-1}^j\\
&= \sum_{i=1}^{n+1}\sum_{j=0}^{i-2} (-1)^{i+j} h \circ (\mathrm{id}_{[0,1]} \times \sigma) \circ (T_n^j \circ f_i^{n+1} - (\mathrm{id}_{[0,1]} \times f_{i-1}^n) \circ T_{n-1}^j)\\
&- \sum_{i=1}^{n+1} h \circ (\mathrm{id}_{[0,1]} \times \sigma) \circ T_n^{i-1} \circ f_i^{n+1} + \sum_{i=0}^{n} h \circ (\mathrm{id}_{[0,1]} \times \sigma) \circ T_n^i \circ f_i^{n+1}\\
&+ \sum_{i=0}^{n+1}\sum_{j=i+1}^{n} (-1)^{i+j} h \circ (\mathrm{id}_{[0,1]} \times \sigma) \circ (T_n^j \circ f_i^{n+1} - (\mathrm{id}_{[0,1]} \times f_i^n) \circ T_{n-1}^{j-1})\\
&\overset{(11)}{=} h \circ (\mathrm{id}_{[0,1]} \times \sigma) \circ T_n^0 \circ f_0^{n+1} - h \circ (\mathrm{id}_{[0,1]} \times \sigma) \circ T_n^n \circ f_{n+1}^{n+1}\\
&\overset{(11)}{=} h \circ (\mathrm{id}_{[0,1]} \times \sigma) \circ i^1 - h \circ (\mathrm{id}_{[0,1]} \times \sigma) \circ i^0 = h(1,-) \circ \sigma - h(0,-) \circ \sigma = g \circ \sigma - f \circ \sigma\\
&= (S_n(g) - S_n(f))(\sigma),
\end{aligned}$$

where we split the first sum into the cases $j < i-1$, $j = i-1$, $j = i$ and $j > i$, the second sum into the cases $j < i$ and $j \geq i$ and shifted the indices $i \mapsto i+1$ and $j \mapsto j+1$ in the second sum to combine the terms. This shows that $S_\bullet(h)$ is indeed a chain homotopy.

2. If f and g are homotopic, then by 1. there is a chain homotopy $S_\bullet(h): S_\bullet(f) \Rightarrow S_\bullet(g)$. By Proposition 2.1.12, this implies $H_n(f) = H_n S_\bullet(f) = H_n S_\bullet(g) = H_n(g)$ for all $n \in \mathbb{N}_0$.

3. If $f: X \to Y$ is a homotopy equivalence, there is a map $g: Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. This implies that $S_\bullet(f): S_\bullet(X) \to S_\bullet(Y)$ is a chain homotopy equivalence, as $S_\bullet(g) \circ S_\bullet(f) = S_\bullet(g \circ f) \sim S_\bullet(\mathrm{id}_X) = \mathrm{id}_{S_\bullet(X)}$ and $S_\bullet(f) \circ S_\bullet(g) = S_\bullet(f \circ g) \sim S_\bullet(\mathrm{id}_Y) = \mathrm{id}_{S_\bullet(Y)}$. By Proposition 2.1.12, 3. $H_n(f) = H_n S_\bullet(f): H_n S_\bullet(X) \to H_n S_\bullet(Y)$ is an isomorphism. □

Corollary 3.2.4: If X is a contractible topological space we have

$$H_n(X) \cong H_n(\bullet) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \in \mathbb{N}. \end{cases}$$

3.3 Subspaces and relative homology

In this subsection and the following subsections, we develop tools to compute the homologies of certain nice topological spaces systematically. In this section, we start by relating the homologies of a topological space X to the homologies of its subspaces $A \subset X$.

The inclusion $\iota_A : A \to X$, $a \mapsto a$ for a subspace $A \subset X$ is continuous by definition of the subspace topology. It induces a chain map $S_\bullet(\iota_A) : S_\bullet(A) \to S_\bullet(X)$ and a group homomorphism $H_n(\iota_A) : H_n(A) \to H_n(X)$. However, this group homomorphism is in general not injective. Consider for instance the inclusion $\iota_{S_1} : S^1 \to D^2$ that realises the circle as the boundary of a 2-disc. Then we have $H_1(S^1) = \mathbb{Z}$ by Theorem 3.1.11 and $H_1(D^2) = 0$ by Corollary 3.2.4, as D^2 is convex and hence contractible. The group homomorphism $H_1(\iota_{S^1}) : \mathbb{Z} \to 0$ is not injective.

Thus, the relation between the homologies of a topological space X and a subspace $A \subset X$ takes a more complicated form. To investigate it, we consider pairs (X, A) of a topological space X and a subspace $A \subset X$ and organise them into a category Top(2). Its morphisms are continuous maps that map the chosen subspaces into each other. Note also that we can interpret a topological space X without a choice of subspace as a pair $(X, \emptyset)$. For a continuous map between such pairs the condition that subspaces are mapped into each other is void.

Definition 3.3.1: The category Top(2) has

- as objects pairs (X, A) of a topological space X and a subspace $A \subset X$,
- as morphisms $f : (X, A) \to (Y, B)$ continuous maps $f : X \to Y$ with $f(A) \subset B$.

We denote by $I : \mathrm{Top} \to \mathrm{Top}(2)$ the inclusion functor that sends

- a topological space X to the pair $(X, \emptyset)$,
- a continuous map $f : X \to Y$ to the morphism $f : (X, \emptyset) \to (Y, \emptyset)$.

We will also need a proper notion of homotopy for morphisms $f : (X, A) \to (Y, B)$ in Top(2) that should reduce to the ordinary notion of homotopy for morphisms $f : (X, \emptyset) \to (Y, \emptyset)$. A sensible condition is that a homotopy $h : [0, 1] \times X \to Y$ between $f, f' : (X, A) \to (Y, B)$ should satisfy $h([0, 1] \times A) \subset B$. Note that this is a weaker condition than a homotopy relative to A, as it is not guaranteed that the restrictions of f and f' to A coincide.

Definition 3.3.2:

1. Two morphisms $f, g : (X, A) \to (Y, B)$ in Top(2) are called **homotopic**, $f \sim_{(X,A)} g$ if there is a homotopy $h : [0, 1] \times X \to Y$ from f to g with $h([0, 1] \times A) \subset B$.
2. A morphism $f : (X, A) \to (Y, B)$ in Top(2) is called a **homotopy equivalence**, if there is a morphism $g : (Y, B) \to (X, A)$ such that $g \circ f \sim_{(X,A)} 1_{(X,A)}$ and $f \circ g \sim_{(Y,B)} 1_{(Y,B)}$.

Remark 3.3.3:

1. Homotopic is an equivalence relation on the morphism sets in Top(2) that is compatible with the composition of morphisms:
   if $f \sim_{(X,A)} f' : (X, A) \to (Y, B)$ and $g \sim_{(Y,B)} g' : (Y, B) \to (Z, C)$ then $g \circ f \sim_{(X,A)} g' \circ f'$.
2. We obtain a homotopy category $\mathcal{K}\mathrm{Top}(2)$ with the same objects as Top(2), whose morphisms are homotopy classes of morphisms in Top(2). Isomorphisms in $\mathcal{K}\mathrm{Top}(2)$ are homotopy equivalences in Top(2).

We will now assign to each object (X, A) of Top(2) a chain complex and to each morphism $f : (X, A) \to (Y, B)$ a chain map. This should define a functor from Top(2) into $\mathrm{Ch}_{\mathrm{Ab}}$ that reduces to the functor from Proposition 3.1.5 for pairs of the form $(X, \emptyset)$ and morphisms

$f : (X, \emptyset) \to (Y, \emptyset)$ between them. In other words, we require that its composite with the inclusion functor I from Definition 3.3.1 coincides with $S_\bullet$.

The key observation is that for any subspace $A \subset X$ the inclusion $\iota_A : A \to X$ is continuous and hence defines a chain map $S_\bullet(\iota_A) : S_\bullet(A) \to S_\bullet(X)$ with injective components. This identifies the singular chain complex of the subspace with a subcomplex $S_\bullet(A) \subset S_\bullet(X)$.

We can therefore consider the associated quotient complex $S_\bullet(X)/S_\bullet(A)$ from Proposition 2.2.4. This amounts to identifying two n-chains in $S_n(X)$, whenever they differ by an n-chain in $S_n(A)$. As $S_n(\emptyset) = 0$ for all $n \in \mathbb{N}_0$ by Example 3.1.7, this yields the usual singular chain complex $S_\bullet(X) = S_\bullet(X)/S_\bullet(\emptyset)$ for pairs $(X, \emptyset)$.

Proposition 3.3.4:

1. There is a functor $S_\bullet^{(2)} : \mathrm{Top}(2) \to \mathrm{Ch}_{\mathrm{Ab}}$, the **relative chain complex functor**, that
   - assigns to (X, A) the chain complex $S_\bullet^{(2)}(X, A)$ with $S_n^{(2)}(X, A) = S_n(X)/S_n(A)$ and
   $$d_n^{(2)} : S_n(X)/S_n(A) \to S_{n-1}(X)/S_{n-1}(A), \quad [\sigma] \mapsto [d_n(\sigma)].$$
   - to a morphism $f : (X, A) \to (Y, B)$ the chain map $S_\bullet^{(2)}(f) : S_\bullet^{(2)}(X, A) \to S_\bullet^{(2)}(Y, B)$
   $$S_n^{(2)}(f) : S_n(X)/S_n(A) \to S_n(Y)/S_n(B), \quad [\sigma] \mapsto [f_n(\sigma)].$$
2. A homotopy $h : [0, 1] \times X \to Y$ from $f : (X, A) \to (Y, B)$ to $g : (X, A) \to (Y, B)$ induces a chain homotopy $S_\bullet^{(2)}(h) : S_\bullet^{(2)}(f) \Rightarrow S_\bullet^{(2)}(g)$.

Proof:
1. The inclusion $\iota_A : A \to X$ defines a chain map $S_\bullet(\iota_A) : S_\bullet(A) \to S_\bullet(X)$ with components $i_n := S_n(\iota_A) : S_n(A) \to S_n(X)$, $\sigma \mapsto \sigma$. Thus $S_\bullet(A) \subset S_\bullet(X)$ is a subcomplex and $S_\bullet(X)/S_\bullet(A)$ the associated quotient complex from Proposition 2.2.4.

Any morphism $f : (X, A) \to (Y, B)$ defines a chain map $S_\bullet(f) : S_\bullet(X) \to S_\bullet(Y)$ that satisfies $S_\bullet(f)|_{S_\bullet(A)} = S_\bullet(f|_A) : S_\bullet(A) \to S_\bullet(B)$ by construction. By Proposition 2.2.4 this defines a chain map $S_\bullet^{(2)}(f) : S_\bullet(X)/S_\bullet(A) \to S_\bullet(Y)/S_\bullet(B)$. Compatibility of the functor with identity morphisms and with the composition of morphisms follows directly from the commuting diagram (5) in Proposition 2.2.4 and the functoriality of $S_\bullet : \mathrm{Top} \to \mathrm{Ch}_{\mathrm{Ab}}$.

2. Let $h : [0, 1] \times X \to Y$ be a homotopy from $f : (X, A) \to (Y, B)$ to $g : (X, A) \to (Y, B)$ with $h([0, 1] \times A) \subset B$. We consider the associated chain homotopy $S_\bullet(h) : S_\bullet(f) \Rightarrow S_\bullet(g)$ given by formula (12) in the proof of Proposition 3.2.3. As $h([0, 1] \times A) \subset B$, its components

$$S_n(h) : S_n(X) \to S_{n+1}(Y), \quad \sigma \mapsto \Sigma_{j=0}^n (-1)^j \, h \circ (\mathrm{id}_{[0,1]} \times \sigma) \circ T_n^j$$

assign to a singular n-simplex $\sigma : \Delta^n \to A$ the singular $(n+1)$-simplex $S_n(h)(\sigma) : \Delta^{n+1} \to B$. It follows that the maps $S_n(h)$ of the chain homotopy restrict to maps $S_n(h) : S_n(A) \to S_{n+1}(B)$. By Proposition 2.2.4 this defines a chain homotopy $S_\bullet^{(2)}(h) : S_\bullet^{(2)}(f) \Rightarrow S_\bullet^{(2)}(g)$. □

Given the functor $S_n^{(2)} : \mathrm{Top}(2) \to \mathrm{Ch}_{\mathrm{Ab}}$ we can now define the homologies of pairs of topological spaces in analogy to Definition 3.1.6, by post-composing with the functor $H_n : \mathrm{Ch}_{\mathrm{Ab}} \to \mathrm{Ab}$ from Proposition 2.1.8. As we already showed that homotopies between morphisms in Top(2) define chain homotopies, we also directly obtain a counterpart of Proposition 3.2.3.

Definition 3.3.5: The **relative homology functor** $H_nS_\bullet : \mathrm{Top}(2) \to \mathrm{Ab}$ assigns to

- a pair (X, A) the homology $H_n(X, A) = H_nS_\bullet^{(2)}(X, A)$,
- a morphism $f : (X, A) \to (Y, B)$ the group homomorphism

$$H_n(f) = H_nS_\bullet(f) : H_nS_\bullet^{(2)}(X, A) \to H_nS_\bullet^{(2)}(Y, B).$$

Corollary 3.3.6:

1. If $f, g : (X, A) \to (Y, B)$ are homotopic, then $H_n(f) = H_n(g) : H_n(X, A) \to H_n(X, B)$ for all $n \in \mathbb{N}_0$.
2. Any homotopy equivalence $f : (X, A) \to (Y, B)$ in Top(2) induces an isomorphism $H_n(f) : H_n(X, A) \xrightarrow{\sim} H_n(X, B)$.

With the relative chain complex functor $S_\bullet^{(2)} : \mathrm{Top}(2) \to \mathrm{Ch_{Ab}}$ from Proposition 3.3.4, we can now clarify how the homologies of a topological space X are related to the homologies of a subspace $A \subset X$. They key observation is in Proposition 2.2.4. It is shown there that subcomplexes and the associated quotients define a short exact sequence of chain complexes. By Proposition 2.2.7 this gives a long exact sequence of homologies.

Theorem 3.3.7 (long exact sequence for relative homology):

1. Every pair (X, A) of topological spaces defines a long exact sequence of relative homologies

$$\ldots \xrightarrow{\partial_2} H_1(A) \xrightarrow{H_1(i)} H_1(X) \xrightarrow{H_1(\pi)} H_1(X, A) \xrightarrow{\partial_1} H_0(A) \xrightarrow{H_0(i)} H_0(X) \xrightarrow{H_0(\pi)} H_0(X, A) \to 0$$

2. Every morphism $f : (X, A) \to (Y, B)$ defines a chain map between long exact sequences

$$\begin{tikzcd}
\ldots \arrow[r, "H_1(i)"] & H_1(X) \arrow[r, "H_1(\pi)"] \arrow[d, "H_1(f^X)"] & H_1(X,A) \arrow[r, "\partial_1"] \arrow[d, "H_1(f)"] & H_0(A) \arrow[r, "H_0(i)"] \arrow[d, "H_0(f^A)"] & H_0(X) \arrow[r, "H_0(\pi)"] \arrow[d, "H_0(f^X)"] & H_0(X,A) \arrow[r] \arrow[d, "H_0(f)"] & 0 \\
\ldots \arrow[r, "H_1(i')"'] & H_1(Y) \arrow[r, "H_1(\pi')"'] & H_1(Y,B) \arrow[r, "\partial_1'"'] & H_0(B) \arrow[r, "H_0(i')"'] & H_0(Y) \arrow[r, "H_0(\pi')"'] & H_0(Y,B) \arrow[r] & 0
\end{tikzcd} \qquad (13)$$

Proof:
Note first that every subspace $A \subset X$ defines two morphisms in Top(2)

$$(A, \emptyset) \xrightarrow{i:a\mapsto a} (X, \emptyset) \xrightarrow{\pi:x\mapsto x} (X, A). \qquad (14)$$

By Proposition 2.2.4 and 3.3.4, the functor $S_\bullet^{(2)} : \mathrm{Top}(2) \to \mathrm{Ch_{Ab}}$ sends them to the following short exact sequence of chain complexes

$$0 \to S_\bullet(A) \xrightarrow{i_\bullet} S_\bullet(X) \xrightarrow{\pi_\bullet} S_\bullet(X)/S_\bullet(A) \to 0.$$

Every morphism $f : (X, A) \to (Y, B)$ in Top(2) induces morphisms $f^A : (A, \emptyset) \to (B, \emptyset)$ and $f^X : (X, \emptyset) \to (Y, \emptyset)$ such that the following diagram in Top(2) commutes

$$\begin{tikzcd}
(A,\emptyset) \arrow[r, "i"] \arrow[d, "f^A"] & (X,\emptyset) \arrow[r, "\pi"] \arrow[d, "f^X"] & (X,A) \arrow[d, "f"] \\
(B,\emptyset) \arrow[r, "i'"'] & (Y,\emptyset) \arrow[r, "\pi'"'] & (Y,B)
\end{tikzcd}$$

Applying the functor $S_\bullet^{(2)} : \mathrm{Top}(2) \to \mathrm{Ch}_{\mathrm{Ab}}$ yields a commuting diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet(A) \arrow[r, "\iota_\bullet"] \arrow[d, "S_\bullet(f^A)"] & S_\bullet(X) \arrow[r, "\pi_\bullet"] \arrow[d, "S_\bullet(f^X)"] & S_\bullet^{(2)}(X,A) \arrow[r] \arrow[d, "S_\bullet^{(2)}(f)"] & 0 \\
0 \arrow[r] & S_\bullet(B) \arrow[r, "\iota'_\bullet"'] & S_\bullet(Y) \arrow[r, "\pi'_\bullet"'] & S_\bullet^{(2)}(Y,B) \arrow[r] & 0
\end{tikzcd}$$

by Proposition 3.3.4 and Proposition 2.2.4. By Proposition 2.2.7 this defines the long exact homology sequence and commuting diagram (13) with exact rows. □

In view of the fact that the relative homologies are functors $H_n : \mathrm{Top}(2) \to \mathrm{Ab}$ one might ask what algebraic structure is given by the connecting homomorphisms. To clarify this, recall that a pair (A, B) of topological spaces always defines three objects in Top(2) and two morphisms between them, namely $(A, \emptyset) \xrightarrow{i} (X, \emptyset) \xrightarrow{\pi} (X, A)$. Consequently, the associated homologies define a triple of functors for each $n \in \mathbb{N}_0$, namely

- the functor $H_n^1 : \mathrm{Top}(2) \to \mathrm{Ab}$ that assigns to (X, A) the homology $H_n(A) = H_n(A, \emptyset)$,
- the functor $H_n^2 : \mathrm{Top}(2) \to \mathrm{Ab}$ that assigns to (X, A) the homology $H_n(X) = H_n(X, \emptyset)$,
- the functor $H_n^3 : \mathrm{Top}(2) \to \mathrm{Ab}$ that assigns to (X, A) the relative homology $H_n(X, A)$.

From this viewpoint, The connecting homomorphisms ∂_n appear as a natural transformations $\partial_n : H_n^3 \Rightarrow H_{n-1}^1$. Their naturality is simply the statement that the squares in diagram (13) that contain connecting homomorphisms commute.

Corollary 3.3.8: For all $n \in \mathbb{N}_0$ the connecting homomorphisms define natural transformations $\partial_n : H_n^3 \Rightarrow H_{n-1}^1$.

Theorem 3.3.7 is a useful tool to compute relative homologies. It works particularly well, if the topological space X or its subspace $A \subset X$ is contractible and for subspaces with special properties, such as retracts or deformation retracts.

Example 3.3.9: Let X be a topological space and $A \subset X$ a subspace with the long exact homology sequence

$$\ldots \xrightarrow{\partial_{n+2}} H_{n+1}(A) \xrightarrow{H_{n+1}(i)} H_{n+1}(X) \xrightarrow{H_{n+1}(\pi)} H_{n+1}(X, A) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(\pi)} \ldots$$

- If X is contractible, then $H_n(X) = 0$ for all $n \in \mathbb{N}$ and $\mathrm{im}\, \partial_{n+1} \cong \ker H_n(\iota) = H_n(A)$ and $\ker \partial_{n+1} \cong \mathrm{im}\, H_{n+1}(\pi) = 0$ for all $n \in \mathbb{N}_0$. This implies
$$H_{n+1}(X, A) \cong H_n(A) \qquad n \in \mathbb{N}.$$
- If A is contractible, then $H_n(A) = 0$ for all $n \in \mathbb{N}$ and $H_0(i) : H_0(A) \to H_0(X)$ is injective. This implies $\mathrm{im}\, H_{n+1}(\pi) \cong \ker \partial_{n+1} = H_{n+1}(X, A)$ and $\ker H_{n+1}(\pi) = \mathrm{im}\, H_{n+1}(i) = 0$ for all $n \in \mathbb{N}_0$. It follows that
$$H_n(X) \cong H_n(X, A) \qquad n \in \mathbb{N}.$$

Example 3.3.10: For $k, n \in \mathbb{N}_0$ the relative homologies of the $(k+1)$-disc D^{k+1} and the k-sphere $S^k = \partial D^{k+1}$ are given by

$$H_n(D^{k+1}, S^k) = \begin{cases} H_{n-1}(S^k) & n \geq 2, k \in \mathbb{N}_0 \\ \mathbb{Z} & n = 1, k = 0 \\ 0 & n = 1, k \in \mathbb{N} \text{ or } n = 0, k \in \mathbb{N}_0. \end{cases}$$

Proof:
For $n \geq 2$ and $k \in \mathbb{N}_0$ the claim follows from Example 3.3.9 for $A = S^k = \partial D^{k+1} \subset D^{k+1} = X$, because the closed $(k+1)$-disc $D^{k+1} = \{x \in \mathbb{R}^{k+1} \mid ||x|| \leq 1\}$ is convex and hence contractible. For $n = 1$ and $k \in \mathbb{N}$, the long exact homology sequence yields

$$\ldots \to \underbrace{H_1(D^{k+1})}_{=0} \to H_1(D^{k+1}, S^k) \xrightarrow{f} \underbrace{H_0(S^k)}_{\cong\mathbb{Z}} \xrightarrow{\mathrm{id}} \underbrace{H_0(D^{k+1})}_{\cong\mathbb{Z}} \xrightarrow{g} H_0(D^{k+1}, S^k) \to 0$$

Hence, f is injective with $\mathrm{im}\, f = \ker \mathrm{id} = 0$ and g surjective with $\ker g = \mathrm{im}\, \mathrm{id} = \mathbb{Z}$, which implies $H_1(D^{k+1}, S^k) = H_0(D^{k+1}, S^k) = 0$. For $n = 1$, $k = 0$, we have the exact sequence

$$\ldots \to \underbrace{H_1(D^1)}_{=0} \to H_1(D^1, S^0) \xrightarrow{f} \underbrace{H_0(S^0)}_{\cong\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{h:(z_1,z_2)\mapsto z_1+z_2} \underbrace{H_0(D^1)}_{\cong\mathbb{Z}} \xrightarrow{g} H_0(D^1, S^0) \to 0.$$

It follows that f is injective with $\mathrm{im}\, f = \ker h \cong \mathbb{Z}$ and g is surjective with $\ker g = \mathrm{im}\, h \cong \mathbb{Z}$, which implies $H_1(D^1, S^0) = \mathbb{Z}$ and $H_0(D^1, S^0) = 0$. $\square$

Another application of Example 3.3.9 arises, when we consider a subspace $A = \{x\} \subset X$ that consists of a single point and hence is contractible by definition. In this case, we can also describe the zeroth relative homology group more explicitly. The resulting relative homologies are called *reduced homologies*. They are convenient, if one works with path connected topological spaces and if one wants to omit the zeroth homology groups that contain no relevant information.

Example 3.3.11: Let $\emptyset \neq X$ a topological space and $x \in X$. For $n \in \mathbb{N}_0$ the **reduced homology groups** of X are the homology groups

$$\tilde{H}_n(X) = H_n(X, \{x\})$$

By Example 3.3.9 we have $\tilde{H}_n(X) \cong H_n(X)$ for all $n \in \mathbb{N}$. The zeroth reduced homology is given by $\tilde{H}_0(X) = \ker \epsilon$, where $\epsilon : H_0(X) \to \mathbb{Z}$ with $\epsilon([x]) = 1$ and $\epsilon([x']) = 0$ if $x' \in X$ is not in the same path component as x. In particular, if X is path-connected, one has $\tilde{H}_0(X) = 0$.

To treat further examples, we consider different notions of retracts and deformation retracts. Recall that for a subspace $A \subset X$ two maps $f, g : X \to Y$ are **homotopic relative to** A, denoted $f \sim_A g$, if $f(a) = g(a)$ for all $a \in A$ and there is a homotopy $h : [0,1] \times X \to Y$ with $h(t, a) = f(a) = g(a)$ for all $a \in A$ and $t \in [0,1]$.

Definition 3.3.12: Let X be a topological space, $A \subset X$ a subspace and $\iota_A : A \to X$ its inclusion. Then A is called a

- **weak retract** of X, if there is a continuous map $r : X \to A$, with $r \circ \iota_A \sim \mathrm{id}_A$,
- **retract** of X, if there is a continuous map $r : X \to A$ with $r \circ \iota_A = \mathrm{id}_A$,
- **deformation retract** of X, if there is a continuous map $r : X \to A$ with $r \circ \iota_A = \mathrm{id}_A$ and $\iota_A \circ r \sim \mathrm{id}_X$,
- **strong deformation retract** of X, if there is a continuous map $r : X \to A$ with $r \circ \iota_A = \mathrm{id}_A$ and $\iota_A \circ r \sim_A \mathrm{id}_X$.

The map $r : X \to A$ is called a **retraction**.

Note that by definition any strong deformation retract is a deformation retract, any deformation retract is a retract, and any retract is a weak retract. Also by definition, any deformation retract is a homotopy equivalence. Some examples are the following.

Example 3.3.13:

1. For any topological space X and any point $x \in X$, the set $\{x\} \subset X$ is a retract of X with retraction $r : X \to \{x\}$, $x \mapsto x$. The subspace $\{x\} \subset X$ is a deformation retract of X, if and only if X is contractible.

2. S^n is a strong deformation retract of $(\mathbb{R}^{n+1})^\times$ with the retraction and homotopy
$$r : (\mathbb{R}^{n+1})^\times \to S^n,\ x \mapsto \frac{x}{||x||} \qquad h : [0,1] \times (\mathbb{R}^{n+1})^\times \to (\mathbb{R}^{n+1})^\times,\ (t,x) \mapsto tx + (1-t)\frac{x}{||x||}.$$

3. For any topological space X the space $X \times \{0\} \cong X$ is a strong deformation retract of the cylinder $X \times [0,1]$ with the retraction and the homotopy
$$r : X \times [0,1] \to X,\ (x,t) \mapsto (x,0),$$
$$h : [0,1] \times (X \times [0,1]) \to X \times [0,1],\ (t,x,s) \mapsto (x,ts).$$

For retracts $A \subset X$ the long exact homology sequence and the relative homologies take a particularly simple form. Whenever a subspace $A \subset X$ is a weak retract of X, the connecting homomorphisms are trivial, and the long exact homology sequence splits into short exact sequences. The homology group $H_n(X)$ is then the direct sum of the homology group $H_n(A)$ and the relative homology group $H_n(X,A)$. If A is even a deformation retract of X, it follows that all relative homology groups are trivial.

Proposition 3.3.14: If $A \subset X$ is a weak retract of X, then for all $n \in \mathbb{N}_0$
$$H_n(X) \cong H_n(A) \oplus H_n(X,A).$$

Proof:
As $A \subset X$ is a weak retract of X there is a retraction $r : X \to A$ with $r \circ \iota_A \sim \mathrm{id}_A$. This implies $H_n(r) \circ H_n(\iota_A) = H_n(r \circ \iota_A) = H_n(\mathrm{id}_A) = \mathrm{id}_{H_n(A)}$, and hence $H_n(\iota_A) : H_n(A) \to H_n(X)$ has a left inverse and is injective. In the long exact homology sequence
$$\ldots \xrightarrow{\partial_{n+2}} H_{n+1}(A) \xrightarrow{H_{n+1}(\iota_A)} H_{n+1}(X) \xrightarrow{H_{n+1}(\pi)} H_{n+1}(X,A) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{H_n(\iota_A)} H_n(X) \xrightarrow{H_n(\pi)} \ldots$$
this yields $0 = \ker H_n(\iota_A) = \mathrm{im}\, \partial_{n+1}$ and $\ker \partial_{n+1} = \mathrm{im}\, H_{n+1}(\pi) = H_{n+1}(X,A)$ for all $n \in \mathbb{N}_0$. The long exact homology sequence splits into short exact sequences
$$0 \to H_n(A) \xrightarrow{H_n(\iota_A)} H_n(X) \xrightarrow{H_n(\pi)} H_n(X,A) \to 0. \tag{15}$$
As the injection $H_n(\iota_A)$ has the left inverse $H_n(r)$, the short exact sequence (15) splits by Exercise 8 and $H_n(X) \cong H_n(A) \oplus H_n(X,A)$ for all $n \in \mathbb{N}_0$. □

Corollary 3.3.15: Let X be a topological space and $A \subset X$ a deformation retract of X. Then we have for all $n \in \mathbb{N}_0$
$$H_n(A) \cong H_n(X) \qquad H_n(X,A) = 0.$$

Proof:
As $A \subset X$ is a deformation retract, its inclusion $\iota_A : A \to X$ is a homotopy equivalence and $H_n(\iota_A) : H_n(A) \to H_n(X)$ is an isomorphism by Proposition 3.2.3. As $A \subset X$ is a retract, we also have the short exact sequence (15). It implies $\ker H_n(\pi) = \mathrm{im}\, H_n(\iota_A) = H_n(X)$. As $H_n(\pi)$ is surjective, it follows that $H_n(X,A) = 0$ for all $n \in \mathbb{N}_0$. □

3.4 Small simplexes and barycentric subdivision

A very useful tool in topology are open covers. An **open cover** of a topological space X is a family $(U_i)_{i\in I}$ of open subsets $U_i \subset X$ such that $X = \cup_{i\in I} U_i$. Open covers are often used to define structures on topological spaces *locally* on the sets U_i, in such a way that they agree on the overlaps $U_i \cap U_j$ of these open sets and then to obtain a global definition. This is a standard procedure when dealing with manifolds. A similar procedure is also used to compute fundamental groups of topological spaces with the Theorem of **Seifert and van Kampen**.

We want to apply similar techniques to compute homologies. For instance, we want to compute the homologies of a topological space X as in the Theorem of Seifert and van Kampen by choosing open subsets $U_1, U_2 \subset X$ with $U_1 \cup U_2 = X$. This requires that we *adapt* singular n-simplexes $\sigma : \Delta^n \to X$ to a given open cover $(U_i)_{i\in I}$ of X. We need to replace them in a controlled and systematic way by simplexes $\tau : \Delta^n \to X$ whose image is contained in at least one of the sets U_i.

The obvious idea how to achieve this is to use a systematic subdivision procedure that is compatible with the boundary operators. As the latter are given by pre-composition with the face maps, it seems sensible to first define this subdivision procedure for affine simplexes between standard simplexes and then to transport it to the topological space X by applying the singular n-simplexes $\sigma : \Delta^n \to X$.

As this procedure should be systematic and compatible with all face maps, it needs to subdivide an affine n-simplex $\sigma : \Delta^n \to \Delta^p$ together with all of its lower dimensional faces. This can be achieved by adding for each k-face of σ a distinguished point, the *barycentre* that is obtained by averaging over its vertices. The resulting procedure is called *barycentric subdivision* and defined inductively.

We start by defining barycentric subdivision for affine simplexes between standard simplexes. As in Section 3.1 we use the notation $\sigma : \Delta^n \to \Delta^p$, $e_i \mapsto v_i$ or $\sigma = [v_0, \ldots, v_n]$ for an affine linear map $\sigma : \Delta^n \to \Delta^p$ with $\sigma(e_i) = v_i \in \Delta^p$. The face maps $f^n_j : \Delta^{n-1} \to \Delta^n$ from Definition 3.1.2 read $f^n_j = [e_0, \ldots, \hat{e}_j, \ldots, e_n]$, and the boundary of an affine simplex $\sigma : \Delta^n \to \Delta^p$ is

$$d_n(\sigma) = [v_1, \ldots, v_n] - [v_0, v_2, \ldots, v_n] + \ldots + (-1)^n [v_0, \ldots, v_{n-1}]$$

As the boundary $d_n(\sigma)$ for an affine n-simplex $\sigma : \Delta^n \to \Delta^p$ is a linear combination of affine $(n-1)$-simplexes, the affine n-chains form a subcomplex $S^{\text{aff}}_\bullet(\Delta^p) \subset S_\bullet(\Delta^p)$.

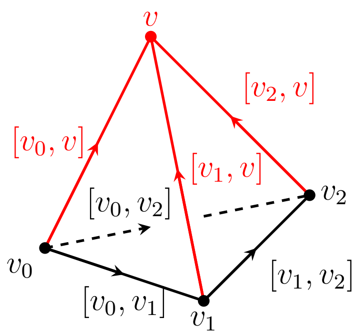
We first describe a procedure that adds an additional vertex v to an ordered affine n-simplex $\sigma : \Delta^n \to \Delta^p$. This is achieved via the *cone maps*. The name is motivated by the fact that this additional vertex v can be viewed as the tip of a cone, whose base is the affine n-simplex σ. We always choose the additional vertex v as the last vertex of the ordered affine simplex, but this is simply a convention that is chosen differently in some references.

Definition 3.4.1: For $v \in \Delta^p$ and $n, p \in \mathbb{N}_0$ the **cone map** $C^v_n : S^{\text{aff}}_n(\Delta^p) \to S^{\text{aff}}_{n+1}(\Delta^p)$ is the group homomorphism

$$C^v_n : S^{\text{aff}}_n(\Delta^p) \to S^{\text{aff}}_{n+1}(\Delta^p), \quad [v_0, \ldots, v_n] \mapsto [v_0, \ldots, v_n, v].$$

Intuitively, it is clear that the boundary of a cone over an n-simplex σ should consist of an n-face opposite to v that corresponds to σ and a number of n-faces that contain v and an

$(n-1)$-face of σ. The latter are themselves cones with tip v over the $(n-1)$-faces of σ. The algebraic counterpart of this statement is the following lemma.



Cone $[v_0, v_1, v_2, v]$ over an affine 2-simplex $[v_0, v_1, v_2]$ with 2-faces $[v_0, v_1, v_2]$, $[v_0, v_1, v]$, $[v_0, v_2, v]$, $[v_1, v_2, v]$.

Lemma 3.4.2: The cone maps satisfy

1. $d_1 \circ C_0^v(\sigma) = v - \sigma$ for all singular 0-simplexes $\sigma : \Delta^0 \to \Delta^p$, where $v : \Delta^0 \to \Delta^p$, $e_0 \mapsto v$.
2. $d_{n+1} \circ C_n^v(\sigma) - C_{n-1}^v \circ d_n(\sigma) = (-1)^{n+1}\sigma$ for all affine simplexes $\sigma : \Delta^n \to \Delta^p$ and $n \in \mathbb{N}$.

Proof:
1. For all 0-simplexes $\sigma = [v_0] : \Delta^0 \to \Delta^p$ we have $d_1 \circ C_0^v(\sigma) = d_1[v_0, v] = [v] - [v_0] = v - \sigma$.

2. For all n-simplexes $\sigma = [v_0, \ldots, v_n] : \Delta^n \to \Delta^p$ we have

$$\begin{aligned}
&d_{n+1} \circ C_n^v(\sigma) - C_{n-1}^v \circ d_n(\sigma) = d_{n+1}[v_0, \ldots, v_n, v] - \Sigma_{j=0}^n (-1)^j C_v^{n-1}[v_0, \ldots, \hat{v}_j, \ldots, v_n] \\
&= \Sigma_{j=0}^n (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, v_n, v] + (-1)^{n+1}[v_0, \ldots, v_n] - \Sigma_{j=0}^n (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, v_n, v] \\
&= (-1)^{n+1}[v_0, \ldots, v_n] = (-1)^{n+1}\sigma.
\end{aligned}$$

□

We will now choose as the tip of a cone over an n-simplex $\sigma = [v_0, \ldots, v_n] : \Delta^n \to \Delta^p$ a special point, namely the barycentre $b(\sigma) = \frac{1}{n+1}(v_0 + \ldots + v_n)$. This has the advantage that it is a canonical choice and always contained in the simplex. For a 1-simplex $\sigma = [v_0, v_1]$ the barycentre defines a subdivision into 1-simplexes $[v_0, b(\sigma)]$ and $[v_1, b(\sigma)]$.

For a 2-simplex $\sigma = [v_0, v_1, v_2]$, we apply this subdivision procedure to all 1-faces and add an additional vertex in the middle that is the barycentre of σ. This subdivides σ into six 2-simplexes, each of which contains the barycentre $b(\sigma)$, one barycentre of a 1-face and one vertex of σ. Likewise, for a 3-simplex $\sigma = [v_0, v_1, v_2, v_3]$ we apply this procedure to all 2-faces and add an additional vertex, the barycentre $b(\sigma)$, in the middle. This subdivides σ into 24 simplexes, each of which contains the barycentre $b(\sigma)$, one barycentre of a 2-face, one barycentre of a 1-face contained in the 2-face and one vertex of σ contained in the 1-face. We extend this procedure inductively to higher-dimensional simplexes.

Definition 3.4.3: Let $p, n \in \mathbb{N}_0$.

1. The **barycentre** of an affine n-simplex $\sigma : \Delta^n \to \Delta^p$ is $b(\sigma) = \frac{1}{n+1}\sum_{k=0}^n \sigma(e_k)$.
2. The **barycentric subdivision** map $B_n : S_n^{\text{aff}}(\Delta^p) \to S_n^{\text{aff}}(\Delta^p)$ is defined inductively by

$$B_0 = \mathrm{id}_{S_0(\Delta^p)} \qquad B_n(\sigma) = (-1)^n C_{n-1}^{b(\sigma)} \circ B_{n-1} \circ d_n(\sigma) \text{ for } n \geq 1, \sigma : \Delta^n \to \Delta^p. \tag{16}$$

Remark 3.4.4:

1. One can show with the inductive definition (Exercise 29) that for any affine-linear simplex $\sigma = [v_0, \ldots, v_n] : \Delta^n \to \Delta^p$ the barycentric subdivision is given by

$$B_n(\sigma)[e_0, \ldots, e_n] = \sum_{\pi \in S_{n+1}} (-1)^n \mathrm{sgn}(\pi)[v_0^\pi, \ldots, v_n^\pi] \qquad v_r^\pi = \frac{1}{r+1}\sum_{j=0}^r v_{\pi(j)}$$

   Each affine n-simplex τ in $B_n(\sigma)$ is of the form $\tau = [b(f_0), b(f_1), \ldots, b(f_{n-1}), b(f_n)]$, where $b(f_k)$ is the barycentre of a k-face f_k of σ and $f_0 \subset f_1 \ldots \subset f_{n-1} \subset f_n$.

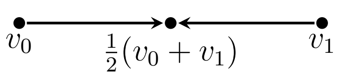
2. It follows from 1. that for each affine n-simplex $\sigma : \Delta^n \to \Delta^p$ we have

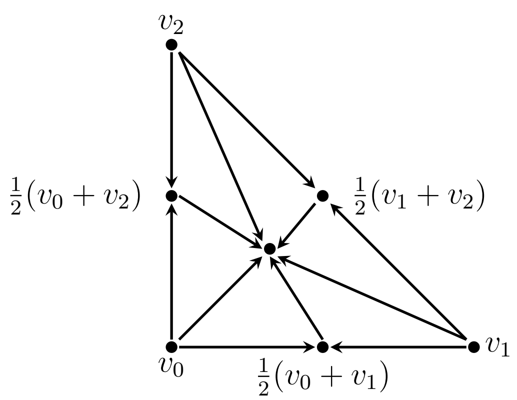
$$B_n(\sigma) = S_n(\sigma) \circ B_n(\mathrm{id}_{\Delta^n}). \tag{17}$$

   The map $S_n(\sigma) : S_n(\Delta^n) \to S_n(\Delta^p)$, $\tau \mapsto \sigma \circ \tau$ acts on an n-simplex τ of $B_n(\mathrm{id}_{\Delta^n})$ by transporting its vertices with σ.

Example 3.4.5: For affine $0, 1, 2$-simplexes we have

$$\begin{aligned}
B_0[v_0] &= [v_0] \\
B_1[v_0, v_1] &= [v_0, \tfrac{1}{2}(v_0+v_1)] - [v_1, \tfrac{1}{2}(v_0+v_1)] \\
B_2[v_0, v_1, v_2] &= [v_0, \tfrac{1}{2}(v_0+v_1), \tfrac{1}{3}(v_0+v_1+v_3)] - [v_0, \tfrac{1}{2}(v_0+v_2), \tfrac{1}{3}(v_0+v_1+v_2)] \\
&\quad + [v_1, \tfrac{1}{2}(v_1+v_2), \tfrac{1}{3}(v_0+v_1+v_3)] - [v_1, \tfrac{1}{2}(v_0+v_1), \tfrac{1}{3}(v_0+v_1+v_2)] \\
&\quad + [v_2, \tfrac{1}{2}(v_0+v_2), \tfrac{1}{3}(v_0+v_1+v_3)] - [v_2, \tfrac{1}{2}(v_1+v_2), \tfrac{1}{3}(v_0+v_1+v_2)]
\end{aligned}$$





Remark 3.4.4 and Example 3.4.5 show that barycentric subdivision gives a systematic way to subdivide an affine n-simplex into smaller simplexes. In fact, we control the size of the simplexes in the barycentric subdivision, if we know the size of the original affine simplex. As the standard n-simplexes Δ^n are compact subspaces of the metric spaces $\mathbb{R}^n$, we can use the diameters of the affine simplexes as a measure of their size.

Recall that for a subspace $M \subset X$ of a metric space (X, d) the diameter of M is defined as $\operatorname{diam} M = \sup\{d(x, y) \mid x, y \in M\}$. Recall also that for a *compact* subspace $M \subset X$ this supremum is attained: there are points $p, q \in M$ with $\operatorname{diam} M = d(p, q)$.

Lemma 3.4.6: Let $\sigma : \Delta^n \to \Delta^p$ be an affine simplex. Then every simplex $\tau : \Delta^n \to \Delta^p$ in the affine n-chain $B_n(\sigma)$ has

$$\operatorname{diam} \tau(\Delta^n) \leq \frac{n}{n+1} \operatorname{diam} \sigma(\Delta^n).$$

Proof:
We use induction over n. For $n = 0$ the claim holds trivially. Suppose it is shown for $n \leq k-1$, let $\sigma : \Delta^k \to \Delta^p$ be an affine k-simplex and $\tau : \Delta^k \to \Delta^p$ a k-simplex in $B_k(\sigma)$. Then $\tau(\Delta^k) \subset \sigma(\Delta^k)$ is compact as the image of the compact space Δ^k under a continuous map, and there are $p, q \in \tau(\Delta^k)$ with $\operatorname{diam} \tau(\Delta^k) = ||p - q||$. By Remark 3.4.4 either (i) there is a face $\rho = \sigma \circ f_i^k : \Delta^{k-1} \to \Delta^p$ of σ such that $p, q \in \rho(\Delta^{k-1})$ or (ii) we can assume that $q = b(\sigma)$.

In case (i) we apply the induction hypothesis to the affine simplex ρ with $\rho(\Delta^{k-1}) \subset \sigma(\Delta^k)$

$$\operatorname{diam} \tau(\Delta^k) = ||p - q|| \leq \tfrac{k-1}{k} \operatorname{diam} \rho(\Delta^{k-1}) \leq \tfrac{k-1}{k} \operatorname{diam} \sigma(\Delta^k) < \tfrac{k}{k+1} \operatorname{diam} \sigma(\Delta^k).$$

For case (ii) note first that we can express any point $p \in \sigma(\Delta^k)$ as $p = \Sigma_{j=0}^k \lambda_j \sigma(e_j)$ with $\lambda_j \in [0, 1]$ and $\Sigma_{j=0}^k \lambda_j = 1$. This yields for any $q \in \sigma(\Delta^k)$

$$||p - q|| = ||\Sigma_{j=0}^k \lambda_j (\sigma(e_j) - q)|| \leq \Sigma_{j=0}^k \lambda_j ||\sigma(e_j) - q|| \leq \max_{j=0,\ldots,k} ||\sigma(e_j) - q||.$$

Setting $q = b(\sigma)$ we then obtain

$$\begin{aligned}
\operatorname{diam} \tau(\Delta^k) &= ||p - q|| = ||p - \tfrac{1}{k+1} \Sigma_{i=0}^k \sigma(e_i)|| \\
&\leq \max_{j=0,\ldots,k} ||\sigma(e_j) - \tfrac{1}{k+1} \Sigma_{i=0}^k \sigma(e_i)|| = \max_{j=0,\ldots,k} ||\tfrac{1}{k+1} \Sigma_{i=0}^k (\sigma(e_j) - \sigma(e_i))|| \\
&\leq \tfrac{1}{k+1} \max_{j=0,\ldots,k} \left(\Sigma_{i=0}^k ||\sigma(e_j) - \sigma(e_i)||\right) = \tfrac{1}{k+1} \max_{j=0,\ldots,k} \left(\Sigma_{i=0, i\neq j}^k ||\sigma(e_j) - \sigma(e_i)||\right) \\
&\leq \tfrac{k}{k+1} \operatorname{diam} \sigma(\Delta^k).
\end{aligned}$$

□

We now define the barycentric subdivision for general singular n-simplexes $\sigma : \Delta^n \to X$. As σ is a continuous map, it induces a chain map $S_\bullet(\sigma) : S_\bullet(\Delta^n) \to S_\bullet(X)$. We can therefore first apply the affine barycentric subdivision map B_n from Definition 3.4.3 to the affine n-simplex $\mathrm{id}_{\Delta^n} : \Delta^n \to \Delta^n$ and then transport the resulting simplexes τ to X by applying $S_n(\sigma)$. This sends each such simplex τ to $\sigma \circ \tau : \Delta^n \to X$. Note in particular that for each *affine* simplex $\sigma : \Delta^n \to \Delta^p$ this yields its barycentric division by Remark 3.4.4, 2.

Definition 3.4.7: Let X be a topological space.
The **barycentric subdivision operator** is the group homomorphism

$$B_n^X : S_n(X) \to S_n(X), \quad \sigma \mapsto B_n^X(\sigma) = S_n(\sigma) \circ B_n(\mathrm{id}_{\Delta^n}).$$

As we already clarified the geometric meaning of the barycentric subdivision operators, we will now focus on their algebraic properties. As they are maps $B_n^X : S_n(X) \to S_n(X)$, it is natural to ask if they are chain maps. A they only subdivide simplexes and carry no additional information, one might also suspect that they could be chain homotopic to the identity map. Finally, as the abelian groups $S_n(X)$ of n-chains are assigned to a topological space X by the singular chain complex functor $S_\bullet : \mathrm{Top} \to \mathrm{Ch}_{\mathrm{Ab}}$ from Proposition 3.1.5, one might ask how the barycentric subdivision operators interact with the chain maps induced by continuous maps. All of these questions are addressed by the following proposition.

Proposition 3.4.8: The barycentric subdivision operators define a natural transformation $B_\bullet : S_\bullet \Rightarrow S_\bullet$ that is naturally chain homotopic to $\mathrm{id}_{S_\bullet}$.

Proof:
1. We show that the group homomorphisms $B_n^X : S_n(X) \to S_n(X)$ define a chain map $B_\bullet^X : S_\bullet(X) \to S_\bullet(X)$:

For this, we compute

$$\begin{aligned}
d_n \circ B_n^X(\sigma) &= d_n \circ S_n(\sigma) \circ B_n(\mathrm{id}_{\Delta^n}) = S_{n-1}(\sigma) \circ d_n \circ B_n(\mathrm{id}_{\Delta^n}) \\
B_{n-1}^X \circ d_n(\sigma) &= B_{n-1}^X(d_n(\sigma)) = S_{n-1}(d_n(\sigma)) \circ B_{n-1}(\mathrm{id}_{\Delta^{n-1}}) \\
&\overset{*}{=} \Sigma_{j=0}^n (-1)^j S_{n-1}(\sigma \circ f_j^n) \circ B_{n-1}(\mathrm{id}_{\Delta^{n-1}}) \\
&\overset{**}{=} \Sigma_{j=0}^n (-1)^j S_{n-1}(\sigma) \circ S_{n-1}(f_j^n) \circ B_{n-1}(\mathrm{id}_{\Delta^{n-1}}) \\
&\overset{***}{=} \Sigma_{j=0}^n (-1)^j S_{n-1}(\sigma) \circ B_{n-1}(f_j^n) \overset{*}{=} S_{n-1}(\sigma) \circ B_{n-1}(d_n(\mathrm{id}_{\Delta^n})) \\
&= S_{n-1}(\sigma) \circ B_{n-1}^n \circ d_n(\mathrm{id}_{\Delta^n}),
\end{aligned}$$

where we used in the first line that $S_\bullet(\sigma)$ is a chain map, in * the definition of the boundary operator, in ** the functoriality of $S_\bullet$ and in *** Remark 3.4.4, 2.

It is therefore sufficient to prove that $d_n \circ B_n = B_{n-1} \circ d_n$ for all $n \in \mathbb{N}_0$ for the affine barycentric subdivision maps from Definition 3.4.3. We show this by induction over n, where we set $B_{n-1} = 0$ for $n \leq 0$. For $n = 0$ we have $d_0 \circ B_0 = d_0 = 0$.

Suppose we showed that $d_k \circ B_k = B_{k-1} \circ d_{k-1}$ for $k \leq n-1$. Then we have for all affine n-simplexes $\sigma : \Delta^n \to \Delta^p$

$$\begin{aligned}
d_n \circ B_n(\sigma) &\overset{(16)}{=} (-1)^n d_n \circ C_{n-1}^{b(\sigma)} \circ B_{n-1} \circ d_n(\sigma) \\
&\overset{3.4.2}{=} B_{n-1} \circ d_n(\sigma) + (-1)^n C_{n-2}^{b(\sigma)} \circ d_{n-1} \circ B_{n-1} \circ d_n(\sigma) \\
&\overset{\mathrm{IH}}{=} B_{n-1} \circ d_n(\sigma) + (-1)^n C_{n-1}^{b(\sigma)} \circ B_{n-2} \circ d_{n-1} \circ d_n(\sigma) = B_{n-1} \circ d_n(\sigma).
\end{aligned}$$

2. We define for each topological space X a chain homotopy $h_\bullet^X : \mathrm{id}_{S_\bullet(X)} \Rightarrow B_\bullet^X$. We first define group homomorphisms $h_n : S_n^{\mathrm{aff}}(\Delta^p) \to S_{n+1}^{\mathrm{aff}}(\Delta^p)$ inductively by

$$\begin{aligned}
&h_0 : S_0(\Delta^p) \to S_1(\Delta^p), && \sigma \mapsto 0 \\
&h_n : S_n(\Delta^p) \to S_{n+1}(\Delta^p), && \sigma \mapsto (-1)^{n+1} C_n^{b(\sigma)}\big(B_n(\sigma) - \sigma - h_{n-1} \circ d_n(\sigma)\big).
\end{aligned} \tag{18}$$

and then post-compose the resulting simplexes with singular simplexes $\sigma : \Delta^n \to X$

$$h_n^X : S_n(X) \to S_{n+1}(X), \quad \sigma \mapsto S_{n+1}(\sigma) \circ h_n(\mathrm{id}_{\Delta^n}). \tag{19}$$

To show that this defines a chain homotopy, note first that for all affine simplexes $\sigma : \Delta^n \to \Delta^p$

$$h_n(\sigma) = S_{n+1}(\sigma) \circ h_n(\mathrm{id}_{\Delta^n}). \tag{20}$$

This follows inductively from the definition of h_n and the corresponding identities for the cone map and the affine barycentric subdivision map in Remark 3.4.4, 2. With this we compute

$$\begin{aligned}
d_{n+1} \circ h_n^X(\sigma) &= d_{n+1} \circ S_{n+1}(\sigma) \circ h_n(\mathrm{id}_{\Delta^n}) = S_n(\sigma) \circ d_n \circ h_n(\mathrm{id}_{\Delta^n}) \qquad (21)\\
h_{n-1}^X \circ d_n(\sigma) &= S_n(d_n(\sigma)) \circ h_{n-1}(\mathrm{id}_{\Delta^{n-1}}) \overset{*}{=} \Sigma_{j=0}^n (-1)^j S_n(\sigma \circ f_j^n) \circ h_{n-1}(\mathrm{id}_{\Delta^{n-1}})\\
&\overset{**}{=} \Sigma_{j=0}^n (-1)^j S_n(\sigma) \circ S_n(f_j^n) \circ h_{n-1}(\mathrm{id}_{\Delta^{n-1}}) \overset{(20)}{=} \Sigma_{j=0}^n (-1)^j S_n(\sigma) \circ h_{n-1}(f_j^n)\\
&\overset{*}{=} S_n(\sigma) \circ h_{n-1} \circ d_n(\mathrm{id}_{\Delta^n}),
\end{aligned}$$

where we used in the first line that $S_\bullet(\sigma)$ is a chain map, in * the definition of the boundary operator and in ** the functoriality of $S_\bullet$. Adding the two terms in (21) yields

$$\begin{aligned}
&(d_{n+1} \circ h_n^X + h_{n-1}^X \circ d_n)(\sigma) + \sigma - B_n^X(\sigma)\\
&= S_n(\sigma) \circ (d_n \circ h_n(\mathrm{id}_{\Delta^n}) + h_{n-1} \circ d_n(\mathrm{id}_{\Delta^n}) + \mathrm{id}_{\Delta^n} - B_n(\mathrm{id}_{\Delta^n})).
\end{aligned}$$

It is therefore sufficient to prove that $(d_{n+1} \circ h_n + h_{n-1} \circ d_n)(\sigma) = B_n(\sigma) - \sigma$ for all affine n-simplexes $\sigma : \Delta^n \to \Delta^p$. This follows again by induction over n.

For $n = 0$ we have $d_1 \circ h_0 = 0 = \sigma - \sigma = B_0(\sigma) - \sigma$ for all singular 0-simplexes $\sigma : \Delta^0 \to \Delta^p$. Suppose we showed that $(d_{k+1} \circ h_k + h_{k-1} \circ d_k)(\sigma) = B_k(\sigma) - \sigma$ for all $k \leq n-1$ and affine k-simplexes $\sigma : \Delta^k \to \Delta^p$. Then we have for $k = n$ and any affine n-simplex $\sigma : \Delta^n \to \Delta^p$

$$\begin{aligned}
&d_{n+1} \circ h_n(\sigma) = (-1)^{n+1} d_{n+1} \circ C_n^{b(\sigma)}(B_n(\sigma) - \sigma - h_{n-1} \circ d_n(\sigma))\\
&\overset{3.4.2}{=} B_n(\sigma) - \sigma - h_{n-1} \circ d_n(\sigma) + (-1)^{n+1} C_{n-1}^{b(\sigma)}\big(d_n B_n(\sigma) - d_n(\sigma) - d_n \circ h_{n-1}(d_n(\sigma))\big)\\
&\overset{*}{=} B_n(\sigma) - \sigma - h_{n-1} \circ d_n(\sigma) + (-1)^{n+1} C_{n-1}^{b(\sigma)}\big(B_{n-1}(d_n(\sigma)) - d_n(\sigma) - d_n \circ h_{n-1}(d_n(\sigma))\big)\\
&\overset{\mathrm{IH}}{=} B_n(\sigma) - \sigma - h_{n-1} \circ d_n(\sigma) + (-1)^{n+1} C_{n-1}^{b(\sigma)}\big(h_{n-2} \circ d_{n-1} \circ d_n(\sigma)\big)\\
&= B_n(\sigma) - \sigma - h_{n-1} \circ d_n(\sigma),
\end{aligned}$$

where we used in * that B_n is a chain map by 1. and in the last step that $d_{n-1} \circ d_n = 0$.

3. We now prove that the chain maps $B_\bullet^X : S_\bullet(X) \to S_\bullet(X)$ as well as the chain homotopies $h_\bullet^X : \mathrm{id}_{S_\bullet(X)} \Rightarrow B_\bullet^X$ are natural in X.

Let $f : X \to Y$ be a continuous map. Then we have for all singular n-simplexes $\sigma : \Delta^n \to X$

$$\begin{aligned}
S_n(f) \circ B_n^X(\sigma) &= S_n(f) \circ S_n(\sigma) \circ B_n(\mathrm{id}_{\Delta^n}) = S_n(f \circ \sigma) \circ B_n(\mathrm{id}_{\Delta^n}) = B_n^Y(f \circ \sigma) = B_n^Y \circ S_n(f)(\sigma)\\
S_{n+1}(f) \circ h_n^X(\sigma) &= S_{n+1}(f \circ \sigma) \circ h_n(\mathrm{id}_{\Delta^n}) = h_n^Y(f \circ \sigma) = h_n^Y \circ S_n(f)(\sigma).
\end{aligned}$$

This shows that the following diagrams commute for all $n \in \mathbb{N}_0$

$$\begin{tikzcd}
S_n(X) \arrow[r, "B_n^X"] \arrow[d, "S_n(f)"'] & S_n(X) \arrow[d, "S_n(f)"] \\
S_n(Y) \arrow[r, "B_n^Y"'] & S_n(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
S_n(X) \arrow[r, "h_n^X"] \arrow[d, "S_n(f)"'] & S_{n+1}(X) \arrow[d, "S_{n+1}(f)"] \\
S_n(Y) \arrow[r, "h_n^Y"'] & S_{n+1}(Y).
\end{tikzcd}$$

$\square$

Remark 3.4.9: The naturality of B_n^X implies for all n-simplexes $\sigma : \Delta^n \to X$ and all $m \in \mathbb{N}_0$

$$(B_n^X)^m(\sigma) = B_n^X \circ \ldots \circ B_n^X(\sigma) = S_n(\sigma) \circ B_n^m(\mathrm{id}_{\Delta^n}) = S_n(\sigma) \circ B_n \circ \ldots \circ B_n(\mathrm{id}_{\Delta^n}). \qquad (22)$$

Proof:
This follows by induction over m. For $m = 0, 1$ it holds by Definition 3.4.7. Suppose the claim is shown for all $m \leq k$. Then we have for every singular n-simplex $\sigma : \Delta^n \to X$

$$\begin{aligned}(B_n^X)^{k+1}(\sigma) &= B_n^X \circ (B_n^X)^k(\sigma) \overset{\text{IH}}{=} B_n^X \circ S_n(\sigma) \circ B_n^k(\mathrm{id}_{\Delta^n}) \overset{\text{nat}}{=} S_n(\sigma) \circ B_n \circ B_n^k(\mathrm{id}_{\Delta^n}) \\ &= S_n(\sigma) \circ B_n^{k+1}(\mathrm{id}_{\Delta^n}).\end{aligned}$$

□

With the barycentric subdivision operator we can now adapt singular simplexes $\sigma : \Delta^n \to X$ and, more generally, singular n-chains in a topological space X to a given open cover of X. In particular, we replace n-cycles $z \in Z_n(X)$ by n-cycles $z' \in Z_n(X)$ in which all n-simplexes have their images in one of the open subsets of the cover. As the barycentric subdivision operator is chain homotopic to the identity map, this does not affect their homologies.

Definition 3.4.10: Let X be a topological space and $\mathcal{U} = (U_i)_{i \in I}$ an open cover of X.

1. An n-chain $x = \sum_{j=0}^k a_j \sigma_j \in S_n(X)$ is called **$\mathcal{U}$-small**, if for every n-simplex $\sigma_j : \Delta^n \to X$ in x there is a $i \in I$ such that $\sigma_j(\Delta^n) \subset U_i$.
2. We denote by $S_n^{\mathcal{U}}(X) \subset S_n(X)$, by $Z_n^{\mathcal{U}}(X) \subset Z_n(X)$ and by $B_n^{\mathcal{U}}(X) \subset B_n(X)$ the subgroups of $\mathcal{U}$-small n-chains, n-cycles and n-boundaries.

Lemma 3.4.11: Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of a topological space X. Then for any n-cycle $z \in Z_n(X)$ there is a $\mathcal{U}$-small n-cycle $z' \in Z_n^{\mathcal{U}}(X)$ with $[z] = [z'] \in H_n(X)$.

Proof:
1. We show first that for each n-simplex $\sigma : \Delta^n \to X$ there is an $r = r(\sigma) \in \mathbb{N}_0$ such that $(B_n^X)^m(\sigma) = B_n^X \circ \ldots \circ B_n^X(\sigma)$ is $\mathcal{U}$-small for every $m \geq r$.

As σ is continuous the sets $\sigma^{-1}(U_i)$ for $i \in I$ form an open cover of the compact metric space $\Delta^n = \sigma^{-1}(X)$. By Lebesgue's lemma, there is an $\epsilon > 0$ such that every subspace $M \subset \Delta^n$ of $\operatorname{diam} M < \epsilon$ is contained in at least one of the sets $\sigma^{-1}(U_i)$.

By Lemma 3.4.6 there is an $r > 0$ such that $\operatorname{diam} \tau(\Delta^n) < \epsilon$ for all n-simplexes τ in $B_n^m(\mathrm{id}_{\Delta^n})$ and $m \geq r$. Hence, every such n-simplex τ is contained in at least one of the sets $\sigma^{-1}(U_i)$. With Remark 3.4.9 this implies that for every n-simplex $\sigma \circ \tau$ in $(B_n^X)^m(\sigma) = S_n(\sigma) \circ B_n^m(\mathrm{id}_{\Delta^n})$ we have $\sigma \circ \tau(\Delta^n) \subset U_i$ for at least one of the sets U_i. Hence, $(B_n^X)^m(\sigma)$ is $\mathcal{U}$-small for all $m \geq r$.

2. Let $z = \Sigma_{j=0}^k a_j \sigma_j \in Z_n(X)$ with singular n-simplexes $\sigma_j : \Delta^n \to X$. By 1. there is an $r \in \mathbb{N}_0$ such that $z' = (B_n^X)^r(z)$ is $\mathcal{U}$-small. By Proposition 3.4.8 $B_\bullet^X : S_\bullet(X) \to S_\bullet(X)$ is chain homotopic to $\mathrm{id}_{S_\bullet}(X)$, and Proposition 2.1.12 implies $H_n(B_\bullet^X) = \mathrm{id}_{H_n(X)} : H_n(X) \to H_n(X)$ for all $n \in \mathbb{N}_0$. This yields for the homology classes in $H_n(X)$

$$[z'] = [(B_n^X)^r(z)] = H_n(B_n^X)^r[z] = \mathrm{id}^r_{H_n(X)}[z] = [z].$$

□

We now consider chain complexes formed by $\mathcal{U}$-small simplexes in a topological space X, a subspace $A \subset X$ and the associated $\mathcal{U}$-small relative chain complexes. The following observations about $\mathcal{U}$-small simplexes are immediate:

- Because $(\sigma \circ f_j^n)(\Delta^{n-1}) \subset \sigma(\Delta^n)$ for any n-simplex $\sigma : \Delta^n \to X$, the boundary $d_n(x)$ of a $\mathcal{U}$-small n-chain x is again $\mathcal{U}$-small. Thus, the $\mathcal{U}$-small n-chains form a subcomplex

$$S_\bullet^{\mathcal{U}}(X) \subset S_\bullet(X) \qquad j_\bullet^X : S_\bullet^{\mathcal{U}}(X) \to S_\bullet(X).$$

- Because $\tau(\Delta^n) \subset \sigma(\Delta^n)$ for any n-simplex τ in the barycentric subdivision $B_n^X(\sigma)$ of an n-simplex $\sigma : \Delta^n \to X$, the barycentric subdivision $B_n^X(x)$ of a $\mathcal{U}$-small n-chain x is again $\mathcal{U}$-small. Barycentric subdivision induces a chain map

$$B_\bullet^X : S_\bullet^{\mathcal{U}}(X) \to S_\bullet^{\mathcal{U}}(X) \qquad \text{with} \qquad B_\bullet^X \circ j_\bullet^X = j_\bullet^X \circ B_\bullet^X.$$

- If $A \subset X$ is a subspace, then $\mathcal{U} = (U_i \cap A)_{i \in I}$ is an open cover of A. This defines a subcomplex and an inclusion

$$S_\bullet^{\mathcal{U}}(A) \subset S_\bullet(A) \qquad j_\bullet^A : S_\bullet^{\mathcal{U}}(A) \to S_\bullet(A) \qquad \text{with} \qquad j_\bullet^X \circ S_\bullet(\iota_A) = S_\bullet(\iota_A) \circ j_\bullet^A,$$

and the barycentric subdivision induces a chain map

$$B_\bullet^A : S_\bullet^{\mathcal{U}}(A) \to S_\bullet^{\mathcal{U}}(A) \qquad \text{with} \qquad B_\bullet^A \circ j_\bullet^A = j_\bullet^A \circ B_\bullet^A, \quad S_\bullet(\iota_A) \circ B_\bullet^A = B_\bullet^X \circ S_\bullet(\iota_A).$$

- This defines a quotient complex and a chain map

$$j_\bullet^{(X,A)} : S_\bullet^{\mathcal{U}}(X)/S_\bullet^{\mathcal{U}}(A) \to S_\bullet(X)/S_\bullet(A)$$

such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet(A) \arrow[r, "S_\bullet(\iota_A)"] & S_\bullet(X) \arrow[r, "\pi_\bullet"] & S_\bullet(X)/S_\bullet(A) \arrow[r] & 0 \\
0 \arrow[r] & S_\bullet^{\mathcal{U}}(A) \arrow[r, "S_\bullet(\iota_A)"'] \arrow[u, "j_\bullet^A"] & S_\bullet^{\mathcal{U}}(X) \arrow[r, "\pi_\bullet"'] \arrow[u, "j_\bullet^X"] & S_\bullet^{\mathcal{U}}(X)/S_\bullet^{\mathcal{U}}(A) \arrow[r] \arrow[u, "j_\bullet^{(X,A)}"] & 0
\end{tikzcd} \tag{23}$$

Definition 3.4.12: Let X be a topological space, $A \subset X$ a subspace and $\mathcal{U} = (U_i)_{i \in I}$ an open cover of X. We denote by $H_n^{\mathcal{U}}(X)$, $H_n^{\mathcal{U}}(A)$ and $H_n^{\mathcal{U}}(X, A)$ the homologies of the chain complexes $S_\bullet^{\mathcal{U}}(X)$, $S_\bullet^{\mathcal{U}}(A)$ and $S_\bullet^{\mathcal{U}}(X, A) = S_\bullet^{\mathcal{U}}(X)/S_\bullet^{\mathcal{U}}(A)$.

We are now able to compare the homologies of the $\mathcal{U}$-small chain complexes $S_\bullet^{\mathcal{U}}(X)$, $S_\bullet^{\mathcal{U}}(A)$ and $S_\bullet^{\mathcal{U}}(X, A)$ to the homologies of $S_\bullet(X)$, $S_\bullet(A)$ and $S_\bullet(X, A)$.

Proposition 3.4.13: Let X be a topological space, $A \subset X$ a subspace and $\mathcal{U} = (U_i)_{i \in I}$ an open cover of X. Then the inclusions of the $\mathcal{U}$-small chain complexes induce isomorphisms

$$H_n(j_\bullet^X) : H_n^{\mathcal{U}}(X) \xrightarrow{\sim} H_n(X) \quad H_n(j_\bullet^A) : H_n^{\mathcal{U}}(A) \xrightarrow{\sim} H_n(A) \quad H_n(j_\bullet^{(X,A)}) : H_n^{\mathcal{U}}(X, A) \xrightarrow{\sim} H_n(X, A).$$

Proof:
1. The group homomorphism $H_n(j_\bullet^X)$ assigns to the homology class $[z]_{\mathcal{U}} \in H_n^{\mathcal{U}}(X)$ of a $\mathcal{U}$-small n-cycle $z \in Z_n^{\mathcal{U}}(X)$ its homology class $[z] \in H_n(X)$. It is surjective by Lemma 3.4.11. We show that $H_n(j_\bullet^X)$ is injective.

Let $z \in Z_n^{\mathcal{U}}(X)$ with $H_n(j_\bullet^X)[z]_{\mathcal{U}} = [z] = 0$. Then there is an $(n+1)$-chain $y \in S_{n+1}(X)$ with $z = d_{n+1}(y)$. By Lemma 3.4.11. there is an $r \in \mathbb{N}_0$ such that $y' = (B_{n+1}^X)^r(y)$ is $\mathcal{U}$-small. As $B_\bullet^X$ is a chain map that is chain homotopic to the identity, we then have

$$\begin{aligned} & d_{n+1}(y') = d_{n+1} \circ (B_{n+1}^X)^r(y) = (B_n^X)^r \circ d_{n+1}(y) = (B_n^X)^r(z). \\ \Rightarrow \quad & 0 = [d_{n+1}(y')]_{\mathcal{U}} = [(B_n^X)^r(z)]_{\mathcal{U}} = H_n(B_n^X)^r[z]_{\mathcal{U}} = [z]_{\mathcal{U}}. \end{aligned}$$

This shows that $\ker H_n(j_\bullet^X) = 0$ and $H_n(j_\bullet^X)$ is an isomorphism. An analogous argument for $\mathcal{U}' = (U_i \cap A)_{i \in I}$ shows that $H_n(j_\bullet^A)$ is an isomorphism.

2. To prove the claim for the relative homologies, we consider the commuting diagram (23) of chain complexes with exact rows. By Proposition 2.2.7 it defines a commuting diagram with exact rows, in which all vertical arrows $H_k(j_\bullet^A)$ and $H_k(j_\bullet^X)$ are isomorphisms by 1.

$$\begin{tikzcd}
\ldots \arrow[r, "\partial_{n+1}"] & H_n(A) \arrow[r, "H_n(\iota_A)"] & H_n(X) \arrow[r, "H_n(\pi_\bullet)"] & H_n(X,A) \arrow[r, "\partial_{n-1}"] & H_{n-1}(A) \arrow[r, "H_{n-1}(\iota_A)"] & H_{n-1}(X) \arrow[r, "H_{n-1}(\pi_\bullet)"] & \ldots \\
\ldots \arrow[r, "\partial_{n+1}^{\mathcal{U}}"'] & H_n^{\mathcal{U}}(A) \arrow[r, "H_n^{\mathcal{U}}(\iota_A)"'] \arrow[u, "H_n(j_\bullet^A)", "\cong"'] & H_n^{\mathcal{U}}(X) \arrow[r, "H_n^{\mathcal{U}}(\pi_\bullet)"'] \arrow[u, "H_n(j_\bullet^X)", "\cong"'] & H_n^{\mathcal{U}}(X,A) \arrow[r, "\partial_{n-1}^{\mathcal{U}}"'] \arrow[u, "H_n(j_\bullet^{(X,A)})"] & H_{n-1}^{\mathcal{U}}(A) \arrow[r, "H_{n-1}^{\mathcal{U}}(\iota_A)"'] \arrow[u, "H_{n-1}(j_\bullet^A)", "\cong"'] & H_{n-1}^{\mathcal{U}}(X) \arrow[r, "H_{n-1}^{\mathcal{U}}(\pi_\bullet)"'] \arrow[u, "H_{n-1}(j_\bullet^X)", "\cong"'] & \ldots
\end{tikzcd}$$

With the 5-Lemma (Exercise 11) it follows that $H_n(j_\bullet^{(X,A)})$ is an isomorphism for all $n \in \mathbb{N}_0$. □

3.5 Excision and the Mayer-Vietoris sequence

We will now apply the results about $\mathcal{U}$-small n-chains from the last section to obtain tools for computing homologies in terms of a cover. The first result is the excision theorem. It states that the relative homologies $H_n(X, A)$ with respect to a subspace $A \subset X$ remain unchanged when a subspace $U \subset A$ with $\overline{U} \subset \mathring{A}$ is removed from A and X.

Theorem 3.5.1 (excision):
Let (X, A) be a pair of topological spaces and $U \subset A$ a subspace with $\overline{U} \subset \mathring{A}$. Then the inclusion $i : (X \setminus U, A \setminus U) \to (X, A)$ induces isomorphisms

$$H_n(i) : H_n(X \setminus U, A \setminus U) \xrightarrow{\sim} H_n(X, A).$$

Proof:
Because $\overline{U} \subset \mathring{A}$ the sets $\mathring{A}$ and $X \setminus \overline{U}$ form an open cover $\mathcal{U} = \{\mathring{A}, X \setminus \overline{U}\}$ of X. All singular simplexes in a $\mathcal{U}$-small n-chain $x \in S_n^{\mathcal{U}}(X)$ have their images in $\mathring{A} \subset A$ or in $X \setminus \overline{U} \subset X \setminus U$. Thus, we can express $S_n^{\mathcal{U}}(X)$ as a (not necessarily direct) sum

$$S_n^{\mathcal{U}}(X) = S_n^{\mathcal{U}}(A) + S_n^{\mathcal{U}}(X \setminus U).$$

By Noether's isomorphism theorem we have for all $n \in \mathbb{N}_0$ a canonical isomorphisms

$$S_n^{\mathcal{U}}(X,A) = \frac{S_n^{\mathcal{U}}(X)}{S_n^{\mathcal{U}}(A)} = \frac{S_n^{\mathcal{U}}(A) + S_n^{\mathcal{U}}(X \setminus U)}{S_n^{\mathcal{U}}(A)} \cong \frac{S_n^{\mathcal{U}}(X \setminus U)}{S_n^{\mathcal{U}}(X \setminus U) \cap S_n^{\mathcal{U}}(A)} = \frac{S_n^{\mathcal{U}}(X \setminus U)}{S_n^{\mathcal{U}}(A \setminus U)} = S_n^{\mathcal{U}}(X \setminus U, A \setminus U).$$

Explicitly, these isomorphisms are given by

$$\phi_n : S_n^{\mathcal{U}}(X \setminus U, A \setminus U) \to S_n^{\mathcal{U}}(X, A), \quad x + S_n^{\mathcal{U}}(A \setminus U) \mapsto x + S_n^{\mathcal{U}}(A),$$

which shows that they define an isomorphism of chain complexes

$$\phi_\bullet : S^{\mathcal{U}}_\bullet(X \setminus U, A \setminus U) \to S^{\mathcal{U}}_\bullet(X, A).$$

and that we have a commuting diagram of chain complexes

$$\begin{tikzcd} S^{\mathcal{U}}_\bullet(X \setminus U, A \setminus U) \arrow[r, "\phi_\bullet", "\cong"'] \arrow[d, "j_\bullet^{(X\setminus U, A\setminus U)}"'] & S^{\mathcal{U}}_\bullet(X, A) \arrow[d, "j_\bullet^{(X,A)}"] \\ S_\bullet(X \setminus U, A \setminus U) \arrow[r, "S_\bullet(i)"'] & S_\bullet(X, A), \end{tikzcd}$$

where $j_\bullet^{(X,A)} : S^{\mathcal{U}}_\bullet(X, A) \to S_\bullet(X, A)$ and $j_\bullet^{(X\setminus U, A\setminus U)} : S^{\mathcal{U}}_\bullet(X \setminus U, A \setminus U) \to S_\bullet(X \setminus U, A \setminus U)$ denote the inclusions for the subcomplexes of $\mathcal{U}$-small chains. Applying the homology functors yields commuting diagrams

$$\begin{tikzcd} H^{\mathcal{U}}_n(X \setminus U, A \setminus U) \arrow[r, "H_n(\phi_\bullet)", "\cong"'] \arrow[d, "H_n(j_\bullet^{(X\setminus U, A\setminus U)})"', "\cong"] & H^{\mathcal{U}}_n(X, A) \arrow[d, "\cong"', "H_n(j_\bullet^{(X,A)})"] \\ H_n(X \setminus U, A \setminus U) \arrow[r, "H_n(i)"'] & H_n(X, A), \end{tikzcd}$$

in which the vertical arrows are isomorphisms by Proposition 3.4.13 and the top arrow is an isomorphism, because $\phi_\bullet$ is an isomorphism of chain complexes. This shows that $H_n(i)$ is an isomorphism as well. □

The excision theorem is very powerful, and we will use it many times in the following. As a first application we address a question arising from the relative homologies in Section 3.3: Given a topological space X and a subspace $A \subset X$, how are the relative homologies $H_n(X, A)$ related to the homologies of the quotient space X/A? This question is difficult to answer in general, but has a simple and intuitive answer under mild assumptions on the pair (X, A).

Definition 3.5.2: A pair of topological spaces (X, A) is called a **good pair**, if there is a subspace $B \subset X$ with $\overline{A} \subset \mathring{B}$ such that A is a strong deformation retract of B.

Suppose now that (X, A) is a good pair and recall that the quotient space X/A is obtained by identifying points of X with the equivalence relation $a \sim a'$ for all $a, a' \in A$. The associated canonical surjection $p : X \to X/A$ sends all points in $A \subset X$ to a single point $p_A = p(A) \in X/A$.

Proposition 3.5.3: If (X, A) is a good pair, then for all $n \in \mathbb{N}_0$ the relative homologies coincide with the reduced homologies of the quotient X/A

$$H_n(X, A) \cong \tilde{H}_n(X/A) := H_n(X/A, \{p_A\}).$$

Proof:
Let $B \subset X$ such that $\overline{A} \subset \mathring{B}$ and A is a strong deformation retract of B. Then we have the following commuting diagram in Top(2)

$$\begin{tikzcd} (X, A) \arrow[r, "j"] \arrow[d, "p'"] & (X, B) \arrow[d, "p"] & (X \setminus A, B \setminus A) \arrow[l, "i"'] \arrow[d, "p''"] \\ (X/A, \{p_A\}) \arrow[r, "j'"] & (X/A, B/A) & ((X/A) \setminus \{p_A\}, (B/A) \setminus \{p_A\}), \arrow[l, "i'"'] \end{tikzcd}$$

where the three morphisms on the vertical arrows are induced by the canonical surjection $p : X \to X/A$ and its restriction to $X \setminus A$, the morphisms j and j' by the identity maps id_X and $\mathrm{id}_{X/A}$ and the morphisms i and i' by the inclusions $\iota : X \setminus A \to X$ and $\iota' : (X/A) \setminus \{p_A\} \to X/A$.

Applying the relative homology functor from Definition 3.3.5 yields the commuting diagram

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "H_n(j)"] \arrow[d, "H_n(p')"] & H_n(X,B) \arrow[d, "H_n(p)"] & H_n(X \setminus A, B \setminus A) \arrow[l, "H_n(i)"'] \arrow[d, "H_n(p'')"] \\
H_n(X/A, \{p_A\}) \arrow[r, "H_n(j')"] & H_n(X/A, B/A) & H_n((X/A) \setminus \{p_A\}, (B/A) \setminus \{p_A\}). \arrow[l, "H_n(i')"']
\end{tikzcd}$$

We claim that all arrows in this diagram are isomorphisms:

- The group homomorphism $H_n(p'')$ is an isomorphism, because p'' is a homeomorphism.

- The group homomorphisms $H_n(i)$ and $H_n(i')$ are isomorphisms by the excision axiom, the latter, because $\overline{A} \subset \mathring{B}$ implies $\overline{\{p_A\}} \subset (B/A)^\circ$.

- This implies that $H_n(p)$ is an isomorphism as well.

- To show that $H_n(j)$ is an isomorphism, apply Theorem 3.3.7 to the morphism of pairs $j : (X, A) \to (Y, B)$. This yields a commuting diagram with exact rows

$$\begin{tikzcd}
\ldots \arrow[r] & H_n(A) \arrow[r, "H_n(\iota)"] \arrow[d, "\cong"] & H_n(X) \arrow[r, "H_n(\pi_\bullet)"] \arrow[d, "\mathrm{id}"] & H_n(X,A) \arrow[r, "\partial_n"] \arrow[d, "H_n(j)"] & H_{n-1}(A) \arrow[r, "H_{n-1}(\iota)"] \arrow[d, "\cong"] & H_{n-1}(X) \arrow[r] \arrow[d, "\mathrm{id}"] & \ldots \\
\ldots \arrow[r] & H_n(B) \arrow[r, "H_n(\iota')"'] & H_n(X) \arrow[r, "H_n(\pi'_\bullet)"'] & H_n(X,B) \arrow[r, "\partial'_n"'] & H_{n-1}(B) \arrow[r, "H_{n-1}(\iota')"'] & H_{n-1}(X) \arrow[r] & \ldots,
\end{tikzcd}$$

  where the vertical arrows between $H_k(A)$ and $H_k(B)$ are isomorphisms, because A is a strong deformation retract of B. The 5-Lemma (Exercise 11) implies that $H_n(j)$ is an isomorphism for $n \in \mathbb{N}$. For $n = 0$ we have an analogous diagram, in which the homologies in the last two columns are zero and obtain that $H_0(j)$ is an isomorphism.

- A retraction $r : B \to A$ and a homotopy $h : [0,1] \times B \to B$ from $\iota_A \circ r$ to id_B with $h(t,a) = a$ for $t \in [0,1]$ and $a \in A$ induce a retraction $r' : B/A \to \{p_A\}$ and homotopy $h' : [0,1] \times B/A \to B/A$ from $\iota_{\{p_A\}} \circ r'$ to $\mathrm{id}_{B/A}$ with $h'(t, p_A) = p_A$ for all $t \in [0,1]$. This implies that $\{p_A\}$ is a strong deformation retract of B/A.

  Applying Theorem 3.3.7 to the morphism of pairs $j' : (X/A, \{p_A\}) \to (X/A, B/A)$ and using the 5-Lemma then shows that $H_n(j')$ is an isomorphism.

- As all other arrows are isomorphisms, $H_n(p')$ is an isomorphism as well.

Hence, we have $H_n(X,A) \cong H_n(X/A, \{p_A\})$ for all $n \in \mathbb{N}_0$. By Example 3.3.11 these are the reduced homologies of X/A. $\square$

The next application of $\mathcal{U}$-small n-chains can be viewed as the homological counterpart of the theorem by **Seifert and van Kampen**. Recall from topology that given an open cover $\mathcal{U} = \{U_1, U_2\}$ of a path connected topological space X such that U_1, U_2 and $U_1 \cap U_2$ are all path connected, the fundamental group $\pi_1(X)$ is given as a quotient $\pi_1(X) = \pi_1(U_1) \star \pi_1(U_2)/N$, where $\star$ denotes the free product of groups and N is a normal subgroup. The normal subgroup $N \subset \pi_1(U_1) \star \pi_1(U_2)$ is generated by the elements $\pi_1(i_1)(\lambda) \cdot \pi_1(i_2)(\lambda)^{-1}$ for $\lambda \in \pi_1(U_1 \cap U_2)$.

If we apply the abelisation functor Ab : Grp $\to$ Ab and use the Huréwicz isomorphism from Theorem 3.1.11 we obtain

$$H_1(X) = \frac{H_1(U_1) \oplus H_1(U_2)}{H_1(U_1 \cap U_2)},$$

where the quotient is given by the map

$$H_1(U_1 \cap U_2) \to H_1(U_1) \oplus H_1(U_2), \quad [z] \mapsto (H_1(i_1)[z], -H_1(i_2)[z]).$$

This suggests that the homologies should be related by a long exact sequence that can be viewed as the homological counterpart of the theorem by Seifert and van Kampen.

Theorem 3.5.4 (Mayer-Vietoris sequence):
Let X be a topological space and $U_1, U_2 \subset X$ open subsets with $X = U_1 \cup U_2$. Set $U_{12} = U_1 \cap U_2$ and denote by $i_k : U_{12} \to U_k$ and $j_k : U_k \to X$ the inclusions.

Then there is an exact sequence, the **Mayer-Vietoris sequence**

$$\ldots \xrightarrow{\partial_{n+1}} H_n(U_{12}) \xrightarrow{(H_n(i_1), -H_n(i_2))} H_n(U_1) \oplus H_n(U_2) \xrightarrow{H_n(j_1)+H_n(j_2)} H_n(X) \xrightarrow{\partial_n} H_{n-1}(U_{12}) \to \ldots$$

Proof:
The sets U_1, U_2 define an open cover $\mathcal{U} = \{U_1, U_2\}$ of X. Every $\mathcal{U}$-small n-chain $x \in S_n^{\mathcal{U}}(X)$ is a sum $x = x_1 + x_2$ of an n-chain $x_1 \in S_n(U_1)$ and $x_2 \in S_n(U_2)$. This implies that the maps

$$S_n(j_1) + S_n(j_2) : S_n(U_1) \oplus S_n(U_2) \to S_n^{\mathcal{U}}(X)$$

are surjective. It follows that the chain maps

$$(S_\bullet(i_1), -S_\bullet(i_2)) : S_\bullet(U_{12}) \to S_\bullet(U_1) \oplus S_\bullet(U_2) \quad S_\bullet(j_1) + S_\bullet(j_2) : S_\bullet(U_1) \oplus S_\bullet(U_2) \to S_\bullet^{\mathcal{U}}(X)$$

define a short exact sequence of chain complexes

$$0 \to S_\bullet(U_{12}) \xrightarrow{(S_\bullet(i_1), -S_\bullet(i_2))} S_\bullet(U_1) \oplus S_\bullet(U_2) \xrightarrow{S_\bullet(j_1)+S_\bullet(j_2)} S_\bullet^{\mathcal{U}}(X) \to 0.$$

Proposition 2.2.7 yields an associated long exact homology sequence

$$\ldots \to H_{n+1}^{\mathcal{U}}(X) \xrightarrow{\partial_{n+1}} H_n(U_{12}) \xrightarrow{(H_n(i_1), -H_n(i_2))} H_n(U_1) \oplus H_n(U_2) \xrightarrow{H_n(j_1)+H_n(j_2)} H_n^{\mathcal{U}}(X) \xrightarrow{\partial_n} H_{n-1}(U_{12}) \to \ldots.$$

By Proposition 3.4.13 we have an isomorphism $H_n(j_\bullet^X) : H_n^{\mathcal{U}}(X) \xrightarrow{\sim} H_n(X)$ for all $n \in \mathbb{N}_0$. $\square$

The Mayer-Vietoris sequence allows one to compute the homologies of a topological space by covering it with two open subsets, which are usually chosen in such a way that their homologies and the homologies of their intersection are known and as simple as possible. A similar procedure also works for relative homologies. In this case, one wants to compute the relative homologies $H_n(X, A)$ by decomposing the subspace A as the union $A = A_1 \cup A_2$ of open subspaces $A_1, A_2 \subset A$ for which the relative homologies are of a particularly simple form. This yields the relative version of the Mayer-Vietoris sequence.

Proposition 3.5.5 (relative Mayer-Vietoris sequence):
Let X be a topological space and $A_1, A_2 \subset X$ subspaces that are open in $A_1 \cup A_2$. Denote by $i_k : (X, A_1 \cap A_2) \to (X, A_k)$ and $j_k : (X, A_k) \to (X, A_1 \cup A_2)$ the inclusions in Top(2).

Then there is an exact sequence, the **relative Mayer-Vietoris sequence**,

$$\ldots \xrightarrow{\partial_{n+1}} H_n(X, A_1 \cap A_2) \xrightarrow{(H_n(i_1), -H_n(i_2))} H_n(X, A_1) \oplus H_n(X, A_2) \xrightarrow{H_n(j_1)+H_n(j_2)} H_n(X, A_1 \cup A_2) \xrightarrow{\partial_n} \ldots$$

Proof:
The set $\mathcal{U} = \{A_1, A_2\}$ is an open cover of $A_1 \cup A_2$. The inclusions $\iota'_n : S_n^{\mathcal{U}}(A_1 \cup A_2) \to S_n(X)$ define chain maps $\iota'_\bullet : S_\bullet^{\mathcal{U}}(A_1 \cup A_2) \to S_\bullet(X)$ and $\pi'_\bullet : S_\bullet(X) \to S_\bullet(X)/S_\bullet^{\mathcal{U}}(A_1 \cup A_2)$, a chain map $\psi_\bullet : S_\bullet(X)/S_\bullet^{\mathcal{U}}(A_1 \cup A_2) \to S_\bullet(X)/S_\bullet(A_1 \cup A_2)$ and a commuting diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet^{\mathcal{U}}(A_1 \cup A_2) \arrow[r, "\iota'_\bullet"] \arrow[d, "j_\bullet^{A_1 \cup A_2}"] & S_\bullet(X) \arrow[r, "\pi'_\bullet"] \arrow[d, "\mathrm{id}"] & S_\bullet(X)/S_\bullet^{\mathcal{U}}(A_1 \cup A_2) \arrow[r] \arrow[d, "\psi_\bullet"] & 0 \\
0 \arrow[r] & S_\bullet(A_1 \cup A_2) \arrow[r, "S_\bullet(\iota)"'] & S_\bullet(X) \arrow[r, "\pi_\bullet"'] & S_\bullet(X)/S_\bullet(A_1 \cup A_2) \arrow[r] & 0.
\end{tikzcd}$$

Proposition 2.2.7 yields a commuting diagram

$$\begin{tikzcd}
\ldots \arrow[r] & H_n^{\mathcal{U}}(A_1 \cup A_2) \arrow[r, "H_n(\iota'_\bullet)"] \arrow[d, "H_n(j_\bullet^{A_1 \cup A_2})", "\cong"'] & H_n(X) \arrow[r, "H_n(\pi'_\bullet)"] \arrow[d, "\mathrm{id}"] & H_n(S_\bullet(X)/S_\bullet^{\mathcal{U}}(A_1 \cup A_2)) \arrow[r, "\partial_n"] \arrow[d, "H_n(\psi_\bullet)"] & H_{n-1}^{\mathcal{U}}(A_1 \cup A_2) \arrow[r, "H_{n-1}(\iota'_\bullet)"] \arrow[d, "H_{n-1}(j_\bullet^{A_1 \cup A_2})", "\cong"'] & H_{n-1}(X) \arrow[r] \arrow[d, "\mathrm{id}"] & \ldots \\
\ldots \arrow[r] & H_n(A_1 \cup A_2) \arrow[r, "H_n(\iota)"] & H_n(X) \arrow[r, "H_n(\pi_\bullet)"] & H_n(X, A_1 \cup A_2) \arrow[r, "\partial_n"] & H_{n-1}(A_1 \cup A_2) \arrow[r, "H_{n-1}(\iota)"] & H_{n-1}(X) \arrow[r] & \ldots
\end{tikzcd}$$

in which the arrows $H_k(j_\bullet^{A_1 \cup A_2})$ are isomorphisms by Proposition 3.4.13. With the 5-Lemma (Exercise 11) it follows that $H_k(\psi_\bullet)$ is an isomorphism for all $k \in \mathbb{N}_0$.

We show that we have a short exact sequence of chain complexes

$$0 \to S_\bullet(X, A_1 \cap A_2) \xrightarrow{i_\bullet} S_\bullet(X, A_1) \oplus S_\bullet(X, A_2) \xrightarrow{p_\bullet} S_\bullet(X)/S_\bullet^{\mathcal{U}}(A_1 \cup A_2) \to 0 \qquad (24)$$

induced by the following diagram in which all columns and the first and second row are exact

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_n(A_1 \cap A_2) \arrow[r, "x \mapsto (x,-x)"] \arrow[d, "S_n(\iota_{12})"] & S_n(A_1) \oplus S_n(A_2) \arrow[r, "(x,y) \mapsto x+y"] \arrow[d, "(S_n(\iota_1), S_n(\iota_2))"] & S_n^{\mathcal{U}}(A_1 \cup A_2) \arrow[r] \arrow[d, "\iota'_n"] & 0 \\
0 \arrow[r] & S_n(X) \arrow[r, "x \mapsto (x,-x)"] \arrow[d, "\pi_n^{12}"] & S_n(X) \oplus S_n(X) \arrow[r, "(x,y) \mapsto x+y"] \arrow[d, "(\pi_n^1, \pi_n^2)"] & S_n(X) \arrow[r] \arrow[d, "\pi'_n"] & 0 \\
0 \arrow[r] & S_n(X, A_1 \cap A_2) \arrow[r, "i_n"] \arrow[d] & S_n(X, A_1) \oplus S_n(X, A_2) \arrow[r, "p_n"] \arrow[d] & S_n(X)/S_n^{\mathcal{U}}(A_1 \cup A_2) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The maps i_n and p_n are induced by the requirement that the diagram commutes, and the exactness of the last row follows by the 9-lemma (Exercise 12).

The short exact sequence (24) induces a long exact homology sequence by Proposition 2.2.7. With the isomorphism $H_n(\psi_\bullet) : H_n(S_\bullet(X)/S_\bullet^{\mathcal{U}}(A_1 \cup A_2)) \to H_n(X, A_1 \cup A_2)$ the claim follows

$$\ldots \xrightarrow{\partial_{n+1}} H_n(X, A_1 \cap A_2) \xrightarrow{H_n(i_\bullet)} H_n(X, A_1) \oplus H_n(X, A_2) \xrightarrow{H_n(p_\bullet)} \underbrace{H_n(S_\bullet(X)/S_\bullet^{\mathcal{U}}(A_1 \cup A_2))}_{\cong H_n(X, A_1 \cup A_2)} \xrightarrow{\partial_n} \ldots$$

□

We now illustrate the Mayer-Vietoris sequence with a number of examples. Its relative version will play an important role in certain proofs later on. The simplest non-trivial homologies that can be computed with the Mayer-Vietoris sequence are homologies of spheres.

Example 3.5.6 (homology groups of spheres):
We consider for $k \in \mathbb{N}$ the k-sphere S^k and the open subsets $U_\pm = S^k \setminus \{\mp e_{k+1}\}$.

- Via the stereographic projection, the subspaces $U_\pm$ are homeomorphic to the k-disc D^k. As D^k is contractible, this gives $H_n(U_\pm) = 0$ for $n \in \mathbb{N}$.
- Because $U_\pm$ is path connected we have $H_0(U_\pm) = \mathbb{Z}$.
- The intersection $U_+ \cap U_- = S^k \setminus \{e_{k+1}, -e_{k+1}\}$ is homotopy equivalent to S^{k-1}. This gives $H_n(U_+ \cap U_-) = H_n(S^{k-1})$ for all $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$.
- $H_0(U_+ \cap U_-) = H_0(S^{k-1}) \cong \mathbb{Z}$ for $k \geq 2$ and $H_0(U_+ \cap U_-) = H_0(S^{k-1}) \cong \mathbb{Z} \oplus \mathbb{Z}$ for $k = 1$, as S^{k-1} has a single path component for $k \geq 2$ and two path components for $k = 1$.

1. For all $n \geq 2$ and $k \in \mathbb{N}$ the exactness of the Mayer-Vietoris sequence implies that the connecting homomorphism $\partial_n : H_n(S^k) \xrightarrow{\sim} H_{n-1}(S^{k-1})$ is an isomorphism.

$$\ldots \to \underbrace{H_n(D^k) \oplus H_n(D^k)}_{=0} \to H_n(S^k) \xrightarrow{\partial_n} H_{n-1}(S^{k-1}) \to \underbrace{H_{n-1}(D^k) \oplus H_{n-1}(D^k)}_{=0} \to \ldots$$

2. The end of the Mayer-Vietoris sequence for $k \geq 2$ is given by

$$\ldots \to \underbrace{H_1(D^k) \oplus H_1(D^k)}_{=0} \to H_1(S^k) \xrightarrow{\partial_1} \underbrace{H_0(S^{k-1})}_{\cong\mathbb{Z}} \xrightarrow{\phi} \underbrace{H_0(D^k) \oplus H_0(D^k)}_{\cong\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{\psi} \underbrace{H_0(S^k)}_{\cong\mathbb{Z}} \to 0$$

with an injective connection homomorphism $\partial_1 : H_1(S^k) \to H_0(S^{k-1})$ and the maps

$$\phi : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z},\ z \to (z, -z) \qquad\qquad \psi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z},\ (z_1, z_2) \mapsto z_1 + z_2.$$

As ϕ is injective, we have $\operatorname{im} \partial_1 = \ker \phi = 0$ and hence $H_1(S^k) \cong \operatorname{im} \partial_1 = 0$.

3. For $k = 1$ the end of the Mayer-Vietoris sequence is

$$\ldots \to \underbrace{H_1(D^k) \oplus H_1(D^k)}_{=0} \to H_1(S^1) \xrightarrow{\partial_1} \underbrace{H_0(S^0)}_{\cong\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{\chi} \underbrace{H_0(D^k) \oplus H_0(D^k)}_{\cong\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{\psi} \underbrace{H_0(S^k)}_{\cong\mathbb{Z}} \to 0.$$

with an injective connecting homomorphism $\partial_1 : H_1(S^1) \to H_0(S^0)$ and the map χ given by $\chi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, $(z_1, z_2) \mapsto (z_1 + z_2, -(z_1 + z_2))$. As $\ker \chi = \{(z, -z) \mid z \in \mathbb{Z}\} \cong \mathbb{Z}$, we obtain $H_1(S^1) \cong \operatorname{im} \partial_1 = \ker \chi \cong \mathbb{Z}$.

Combining these results and using the results from Example 3.3.10 yields

$$H_n(S^k) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = k = 0 \\ \mathbb{Z} & n = k \in \mathbb{N} \text{ or } n = 0, k \in \mathbb{N} \\ 0 & \text{else.} \end{cases} \qquad\qquad H_n(D^k, S^{k-1}) = \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k. \end{cases}$$

Example 3.5.6 already demonstrates the usefulness of the Mayer-Vietoris sequence in the computation of homologies. Another example that shows this is the wedge sum of pointed topological spaces. The wedge sum of a family $(X_i)_{i\in I}$ of topological spaces X_i with chosen basepoints $x_i \in X_i$ is obtained by taking their disjoint union $\amalg_{i\in I}X_i$ with the inclusions $\iota_j : X_j \to \amalg_{i\in I}X_i$ and then identifying all points $\iota_j(x_j)$ to a single point. It is thus given as a quotient $\vee_{i\in I}X_i = \amalg_{i\in I}X_i/\sim$ and equipped with a canonical surjection $\pi : \amalg_{i\in I}X_i \to \vee_{i\in I}X_i$.

Definition 3.5.7: Let $(X_i)_{i\in I}$ be a family of topological spaces with basepoints $x_i \in X_i$. Their **wedge sum** is the quotient space

$$\vee_{i\in I}X_i = (\amalg_{i\in I}X_i)/\sim \qquad \iota_i(x_i) \sim \iota_j(x_j) \quad \forall i, j \in I.$$

Remark 3.5.8:

1. The wedge sum has the following **universal property** (Exercise 34):
   The maps $i_j = \pi \circ \iota_j : X_j \xrightarrow{\iota_j} \amalg_{i\in I}X_i \xrightarrow{\pi} \vee_{i\in I}X_i$ are continuous for all $j \in I$.
   For every family $(f_i)_{i\in I}$ of continuous maps $f_i : X_i \to Y$ with $f_i(x_i) = f_j(x_j)$ for all $i, j \in I$ there is a unique continuous map $f : \vee_{i\in I}X_i \to Y$ with $f \circ i_j = f_j$ for all $j \in I$.

2. The universal property of the wedge sum defines continuous projection maps

$$p_j : \vee_{i\in I}X_i \to X_j \quad \text{with} \quad p_j \circ i_j = \mathrm{id}_{X_j} \text{ and } p_j \circ i_k = x_j : X_k \to X_j, x \mapsto x_j \text{ for } j \neq k.$$

Given a finite family of topological spaces X_i with basepoints $x_i \in X_i$, we can compute the homologies of their wedge sum via the Mayer-Vietoris sequence, as long as they satisfy a mild regularity condition. This regularity condition states that every basepoint $x_i \in X_i$ must have a neighbourhood $x_i \in U_i \subset X_i$ that deformation retracts to a point. Note that this condition is always satisfied for subsets of $\mathbb{R}^n$, where we can choose $U_i = B_\epsilon(x_i)$ for a sufficiently small $\epsilon > 0$ and then use the fact that $B_\epsilon(x_i)$ is convex to construct a retraction and a homotopy.

Proposition 3.5.9: Let $X_1, \ldots, X_k$ be topological spaces with basepoints $x_i \in X_i$. Suppose that (X_i, x_i) is **well-pointed**: every basepoint x_i has a neighbourhood $U_i \subset X_i$ such that $\{x_i\}$ is a strong deformation retract of U_i. Then we have for all $n \in \mathbb{N}$ mutually inverse isomorphisms

$$\begin{aligned}
&(H_n(p_1), \ldots, H_n(p_k)) : H_n(X_1 \vee \ldots \vee X_n) \xrightarrow{\sim} H_n(X_1) \oplus \ldots \oplus H_n(X_k)\\
&H_n(i_1) + \ldots + H_n(i_k) : H_n(X_1) \oplus \ldots \oplus H_n(X_k) \xrightarrow{\sim} H_n(X_1 \vee \ldots \vee X_k).
\end{aligned}$$

Proof:
The claim follows by induction over k. We prove the case $k = 2$. Let $x_1 \in U_1 \subset X_1$ and $x_2 \in U_2 \subset X_2$ neighbourhoods such that $\{x_i\}$ is a deformation retract of U_i. Set $V_1 = X_1 \vee U_2$ and $V_2 = U_1 \vee X_2$. Then we have $X := X_1 \vee X_2 = V_1 \cup V_2$ and $V_{12} := V_1 \cap V_2 = U_1 \vee U_2$.

- The inclusions, retractions and homotopies for x_i and U_i induce inclusions, retractions and homotopies for $U_1 \vee U_2$, which implies $V_1 \cap V_2 = U_1 \vee U_2 \simeq \bullet$.
- The inclusions, retractions and homotopies for x_i and U_i induce inclusions, retractions and homotopies for V_1 and V_2, which implies $V_1 \simeq X_1$ and $V_2 \simeq X_2$.

1. The Mayer-Vietoris sequence then gives for all $n \geq 2$

$$\begin{tikzcd}
\ldots \arrow[r, "\partial_{n+1}"] & H_n(V_{12}) \arrow[r] \arrow[d, "\cong"'] & H_n(V_1) \oplus H_n(V_2) \arrow[r] \arrow[d, "\cong"'] & H_n(X) \arrow[r, "\partial_n"] \arrow[d, "\mathrm{id}"] & H_{n-1}(V_{12}) \arrow[r] \arrow[d, "\cong"'] & \ldots \\
\ldots \arrow[r, "\partial_{n+1}"'] & 0 \arrow[r] & H_n(X_1) \oplus H_n(X_2) \arrow[r, "H_n(i_1)+H_n(i_2)"'] & H_n(X) \arrow[r, "\partial_n"'] & 0 \arrow[r] & \ldots
\end{tikzcd}$$

and implies the group homomorphism $H_n(i_1) + H_n(i_2) : H_n(X_1) \oplus H_n(X_2) \to H_n(X_1 \vee X_2)$ is an isomorphism. For $n = 1$ we obtain

$$\begin{tikzcd}
\ldots \arrow[r, "\partial_{n+1}"] & H_1(V_{12}) \arrow[r] \arrow[d, "\cong"'] & H_1(V_1) \oplus H_1(V_2) \arrow[r] \arrow[d, "\cong"'] & H_1(X) \arrow[r, "\partial_n"] \arrow[d, "\mathrm{id}"] & H_0(V_{12}) \arrow[r] \arrow[d, "\cong"'] & \ldots \\
\ldots \arrow[r, "\partial_{n+1}"'] & 0 \arrow[r] & H_1(X_1) \oplus H_1(X_2) \arrow[r, "H_1(i_1)+H_1(i_2)"'] & H_1(X) \arrow[r, "\partial_n"'] & \mathbb{Z} \arrow[r, "{[p]\mapsto([p],-[p])}"'] & \ldots
\end{tikzcd}$$

As $H_0(V_{12}) = \mathbb{Z}$ and the map $H_0(V_{12}) \to H_0(V_1) \oplus H_0(V_2)$, $[p] \mapsto ([p], -[p])$ is injective, it follows that $H_1(i_1) + H_1(i_2)$ is an isomorphism as well.

2. That $(H_n(p_1), H_n(p_2)) : H_n(X) \to H_n(X_1) \oplus H_n(X_2)$ is inverse to $H_n(i_1) + H_n(i_2)$ follows directly from the definitions, which imply $H_n(p_j) \circ H_n(i_j) = H_n(\mathrm{id}_{X_j}) = \mathrm{id}_{H_n(X_j)}$ for $j = 1, 2$ and $H_n(p_j) \circ H_n(i_k) = H_n(x \mapsto x_j) = 0 : H_n(X_k) \to H_n(X_j)$ for $j \neq k$. □

3.6 Homologies of spheres: topological applications

In this section, we show how results on the homologies of spheres can be applied to topological questions. We will first prove a number of elementary topological applications of Example 3.5.6. Recall from Example 3.5.6 that the homologies of the n-spheres are given by

$$H_n(S^k) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = k = 0 \\ \mathbb{Z} & n = k \in \mathbb{N} \text{ or } n = 0, k \in \mathbb{N} \\ 0 & \text{else.} \end{cases}$$

As an immediate consequence, we have that for ≥ 1 the n-sphere S^n cannot be homotopy equivalent to a point, as $H_n(S^n) \cong \mathbb{Z}$ and $H_n(\bullet) = 0$. As homotopy equivalent spaces have isomorphic homology groups, it follows that S^n and $\bullet$ are not homotopy equivalent. For $n = 0$, this follows already with elementary topology as S^0 has two connected components and the one point space $\bullet$ only one. Likewise, for $m < n$ we have $H_n(S^m) = 0 \ncong \mathbb{Z} = H_n(S^n)$ and hence S^m and S^n cannot be homotopy equivalent. This fact has interesting consequences.

Corollary 3.6.1:

1. The n-sphere S_n is not contractible for any $n \in \mathbb{N}_0$.
2. For $n \neq m$ the n-spheres S^n and S^m are not homotopy equivalent.

Corollary 3.6.2: For $n \neq m$ the spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.

Proof:
A homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^m$ induces a homeomorphism $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \setminus \{f(0)\}$. As $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$ and $\mathbb{R}^m \setminus \{f(0)\} \simeq S^{m-1}$ this would imply that S^{n-1} and S^{m-1} are homotopy equivalent, a contradiction to Corollary 3.6.1. $\square$

Corollary 3.6.3: The $(n-1)$-sphere $S^{n-1} = \partial D^n$ is not a retract of D^n.

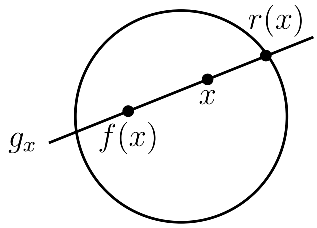
Proof:
For $n = 1$ this follows from the the fact that any continuous map $r : D^1 = [-1, 1] \to \{1, -1\} = S^0$ is constant, as D^1 is connected. This contradicts the condition $r \circ \iota = \mathrm{id}_{S^0}$. For $n \geq 2$ let $\iota : S^{n-1} \to D^n$ be the inclusion and suppose that there is a retraction $r : D^n \to S^{n-1}$ with $r \circ \iota = \mathrm{id}_{S^{n-1}}$. Then we have the following commuting diagram, with the identity on the bottom arrow and the other two arrows composing to the zero map - a contradiction

$$\begin{tikzcd} & H_{n-1}(D^n) = 0 \arrow[dr, "H_{n-1}(r)"] & \\ H_{n-1}(S^{n-1}) \cong \mathbb{Z} \arrow[ur, "H_{n-1}(\iota)"] \arrow[rr, "H_{n-1}(\mathrm{id})=\mathrm{id}"'] & & H_{n-1}(S^{n-1}) \cong \mathbb{Z}. \end{tikzcd}$$

$\square$

Corollary 3.6.4 (Brouwer's fixed point theorem):
For $n \in \mathbb{N}$ any continuous map $f : D^n \to D^n$ has a fixed point.

Proof:
For $n = 1$, the claim follows from the midpoint theorem. The continuous map $g : D^1 \to \mathbb{R}$, $x \mapsto f(x) - x$ satisfies $g(-1) \geq 0$ and $g(1) \leq 0$, which implies that there is an $x \in [-1, 1]$ with $g(x) = 0$, a fixed point of f.



Suppose $n \geq 2$ and $f : D^n \to D^n$ is continuous without a fixed point. Then for all $x \in D^n$ there is a unique straight line g_x through x and $f(x)$. By assigning to $x \in D^n$ the intersection point of g_x with $\partial D^n = S^{n-1}$ that is closer to x than to $f(x)$, we obtain a continuous map $r : D^n \to S^{n-1}$ with $r \circ \iota = \mathrm{id}_{S^{n-1}}$ for the inclusion $\iota : S^{n-1} \to D^n$. Thus, g is a retraction from D^n to S^{n-1}, in contradiction to Corollary 3.6.3. $\square$

Further interesting topological applications arise, when we consider continuous maps between n-spheres. By Definition 3.1.6 any continuous map $f : S^n \to S^n$ induces a group homomorphism $H_n(f) : H_n(S^n) \to H_n(S^n)$. As $H_n(S^n) \cong \mathbb{Z}$, we can fix an isomorphism $\phi : H_n(S^n) \to \mathbb{Z}$ and assign to a continuous map $f : S^n \to S^n$ the group homomorphism $\phi \circ H_n(f) \circ \phi^{-1} : \mathbb{Z} \to \mathbb{Z}$. It is a basic fact from algebra that any group homomorphism $\alpha : \mathbb{Z} \to \mathbb{Z}$ is of the form $\alpha : \mathbb{Z} \to \mathbb{Z}$, $z \mapsto m \cdot z$ for some integer $m \in \mathbb{Z}$.

Definition 3.6.5:
For $n \in \mathbb{N}$ the **mapping degree** of a continuous map $f : S^n \to S^n$ is the number $\deg(f) \in \mathbb{Z}$ defined by the following commuting diagram for any group isomorphism $\phi : H_n(S^n) \to \mathbb{Z}$

$$\begin{tikzcd} H_n(S^n) \arrow[r, "H_n(f)"] \arrow[d, "\phi"', "\cong"] & H_n(S^n) \arrow[d, "\cong"', "\phi"] \\ \mathbb{Z} \arrow[r, "z \mapsto \deg(f)\cdot z"'] & \mathbb{Z}. \end{tikzcd}$$

Clearly, the mapping degree does not depend on the choice of the isomorphism $\phi : H_n(S^n) \to \mathbb{Z}$. If $\psi : H_n(S^n) \to \mathbb{Z}$ is another isomorphism, then composing them yields an isomorphism $\psi \circ \phi^{-1} : \mathbb{Z} \to \mathbb{Z}$, which is given by multiplication with 1 or -1. Applying this isomorphism twice does not change the degree.

Lemma 3.6.6: The mapping degree has the following properties:

1. Homotopic maps have the same degree: if $f \sim g$, then $\deg(f) = \deg(g)$.
2. The identity has degree $\deg(\mathrm{id}_{S_n}) = 1$.
3. The degree is multiplicative: $\deg(g \circ f) = \deg(g) \cdot \deg(f)$.
4. If f is not surjective, then $\deg(f) = 0$.

Proof:
1. follows directly from the fact that $f \sim g$ implies $H_n(f) = H_n(g)$ by Proposition 3.2.3. Claims 2. and 3. follow from the commuting diagrams

$$\begin{tikzcd} H_n(S^n) \arrow[r, "H_n(\mathrm{id}_{S^n})=\mathrm{id}_{H_n(S^n)}"] \arrow[d, "\phi"', "\cong"] & H_n(S^n) \arrow[d, "\cong"', "\phi"] \\ \mathbb{Z} \arrow[r, "z \mapsto z"'] & \mathbb{Z}. \end{tikzcd}
\qquad
\begin{tikzcd} H_n(S^n) \arrow[r, "H_n(f)"'] \arrow[rr, bend left, "H_n(g\circ f)"] \arrow[d, "\cong"', "\phi"] & H_n(S^n) \arrow[r, "H_n(g)"'] \arrow[d, "\cong"', "\phi"] & H_n(S^n) \arrow[d, "\cong"', "\phi"] \\ \mathbb{Z} \arrow[r, "z\mapsto\deg(f)\cdot z"] \arrow[rr, bend right, "z\mapsto\deg(g)\cdot\deg(f)\cdot z"'] & \mathbb{Z} \arrow[r, "z\mapsto\deg(g)\cdot z"] & \mathbb{Z} \end{tikzcd}$$

4. If $f : S^n \to S^n$ is not surjective, then there is a point $p \in S^n \setminus f(S^n)$, and f corestricts to a map $f' : S^n \to S^n \setminus \{p\}$. We then have $\iota \circ f' = f : S^n \to S^n$ for the inclusion $\iota : S^n \setminus \{p\} \to S^n$. As $S^n \setminus \{p\} \cong D^n \simeq \bullet$, we have $H_n(S^n \setminus \{p\}) = 0$ and the commuting diagram

$$\begin{tikzcd} H_n(S^n) \arrow[r, "H_n(f')"'] \arrow[rr, bend left, "H_n(f)"] \arrow[d, "\cong"', "\phi"] & H_n(S^n \setminus \{p\}) \arrow[r, "H_n(\iota)"'] \arrow[d, "\cong"'] & H_n(S^n) \arrow[d, "\cong"', "\phi"] \\ \mathbb{Z} \arrow[r] \arrow[rr, bend right, "z\mapsto 0"'] & 0 \arrow[r] & \mathbb{Z}. \end{tikzcd}$$

□

We now construct continuous maps $f : S^n \to S^n$ with other mapping degrees than 0,1. Note that by Lemma 3.6.6 any such map is a surjective non-identity map.

We first consider the reflection on an n-dimensional linear subspace in $\mathbb{R}^{n+1}$. It is the identity on every vector contained in the subspace and reverses the direction of very vector orthogonal to the subspace. It also restricts to a continuous map from S^n to S^n of mapping degree -1.

Lemma 3.6.7: For all $n \in \mathbb{N}$ the reflection $\phi_n : S^n \to S^n$, $(x_1, \ldots, x_{n+1}) \mapsto (-x_1, \ldots, x_{n+1})$ has mapping degree $\deg(\phi_n) = -1$.

Proof:
For $0 < \epsilon < 1$ we consider the open subsets

$$U_+ = \{x \in S^n \mid x_{n+1} > -\epsilon\} \qquad U_- = \{x \in S^n \mid x_{n+1} < \epsilon\}.$$

with $U_\pm \simeq D^n \simeq \bullet$ and $U_+ \cap U_- = \{x \in S^n \mid -\epsilon < x_{n+1} < \epsilon\} \simeq S^{n-1}$. More specifically, a homotopy equivalence is given by the maps

$$i_n : S^{n-1} \to U_+ \cap U_-, \quad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$$
$$r_n : U_+ \cap U_- \to S^{n-1}, \quad (x_1, \ldots, x_{n+1}) \mapsto (1 - x_{n+1}^2)^{-1/2}(x_1, \ldots, x_n)$$

satisfying $r_n \circ \phi_n = \phi_{n-1} \circ r_n$ and $\phi_n \circ i_n = i_n \circ \phi_{n-1}$. With the Mayer-Vietoris sequence we obtain for $n \geq 2$ the following commuting diagram, in which $H_n(U_\pm) = H_{n-1}(U_\pm) = 0$ and, consequently, ∂_n is an isomorphism

$$\begin{tikzcd}
\ldots \arrow[r] & H_n(U_+) \oplus H_n(U_-) \arrow[r] \arrow[d] & H_n(S^n) \arrow[r, "\partial_n"] \arrow[d, "H_n(\phi_n)"] & H_{n-1}(S^{n-1}) \arrow[r] \arrow[d, "H_n(\phi_{n-1})"] & H_{n-1}(U_+) \oplus H_{n-1}(U_-) \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & H_n(U_+) \oplus H_n(U_-) \arrow[r] & H_n(S^n) \arrow[r, "\partial_n"] & H_{n-1}(S^{n-1}) \arrow[r] & H_{n-1}(U_+) \oplus H_{n-1}(U_-) \arrow[r] & \ldots
\end{tikzcd}$$

This gives $\deg(\phi_n) = \deg(\phi_{n-1}) \ldots = \deg(\phi_1)$ for all $n \in \mathbb{N}$.

For $n = 1$ The Huréwicz Theorem 3.1.11 implies that $H_1(S^1)$ is generated by the homology class of the cycle

$$\sigma : [0, 1] \to S^1, \quad t \mapsto (\sin(2\pi t), -\cos(2\pi t)).$$

As we have $\phi_1 \circ \sigma = \bar{\sigma} : [0, 1] \to S^1$ and hence $H_1(\phi_1)[\sigma] = -[\sigma]$ by the Huréwicz isomorphism. This gives $\deg(\phi_n) = \deg(\phi_1) = -1$ for all $n \in \mathbb{N}$. □

Corollary 3.6.8: The antipodal map $a : S^n \to S^n$, $x \mapsto -x$ has degree $\deg(a) = (-1)^{n+1}$.

Proof:
The antipodal map is the composite $a = \phi_n^{(n+1)} \circ \ldots \circ \phi_n^{(1)}$ of the reflections

$$\phi_n^{(i)} : S^n \to S^n, \quad (x_1, \ldots, x_{n+1}) \to (x_1, \ldots, -x_i, \ldots, x_{n+1}).$$

As in the proof of Lemma 3.6.7, one can show that $\deg(\phi_n^{(i)}) = -1$. With the multiplicativity of the degree from Lemma 3.6.6 this gives

$$\deg(a) = \deg(\phi_n^{(1)}) \cdots \deg(\phi_n^{(n+1)}) = (-1)^{n+1}.$$

□

Corollary 3.6.9: The restriction of any orthogonal linear map $M \in \mathrm{O}(n+1)$ to $S^n \subset \mathbb{R}^{n+1}$ has degree $\deg(M) = \det(M)$.

Proof:
Every orthogonal map $M \in O(n+1)$ can be written as a product $M = \psi_1 \ldots \circ \psi_k$ of reflections $\psi_j : S^n \to S^n$ on planes of dimension n. As in Lemma 3.6.7 one can show that each of them has degree $\deg(\psi_j) = -1$, and from linear algebra we know that $\det(\psi_j) = -1$. This gives $\deg(M) = (-1)^k = \det(M)$. □

To construct continuous maps $f : S^n \to S^n$ with mapping degrees other than 0,1,-1, we require a construction that *adds* the mapping degrees. This is achieved by considering wedge sums of spheres. For this, we equip the n-sphere S^n with the basepoint $e_{n+1} \in S^n$ and consider the wedge sum $S^n \vee S^n$ with respect to these basepoints. We denote for $j = 1, 2$ by $i_j : S^n \to S^n \vee S^n$ and $p_j : S^n \vee S^n \to S^n$ the inclusion and projection maps from Remark 3.5.8.

By the universal property of the wedge sum, the identity maps $\mathrm{id}_{S^n} : S^n \to S^n$ induce a unique map $F : S^n \vee S^n \to S^n$ with $F \circ i_1 = F \circ i_2 = \mathrm{id}_{S^n}$, the *fold map*. Likewise, we have a canonical map $P : S^n \to S^n \vee S^n$ that collapses the equator $S^{n-1} \times \{0\} = \{x \in S^n \mid x_{n+1} = 0\} \cong S^{n-1}$ to a point. This defines a quotient space S^n/S^{n-1} that is homeomorphic to $S^n \vee S^n$ (Exercise). The associated canonical surjection $P : S^n \to S^n/S^{n-1} \cong S^n \vee S^n$ is called the *pinch map* or *pinching map*, because it pinches the sphere S^n in the middle.

Definition 3.6.10:

1. The **pinch map** is the canonical surjection $P : S^n \to S^n/S^{n-1} \cong S^n \vee S^n$,
2. The **fold map** is the continuous map $F : S^n \vee S^n \to S^n$ induced by id_{S^n}.

Let $f_1, f_2 : S^n \to S^n$ be continuous maps with $f_1(e_{n+1}) = f_2(e_{n+1}) = e_{n+1}$. By the universal property of the wedge sum, there is a unique continuous map $f_1 \vee f_2 : S^n \vee S^n \to S^n \vee S^n$, the *wedge sum* of f_1 and f_2, such that

$$p_j \circ (f_1 \vee f_2) \circ i_j = f_j, \qquad p_2 \circ (f_1 \vee f_2) \circ i_1 = p_1 \circ (f_1 \vee f_2) \circ i_2 = e_{n+1} : x \mapsto e_{n+1}. \tag{25}$$

By applying first the pinch map, then the wedge sum of f_1 and f_2 and then the fold map, we obtain a continuous map $F \circ (f_1 \vee f_2) \circ P : S^n \to S^n$.

Lemma 3.6.11: For all continuous maps $f_1, f_2 : S^n \to S^n$ with $f_1(e_{n+1}) = f_2(e_{n+1}) = e_{n+1}$

$$\deg(F \circ (f_1 \vee f_2) \circ P) = \deg(f_1) + \deg(f_2).$$

Proof:
As the fold map is the unique continuous map $F : S^n \vee S^n \to S^n$ with $F \circ i_1 = F \circ i_2 = \mathrm{id}_{S^n}$ from Remark 3.5.8, we have from Remark 3.5.8 and Proposition 3.5.9

$$H_n(F) \circ (H_n(i_1) + H_n(i_2)) : H_n(S^n) \oplus H_n(S^n) \to H_n(S^n), \quad ([z_1], [z_2]) \mapsto [z_1] + [z_2]. \tag{26}$$

The pinch map $P : S^n \to S^n/S^{n-1} \cong S^n \vee S^n$ can be described in coordinates as follows. We parametrise points $x \in S^n$ in terms of a vector $v \in S^{n-1}$ and $t \in [0, 1]$ as $x = (\sin(\pi t)v, \cos(\pi t))$. Then the composites of P with the maps $p_j : S^n \vee S^n \to S^n$ from Remark 3.5.8 are given by

$$p_1 \circ P : S^n \to S^n, \quad (\sin(\pi t)v, \cos(\pi t)) \mapsto \begin{cases} (\sin(2\pi t)v, \cos(2\pi t)) & t \in [0, \frac{1}{2}] \\ -e_{n+1} & t \in [\frac{1}{2}, 1] \end{cases}$$

$$p_2 \circ P : S^n \to S^n, \quad (\sin(\pi t)v, \cos(\pi t)) \mapsto \begin{cases} e_{n+1} & t \in [0, \frac{1}{2}] \\ (\sin(2\pi t - \pi)v, \cos(2\pi t - \pi)) & t \in [\frac{1}{2}, 1] \end{cases}$$

Both, $p_1 \circ P$ and $p_2 \circ P$ are homotopic to the identity map id_{S^n}. Thus, the induced map on the homologies is given by

$$(H_n(p_1), H_n(p_2)) \circ H_n(P) : H_n(S^n) \to H_n(S^n) \oplus H^n(S^n), \quad [z] \mapsto ([z],[z]). \tag{27}$$

The map $f_1 \vee f_2 : S^n \vee S^n \to S^n \vee S^n$ is given by (25), and this implies

$$\begin{aligned}(H_n(p_1), H_n(p_2)) \circ H_n(f_1 \vee f_2) \circ (H_n(i_1) + H_n(i_2)) : H_n(S^n) \oplus H_n(S^n) \to H_n(S^n) \oplus H_n(S^n), \\ ([z_1],[z_2]) \mapsto ([f_1(z_1)],[f_2(z_2)]). \qquad (28)\end{aligned}$$

Combining (26), (27) and (28) and using that $(H_n(i_1)+H_n(i_2))\circ(H_n(p_1), H_n(p_2)) = \mathrm{id}_{H_n(S^n \vee S^n)}$ by Proposition 3.5.9, we obtain

$$\begin{aligned}H_n(F \circ (f_1 \vee f_2) \circ P) =& H_n(F) \circ H_n(f_1 \vee f_2) \circ H_n(P) = H_n(f_1) + H_n(f_2) : H_n(S^n) \to H_n(S^n) \\ & [z] \mapsto ([z],[z]) \mapsto ([f_1(z)],[f_2(z)]) \mapsto [f_1(z)] + [f_2(z)].\end{aligned}$$

This yields $\deg(F \circ (f_1 \vee f_2) \circ P) = \deg(f_1) + \deg(f_2)$. □

Corollary 3.6.12:
For all $n \in \mathbb{N}_0$, $k \in \mathbb{Z}$ there is a continuous map $f_n^{(k)} : S^n \to S^n$ with $\deg(f_n^{(k)}) = k$.

Proof:
Setting $f_n^{(1)} = \mathrm{id}_{S^n}$ and $f_n^{(k)} = F \circ (f_n^{(k-1)} \vee \mathrm{id}_{S_n}) \circ P$ yields $\deg(f_n^{(k)}) = k \in \mathbb{N}$. Composing them with a reflection ϕ_n yields $f_n^{(-k)} = \phi_n \circ f_n^{(k)}$ with $\deg(f_n^{(-k)}) = \deg(\phi_n) \cdot \deg(f_n^{(k)}) = -k$. Any non-surjective continuous map $f_n^{(0)} : S^n \to S^n$ has degree $\deg(f_n^{(0)}) = 0$ by Lemma 3.6.6, 4. □

We will now apply the results on mapping degrees to prove some deceptively simple statements about continuous maps on spheres that are difficult to prove with other methods. Many of them are concerned with fixed points or antipodal points. There is also the famous statement about even-dimensional hedgehogs: Combing such a hedgehog always leads to a bald spot.

Corollary 3.6.13: Let $f, g : S^n \to S^n$ with $f(x) \neq g(x)$ for all $x \in S^n$. Then f is homotopic to $a \circ g$ and $\deg(f) = (-1)^{n+1} \deg(g)$.

Proof:
If $f(x) \neq g(x)$ for all $x \in S^n$ then $0 \neq (1-t)f(x) - tg(x)$ for all $x \in S^n$ and $t \in [0,1]$. This yields the following homotopy from f to $-g = a \circ g$

$$h : [0,1] \times S^n \to S^n, \quad (t,x) \mapsto \frac{(1-t)f(x) - tg(x)}{||(1-t)f(x) - tg(x)||}$$

and implies $\deg(f) = \deg(a \circ g) = (-1)^{n+1} \deg(g)$. □

Corollary 3.6.14: Any continuous map $f : S^n \to S^n$ with $\deg(f) = 0$ has a fixed point $x_+ \in S^n$ with $f(x_+) = x_+$ and an antipodal point $x_- \in S^n$ with $f(x_-) = -x_-$.

Proof:
If $f(x) \neq x = \mathrm{id}_{S^n}(x)$ for all $x \in S^n$, then $\deg(f) = \deg(a) \cdot \deg(\mathrm{id}_{S^n}) = \deg(a) = (-1)^{n+1} \neq 0$ by Corollary 3.6.13. So there is a point $x_+ \in S^n$ with $f(x_+) = x_+$. Likewise, if $f(x) \neq -x = a(x)$ for all $x \in S^n$, then by Corollary 3.6.13 we have $\deg(f) = \deg(a \circ a) = \deg(\mathrm{id}_{S_n}) = 1 \neq 0$. Hence, there is a point $x_- \in S^n$ with $f(x_-) = -x_-$. □

Corollary 3.6.15: For even $n \in \mathbb{N}$ any continuous map $f : S^n \to S^n$ has a fixed point $x_+ \in S^n$ with $f(x_+) = x_+$ or an antipodal point $x_- \in S^n$ with $f(x_-) = -x_-$.

Proof:
As in the proof of Corollary 3.6.14 $f(x) \neq x$ for all $x \in S^n$ implies $\deg(f) = (-1)^{n+1}$ and $f(x) \neq -x$ for all $x \in S^n$ implies $\deg(f) = 1$. As n is even, only one of them can be true. □

Corollary 3.6.16 (combing the hedgehog):
Any tangential vector field to an even-dimensional sphere S^n vanishes in at least one point.

Proof:
In terms of the euclidean scalar product $\langle\ ,\ \rangle$ on $\mathbb{R}^{n+1}$ the tangent space at $x \in S^n$ is given by $T_xS^n = x^\perp = \{y \in \mathbb{R}^{n+1} \mid \langle x, y\rangle = 0\}$. Suppose $V \in TS^n$ is a vector field on S^n with $0 \neq V(x) \in T_xS^n$ for all $x \in X$. Then rescaling V yields a continuous map

$$f : S^n \to S^n, \quad x \mapsto \frac{V(x)}{||V(x)||}.$$

As n is even, by Corollary 3.6.15 there is a point $x_+ \in S^n$ with $f(x_+) = x_+$ or a point $x_- \in S^n$ with $f(x_-) = -x_-$. This would imply $V(x_+) = ||V(x_+)||\, x_+$ or $V(x_-) = -||V(x_-)||\, x_-$, in contradiction to $V(x) \in T_xS^n = x^\perp$ for all $x \in S^n$. □

3.7 The Eilenberg-Steenrod Axioms

In this section we summarise our results on homologies in terms of an axiomatic definition of a homology theory due to Eilenberg and Steenrod. These axioms characterise a homology theory *abstractly*, and we show that singular homology satisfies these axioms.

Recall that singular homology defines

- **functors $H_n : \mathrm{Top}(2) \to \mathrm{Ab}$ for $n \in \mathbb{Z}$:**
  By Definition 3.3.5 we have singular homology functors $H_n : \mathrm{Top}(2) \to \mathrm{Ab}$ from the category of pairs of topological spaces and morphisms of pairs into the category of abelian groups and group homomorphisms. This also covers also the non-relative homologies, as we can view a topological space X as a pair $(X, \emptyset)$ and a continuous map $f : X \to Y$ as a morphism of pairs $f : (X, \emptyset) \to (Y, \emptyset)$.

- **connecting homomorphisms as natural transformations:**
  Theorem 3.3.7 and Corollary 3.3.8 imply that each pair of topological spaces yields in fact three homology functors for each $n \in \mathbb{N}_0$, the functors $H^1_n, H^2_n, H^3_n : \mathrm{Top}(2) \to \mathrm{Ab}$ that assign to a pair (X, A) the homology $H_n(A)$, $H_n(X)$ and the relative homology $H_n(X, A)$, respectively. The connecting homomorphisms define natural transformations $\partial_n : H^3_n \Rightarrow H^1_{n-1}$.

These functors and natural transformations have the following properties:

1. **Long exact homology sequence:** By Theorem 3.3.7 for each pair (X, A) of topological spaces, the homology functors and the connecting homomorphisms organise into a long exact homology sequence.

2. **Homotopy invariance:** By Corollary 3.3.6 $H_n(f) = H_n(f') : H_n(X, A) \to H_n(X, B)$ for all morphisms $f, f' : (X, A) \to (Y, B)$ that are homotopic in Top(2).

3. **Excision:** By Theorem 3.5.1 the relative homologies of a pair (X, A) do not change, if a subspace $U \subset A$ with $\overline{U} \subset \mathring{A}$ is removed from X and from A. One says that the singular homologies *satisfy excision.*

4. **Additivity:** In Exercises 22 we proved that the relative homology groups of a topological sum $(\amalg_{i\in I} X_i, \amalg_{i\in I} A_i)$ for a family of pairs $(X_i, A_i)_{i\in I}$ are given by the direct sum of the relative homology groups of these pairs.

5. **Dimension:** By Example 3.1.8, we have $H_n(\bullet) = 0$ for $n \in \mathbb{Z} \setminus \{0\}$.

These properties of singular homology are exactly the axioms one imposes on a homology theory, the famous *Eilenberg-Steenrod axioms.* They were introduced by Eilenberg and Steenrod in 1964 to unify different homology theories that were known at that time.

Definition 3.7.1: An (ordinary) **homology theory** with values in Ab is a

- collection of functors $H_n : \mathrm{Top}(2) \to \mathrm{Ab}$ for each $n \in \mathbb{Z}$,
- collection of natural transformations $\partial_n : H^3_n \Rightarrow H^1_{n-1}$ for all $n \in \mathbb{Z}$,

that satisfy the **Eilenberg-Steenrod axioms**:

1. **Long exact sequence:** for every pair (X, A) there is a long exact sequence

$$\ldots \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(\pi)} H_n(X, A) \xrightarrow{\partial_n} H_{n-1}(A) \xrightarrow{H_{n-1}(i)} \ldots$$

2. **Homotopy invariance:** If $f, g : (X, A) \to (Y, B)$ are homotopic, then for all $n \in \mathbb{Z}$

$$H_n(f) = H_n(g) : H_n(X, A) \to H_n(Y, B).$$

3. **Excision:** For every pair (X, A) and every open subset $U \subset A$ with $\overline{U} \subset \mathring{A}$ the inclusions $\iota : (X \setminus U, A \setminus U) \to (X, A)$ induce isomorphisms

$$H_n(\iota) : H_n(X \setminus U, A \setminus U) \xrightarrow{\sim} H_n(X, A).$$

4. **Additivity:** For any family $(X_i, A_i)_{i\in I}$ of pairs of topological spaces and all $n \in \mathbb{Z}$

$$H_n(\amalg_{i\in I} X_i, \amalg_{i\in I} A_i) \cong \oplus_{i\in I} H_n(X_i, A_i).$$

5. **Dimension axiom:** $H_n(\bullet) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$.

The abelian group $H_0(\bullet)$ is called the **coefficient group** of the homology theory.

Corollary 3.7.2: Singular homology is a homology theory with coefficient group $\mathbb{Z}$.

Note that the homotopy invariance axiom implies directly that homotopy equivalent topological spaces have isomorphic homologies, see the proof of Proposition 3.2.3. Together with the dimension axiom, this implies that for contractible topological spaces X all homology groups except $H_0(X)$ vanish, and $H_0(X)$ is isomorphic to the coeficient group.

Many results, such as the Mayer-Vietoris sequence and the homology groups of spheres can be derived directly from the Eilenberg-Steenrod axioms. For the Mayer-Vietoris sequence, this is shown in Exercise 41. As the computation of the homologies of spheres in Example 3.5.6 only makes use of the Mayer-Vietoris sequence and the fact that discs are contractible, the homologies of spheres in an ordinary homology theory are determined uniquely by its coefficient group. Eilenberg and Steenrod proved that this does not only hold for spheres, but for all finite-dimensional CW complexes. The precise statement of this is the following theorem.

Theorem 3.7.3 (Eilenberg-Steenrod)**:** If $(H_n, \partial_n)_{n\in\mathbb{Z}}$ and $(H_n', \partial_n)_{n\in\mathbb{Z}}$ are ordinary homology theories and $(T^n)_{n\in\mathbb{Z}}$ is a family of natural transformations $T^n : H_n \Rightarrow H_n'$ such that

(i) $T^n_{(S^k,\emptyset)} : H_n(S^k) \to H_n'(S^k)$ is an isomorphism for all $k \in \mathbb{N}_0$,
(ii) the following diagram commutes for all pairs (X, A)

$$\begin{tikzcd} H_n(X,A) \arrow[r, "T^n_{(X,A)}"] \arrow[d, "\partial_n"'] & H_n'(X,A) \arrow[d, "\partial_n'"] \\ H_{n-1}(A) \arrow[r, "T^{n-1}_{(X,A)}"'] & H_{n-1}'(A), \end{tikzcd}$$

then $T^n_{(X,\emptyset)} : H_n(X) \to H_n'(X)$ is an isomorphism on all finite CW complexes X.

As the homologies of spheres are determined by the Eilenberg-Steenrod axioms, this states that all homologies of finite CW complexes are determined uniquely by their coefficient groups. We will investigate the homologies of finite and non-finite CW complexes in the next section.

Homology theories that satisfy all axioms except the dimension axiom are called **extraordinary homology theories**. They are more difficult to handle than ordinary homology theories. Important examples are topological K-theory and bordism theories.

4 Homologies of CW complexes

4.1 CW complexes

CW complexes are topological spaces that are built up from simple pieces, closed discs D^n for $n \in \mathbb{N}_0$, that are glued together along their boundaries $\partial D^n = S^{n-1}$ to form a topological space. In this section we first investigate the construction and topological properties of CW complexes and then derive a way to systematically compute their homologies. We will see that this leads to simple formulas for the homology groups that are obtained by just counting the number of discs in each dimension.

We first introduce the concept that describes the gluing of topological spaces. Intuitively, gluing two topological spaces means considering them together, which amounts to taking a topological sum, and then identifying points of one space with the other, which amounts to taking a quotient. This combination of a topological sum and a quotient is encoded in the concept of a pushout. As topological sums and quotients are final topologies, this also holds for pushouts.

Definition 4.1.1: Let $f_1 : A \to X_1$ and $f_2 : A \to X_2$ be continuous maps.
The **pushout** of f_1 and f_2 is the topological space

$$X_1 \amalg_A X_2 = (X_1 \amalg X_2)/\sim \qquad \iota_1 \circ f_1(a) \sim \iota_2 \circ f_2(a) \text{ for all } a \in A.$$

Remark 4.1.2: The inclusions $\iota_j : X_j \to X_1 \amalg X_2$ for the topological sum and the canonical surjection $\pi : X_1 \amalg X_2 \to X_1 \amalg_A X_2$ define continuous maps $i_j = \pi \circ \iota_j : X_j \to X_1 \amalg_A X_2$.
The space $X_1 \amalg_A X_2$ is equipped with the final topology induced by the maps $i_j : X_j \to X_1 \amalg_A X_2$: A subset $U \subset X_1 \amalg_A X_2$ is open if and only if $i_1^{-1}(U) \subset X_1$ and $i_2^{-1}(U) \subset X_2$ are open.

The pushout can also be described more abstractly and efficiently via its universal property. This universal property allows one to characterise continuous maps $g : X_1 \amalg_A X_2 \to Y$ from a pushout in terms of continuous maps $g_1 : X_1 \to Y$ and $g_2 : X_2 \to Y$ with $g_1 \circ f_1 = g_2 \circ f_2$. Continuous maps into a pushout can be constructed directly by considering continuous maps into the spaces X_1 and X_2 and composing them with the maps i_1 and i_2.

Lemma 4.1.3: The pushout of continuous maps $f_1 : A \to X_1$ and $f_2 : A \to X_2$ has the following **universal property**:

The maps $i_1 : X_1 \to X_1 \amalg_A X_2$ and $i_2 : X_2 \to X_1 \amalg_A X_2$ are continuous with $i_1 \circ f_1 = i_2 \circ f_2$. For every pair of continuous maps $g_1 : X_1 \to Y$ and $g_2 : X_2 \to Y$ with $g_1 \circ f_1 = g_2 \circ f_2$, there is a unique continuous map $g : X_1 \amalg_A X_2 \to Y$ with $g \circ i_1 = g_1$ and $g \circ i_2 = g_2$.

$$\begin{tikzcd}
Y & & \\
& X_1 \amalg_A X_2 \arrow[ul, dashed, "\exists! g"'] & X_1 \arrow[l, "i_1"'] \arrow[ull, bend right, "g_1"'] \\
& X_2 \arrow[u, "i_2"] \arrow[uul, bend left, "g_2"] & A. \arrow[l, "f_2"] \arrow[u, "f_1"']
\end{tikzcd}$$

Proof:
The maps i_1 and i_2 are continuous as composites of continuous maps and satisfy $i_1 \circ f_1 = i_2 \circ f_2$ by construction.

Let $g_1 : X_1 \to Y$ and $g_2 : X_2 \to Y$ be continuous maps with $g_1 \circ f_1 = g_2 \circ f_2$. By the universal property of the direct sum there is a unique continuous map $g' : X_1 \amalg X_2 \to Y$ with $g' \circ \iota_1 = g_1$ and $g' \circ \iota_2 = g_2$. By construction, it satisfies $g' \circ \iota_1 \circ f_1 = g_1 \circ f_1 = g_2 \circ f_2 = g' \circ \iota_2 \circ f_2$. By the universal property of the quotient, there is a unique continuous map $g : X_1 \amalg_A X_2 \to Y$ with $g \circ \pi = g'$. This implies $g \circ i_j = g \circ \pi \circ \iota_j = g' \circ \iota_j = g_j$ for $j = 1, 2$.

If $g'' : X_1 \amalg_A X_2 \to Y$ is another continuous map with $g'' \circ i_j = g'' \circ \pi \circ \iota_j = g_j$, then by the universal property of the topological sum $g'' \circ \pi = g \circ \pi$, and surjectivity of π implies $g'' = g$. $\square$

It is clear that the universal property of the pushout in Lemma 4.1.3 generalises to other categories, if we replace continuous maps by morphisms in the relevant category. The description of a pushout in terms of a topological sum and a quotient is a special case of the construction of a pushout from coproducts and coequalisers. As a categorical construction characterised by a universal property, a pushout is determined uniquely up to unique isomorphism by its universal property (Exercise 42).

Example 4.1.4:

1. **topological sums:**
If we take $A = \emptyset$, then $f_1 : \emptyset \to X_1$ and $f_2 : \emptyset \to X_2$ are the empty maps. Then the resulting pushout is the topological sum $X_1 \amalg_A X_2 = X_1 \amalg X_2$ with the injective inclusions $\iota_1 = i_1 : X_1 \to X_1 \amalg X_2$ and $\iota_2 = i_2 : X_2 \to X_1 \amalg X_2$.
2. **wedge sums:**
If we take $A = \bullet$, then a continuous map $f_j : A \to X_j$ amounts to the choice of a basepoint $x_j = f_j(\bullet) \in X_j$. By comparing with Definition 3.5.7, we see that $X_1 \amalg_\bullet X_2 = X_1 \vee X_2$. Again, the maps $i_1 : X_1 \to X_1 \amalg_A X_2$ and $i_2 : X_2 \to X_1 \amalg_A X_2$ are injective.
3. **collapsing a subspace:**
If we take a subspace $A \subset X_1$ with the inclusion $f_1 = \iota_A : A \to X_1$ and the terminal map $f_2 : A \to \bullet$, then we identify $\iota_2(\bullet) \sim \iota_1(a)$ in $X_1 \amalg \bullet$ for all $a \in A$. The pushout the quotient space $X_1 \amalg_A \bullet \cong X_1/A$ obtained by collapsing the subspace $A \subset X_1$. In this case, $i_1 = \pi : X_1 \to X_1/A$ is the canonical surjection.
4. **attaching topological spaces:**
If we take again a subspace $A \subset X_1$ with the inclusion $f_1 = \iota_A : A \to X_1$ and a general continuous map $f_2 : A \to X_2$, then one says the pushout $X_1 \amalg_A X_2$ is obtained by **gluing** or **attaching** X_1 to X_2 with the **attaching map** $f_2 : A \to X_2$.

   The maps $i_2 : X_2 \to X_1 \amalg_A X_2$ and $i_1|_{X_1 \setminus A} : X_1 \setminus A \to X_1 \amalg_A X_2$ are always injective. The map $i_1 : X_1 \to X_1 \amalg_A X_2$ is injective if and only if f_2 is injective.
5. **attaching n-discs:** If $X_1 = \amalg_{i \in I} D^n$ and $A = \amalg_{i \in I} S^{n-1} \subset X_1$, then an attaching map $f_2 : \amalg_{i \in I} S^{n-1} \to X_2$ corresponds to a family $(f_2^i)_{i \in I}$ of continuous maps $f_2^i : S^{n-1} \to X_2$.

   Likewise, by the universal property of the topological sum, the continuous map $i_1 : X_1 \to X_1 \amalg_A X_2$ corresponds to a family $(c_i)_{i \in I}$ of continuous maps $c_i : D^n \to X_1 \amalg_A X_2$.

   On then says the pushout $X_1 \amalg_A X_2$ is obtained by **attaching n-discs** to X_2 with the **attaching maps** $f_2^i : S^{n-1} \to X_2$ and **characteristic maps** $c_i : D^n \to X_1 \amalg_A X_2$.

A *CW complex* is a topological space that is built inductively, starting out with a discrete set $X_0 = \amalg_{i\in I} D^0$ of 0-discs or points. In the first step, one attaches intervals $D^1 = [-1, 1]$ to these points. In the second step one attaches 2-discs D^2 to the topological space constructed in the first step, in the third step 3-discs D^3 to the space from the second step, in the nth step n-discs D^n to the topological space constructed in step $n-1$.

As we work with pairs of topological spaces, we also need a concept of a *relative CW complex*. In this case, one starts with an arbitrary topological space X_{-1}. Attaching points or 0-discs to X_{-1} means taking the topological sum of X_{-1} and a discrete set of points. The procedure then continues as in the case of a CW complex, by first attaching intervals to this topological sum, then 2-discs to the resulting topological space.

A CW-complex can thus be viewed as a relative CW complex, where one starts out with the empty set. In both cases, it is also allowed not to attach any discs in a given step, and the number of discs attached in each step need not be finite. Also, the attaching procedure does not need to stop after a finite number of steps, but can continue indefinitely.

Definition 4.1.5: A **relative CW complex** (X, X_{-1}) is a topological space X with a subspace $X_{-1} \subset X$ and a filtration $X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots \subset X$ such that

(CW1) $X = \cup_{n=-1}^{\infty} X_n$,

(CW2) X_n is obtained from X_{n-1} by attaching n-discs for all $n \in \mathbb{N}_0$

$$\begin{tikzcd} X_n & X_{n-1} \arrow[l, "i_{n-1,n}"'] \\ \amalg_{i\in I_n} D^n \arrow[u, "(c_i)_{i\in I_n}"] & \amalg_{i\in I_n} S^{n-1} \arrow[l, "(\iota)_{i\in I_n}"] \arrow[u, "(f_i)_{i\in I_n}"'] \end{tikzcd}$$

(CW3) X is equipped with the final topology induced by the inclusions $i_n : X_n \to X$:
A subset $U \subset X$ is open if and only if $U \cap X_n$ is open for all $n \geq -1$.

The spaces X_n are called the n-**skeleta**, the maps $f_i : S^{n-1} \to X_n$ the **attaching maps**, the maps $c_i : D^n \to X_n$ the **characteristic maps** and the spaces $\sigma_i = c_i(\mathring{D}^n) \subset X$ the n-**cells**. If $X_{-1} = \emptyset$, then $X = (X, \emptyset)$ is called a **CW complex**.

Remark 4.1.6: Let (X, X_{-1}) be a relative CW complex.

1. Axiom (CW2) for $n = 0$ states that X_0 is the topological sum of X_{-1} and a discrete set of points $\amalg_{i\in I_0} \bullet$, as the pushout is over the empty space, cf. Example 4.1.4, 1. If $X_{-1} = \emptyset$, then X_0 is a discrete set of points.

2. By Remark 4.1.2, a subset $U \subset X_n$ is open (closed) in X_n if and only if $c_i^{-1}(U)$ is open (closed) in D^n for all $i \in I_n$ and $U \cap X_{n-1}$ is open (closed) in X_{n-1}.

3. This implies that a subset $U \subset X$ is open (closed) if and only if $U \cap X_{-1}$ is open (closed) in X_{-1} and for each $n \in \mathbb{N}_0$ and $i \in I_n$ the set $c_i^{-1}(U)$ is open (closed) in D^n.

4. In particular, all skeleta X_n for $n \geq -1$ are closed in X.

   This follows, because $c_i^{-1}(X_{n-1}) = f_i^{-1}(X_{n-1}) = S^{n-1}$ is closed in D^n for all $i \in I_n$ and X_{n-1} is closed in itself. Hence, X_{n-1} is closed in X_n for all $n \in \mathbb{N}_0$, and this implies inductively that X_n closed in X_k for all $k \geq n$.

5. Each point $x \in X$ is contained either in X_{-1} or in a unique cell. This defines a **relative cell decomposition** of X, a decomposition of the *set* X as a disjoint union

$$X = X_{-1}\dot\cup \left(\dot\cup_{n=0}^\infty \dot\cup_{i\in I_n} c_i(\mathring{D}^n)\right).$$

   If $X_{-1} = \emptyset$, this is called a **cell decomposition** of X.

6. By 3. and 5. the restrictions $c_i|_{\mathring{D}^n} : \mathring{D}^n \to c_i(\mathring{D}^n)$ are homeomorphisms or all $i \in I_n$.
7. Continuous maps $f : X \to Y$ correspond to families $(f_n)_{n\in\mathbb{Z},n\geq -1}$ of continuous maps $f_n : X_n \to Y$ with $f_n|_{X_{n-1}} = f_{n-1}$ for all $n \in \mathbb{N}_0$ (Exercise 48).

One says a topological space X *has a CW structure* or, more informally, *is a CW complex* if it is homeomorphic to the topological space underlying a CW complex. Note that such a decomposition and a CW structure on a given topological space X need not be unique, as we will see in the following. A topological space may have several CW structures or none.

To show that a topological space is homeomorphic to a CW complex, one usually constructs a homeomorphism from the pushout into the space X via its universal property. This homeomorphism can often be deduced from a cell decomposition of the space X as in Example 4.1.6, 5. that satisfies certain additional conditions. In fact, cell decompositions are the starting point of the original definition of a CW complex, which did not make use of pushouts. The conditions on cell complexes in this older definition motivate the name CW complex: C stands for *closure finiteness* and W for *weak topology.*

Remark 4.1.7: A Hausdorff space X with a decomposition $X = \dot\cup_{n\in\mathbb{N}_0}\dot\cup_{i\in I_n}\sigma_i$ such that $\sigma_i \cong \mathring{D}^n$ for all $i \in I_n$ is a CW complex, if and only if

1. **characteristic maps:** there are continuous maps $c_i : D^n \to X$ for $i \in I_n$, $n \in \mathbb{N}_0$ that restrict to homeomorphisms $c_i|_{\mathring{D}^n} : \mathring{D}^n \to \sigma_i$ and such that $c_i(\partial D^n)$ is contained in cells of dimension at most $n-1$ for all $i \in I$.
2. **closure finiteness:** $\overline{\sigma}_i$ intersects only finitely many cells of X for all $i \in I_n$, $n \in \mathbb{N}_0$.
3. **weak topology:** $A \subset X$ is open if and only if $A \cap \overline{\sigma}_i$ is open in $\overline{\sigma}_i$ for all $i \in I_n$, $n \in \mathbb{N}_0$.

We will not prove that the conditions in Remark 4.1.7 are equivalent to our definition of the CW complex, but we will show that CW complexes according to our definition give raise to cell decompositions that satisfy these conditions. We start by considering examples.

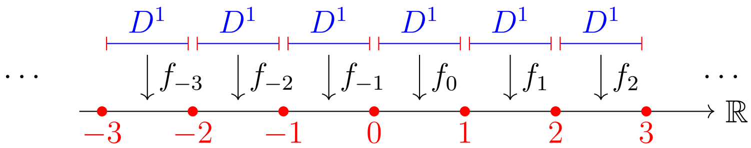
Example 4.1.8:

1. **Real space** $\mathbb{R}$ has the CW complex structure $\emptyset \subset \mathbb{Z} \subset \mathbb{R} \subset \mathbb{R} \subset \ldots$ given by

$$\begin{tikzcd} X_1 = \mathbb{R} & X_0 = \mathbb{Z} \arrow[l, "\iota_\mathbb{Z}"'] \\ \amalg_\mathbb{Z} D^1 \arrow[u, "(c_i)_{i\in\mathbb{Z}}"] & \amalg_\mathbb{Z} S^0 \arrow[l, "(\iota)_{i\in\mathbb{Z}}"] \arrow[u, "(f_i)_{i\in\mathbb{Z}}"'] \end{tikzcd}$$

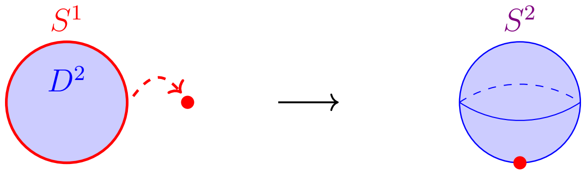
   with the attaching maps and characteristic maps

$$f_i : \{-1, 1\} \to \mathbb{Z},\ x \mapsto i + \tfrac{1}{2}(x + 1) \qquad c_i : [-1, 1] \to \mathbb{R},\ x \mapsto i + \tfrac{1}{2}(x + 1)$$



2. **The n-sphere** S^n has a CW complex structure $\emptyset \subset \bullet \subset \ldots \subset \bullet \subset S^n \subset S^n \subset \ldots$ with $X_k = \bullet$ for $0 < k < n$ and $X_k = S^n$ for $k \geq n$. It is obtained by attaching an n-disc to a point $\bullet$ with the terminal map $f : S^{n-1} \to \bullet$. Its pushout describes the construction of $S^n = D^n/S^{n-1}$ by collapsing $S^{n-1} = \partial D^n$

$$\begin{tikzcd}
X_n = S^n & X_{n-1} = \{\bullet\} \arrow[l] \\
D^n \arrow[u, "\pi"] & S^{n-1}. \arrow[l, "\iota"] \arrow[u, "f"']
\end{tikzcd}$$

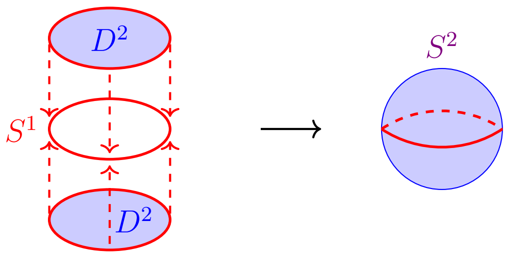


3. **The n-sphere** S^n has a CW complex structure $\emptyset \subset S^0 \subset S^1 \subset S^2 \subset \ldots \subset S^n \subset S^n \subset \ldots$ with $X_k = S^k$ for $0 \leq k \leq n$ and $X_k = S^n$ for $k \geq n$. Here, S^k is obtained from S^{k-1} by attaching two k-discs with the map $\mathrm{id} : S^{k-1} \to S^{k-1}$.

$$\begin{tikzcd}
S^k & S^{k-1} \arrow[l, "x \mapsto (x,0)"'] \\
D^k \amalg D^k \arrow[u, "(c_+, c_-)"] & S^{k-1} \amalg S^{k-1} \arrow[l, "(\iota,\iota)"] \arrow[u, "(\mathrm{id},\mathrm{id})"']
\end{tikzcd} \tag{29}$$

More explicitly, we can identify S^{k-1} with the equator $S^{k-1} \cong S^{k-1} \times \{0\} \subset S^k$ and attach the discs D^k as the upper and lower hemisphere $S^k_\pm = \{x \in S^k \mid \pm x_{k+1} \geq 0\}$. Then the attaching maps and characteristic maps are given by

$$f_\pm = f : S^{k-1} \to S^{k-1} \times \{0\}, \quad (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0)$$
$$c_\pm : D^k \to S^k, \quad (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, \pm\sqrt{1 - x_1^2 - \ldots - x_k^2})$$



4. **Real projective space** $\mathbb{RP}^n$ is the space of real lines through the origin in $\mathbb{R}^{n+1}$ and can be realised as the quotient $\mathbb{RP}^n = S^n/\sim$, where $x \sim -x$ for all $x \in S^n$. As any point in the lower hemisphere S^n_- is identified with a point in the upper hemisphere S^n_+, we obtain a CW structure $\emptyset \subset \mathbb{RP}^1 \subset \mathbb{RP}^2 \subset \ldots \subset \mathbb{RP}^n \subset \mathbb{RP}^n \subset \ldots$ with $X_k = \mathbb{RP}^k$ for $0 \leq k \leq n$

and $X_k = \mathbb{R}\mathrm{P}^n$ for $k \geq n$ from 3. Here, $\mathbb{R}\mathrm{P}^k$ is obtained from $\mathbb{R}\mathrm{P}^{k-1}$ by attaching a k-disc with the canonical surjection $\pi : S^{k-1} \to \mathbb{R}\mathrm{P}^{k-1}$, $x \mapsto [x]$

$$\begin{tikzcd} \mathbb{R}\mathrm{P}^k & \mathbb{R}\mathrm{P}^{k-1} \arrow[l, "{[x]\mapsto[(x,0)]}"'] \\ D^k \arrow[u, "c"] & S^{k-1}, \arrow[l, "\iota"] \arrow[u, "\pi"'] \end{tikzcd} \tag{30}$$

5. **Complex projective space** $\mathbb{C}\mathrm{P}^n$ is the space of complex lines through the origin in $\mathbb{C}^{n+1}$. If we identify $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ via the map

$$\phi : \mathbb{R}^{2n+2} \to \mathbb{C}^{n+1}, \ (x_1, \ldots, x_{2n+2}) \mapsto (x_1 + \mathrm{i}x_2, \ldots, x_{2n+1} + \mathrm{i}x_{2n+2}),$$

we can express it as the quotient $\mathbb{C}\mathrm{P}^n = S^{2n+1}/\sim$ with

$$(x_1, \ldots, x_{2n+2}) \sim (R(x_1, x_2), \ldots, R(x_{2n+1}, x_{2n+2})) \qquad \forall R \in \mathrm{SO}(2).$$

This yields the skeleton filtration

$$\emptyset \subset \bullet \subset \bullet \subset \mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^1 \subset \ldots \subset \mathbb{C}\mathrm{P}^{n-1} \subset \mathbb{C}\mathrm{P}^{n-1} \subset \mathbb{C}\mathrm{P}^n \subset \mathbb{C}\mathrm{P}^n \subset \mathbb{C}\mathrm{P}^n \ldots,$$

with $X_{2k} = X_{2k+1} = \mathbb{C}\mathrm{P}^k$ for $0 \leq k \leq n$ and $X_k = \mathbb{C}\mathrm{P}^n$ for $k \geq 2n$, where $\mathbb{C}\mathrm{P}^k$ is obtained from $\mathbb{C}\mathrm{P}^{k-1}$ by attaching a $2k$-disc D^{2k} with the canonical surjection $\pi : S^{2k-1} \to \mathbb{C}\mathrm{P}^{k-1}$

$$\begin{tikzcd} \mathbb{C}\mathrm{P}^k & \mathbb{C}\mathrm{P}^{k-1} \arrow[l, "{[x]\mapsto[(x,0,0)]}"'] \\ D^{2k} \arrow[u, "c"] & S^{2k-1}. \arrow[l, "\iota"] \arrow[u, "\pi"'] \end{tikzcd} \tag{31}$$

6. In the last three examples the attaching need not be stopped at a fixed skeleton, but can be continued indefinitely. This yields CW complexes
   - $X = S^\infty$ with skeleta $X_k = S^k$ for all $k \in \mathbb{N}_0$, where S^k is obtained from S^{k-1} by attaching two k-discs as in (29),
   - $X = \mathbb{R}\mathrm{P}^\infty$ with skeleta $X_k = \mathbb{R}\mathrm{P}^k$ for all $k \in \mathbb{N}_0$, where $\mathbb{R}\mathrm{P}^k$ is obtained from $\mathbb{R}\mathrm{P}^{k-1}$ by attaching a single k-disc as in (30),
   - $X = \mathbb{C}\mathrm{P}^\infty$ with skeleta $X_{2k} = X_{2k+1} = \mathbb{C}\mathrm{P}^k$ for all $k \in \mathbb{N}_0$, where $\mathbb{C}\mathrm{P}^k$ is obtained from $\mathbb{C}\mathrm{P}^{k-1}$ by attaching a single $2k$-disc as in (30).

7. It is shown in [H, Corollary A.12] that every compact topological manifold is homotopy equivalent to a CW complex.

   It is known that every compact smooth manifold is homeomorphic to a CW complex and that every compact topological manifold of dimension $d \neq 4$ is homeomorphic to a CW complex. In dimension 4 the latter is still an opn question.

8. Examples of *relative* CW complexes arise from CW complexes. For every CW complex X the pairs (X, X_k) for $k \in \mathbb{N}_0$ and (X_n, X_k) for $k \leq n$ are relative CW complexes.

Some of the CW complexes in Example 4.1.8 contain only cells up to a fixed dimension, whereas the ones in Example 4.1.8, 6. contain cells of arbitrarily large dimensions. Note also that even if the dimension of cells is restricted, there can still be infinitely many cells of a given dimension as in Example 4.1.8, 1. It is also apparent that all examples of CW complexes in Example 4.1.8 with a *finite* number of cells are compact. This is not a coincidence.

Definition 4.1.9: A CW complex X is called

- **finite-dimensional** of dimension d, if $d = \inf\{k \in \mathbb{N}_0 \mid X = X_k\} \in \mathbb{N}_0$,
- **of finite type**, if each set I_n of n-cells is finite,
- **finite**, if it is finite-dimensional and of finite type.

Corollary 4.1.10: Every finite CW complex is compact.

Proof:
This follows by induction over the n-skeleta. The space X_0 is compact, as it is a finite discrete set. If X_{n-1} is compact and I_n is finite, then the topological sum $X_{n-1} \amalg (\amalg_{i\in I_n} D^n)$ is compact as a finite topological sum of compacta. The space X_n is a quotient of this topological sum and hence compact as a quotient of a compact space. □

This shows in particular that it is futile to look for a *finite* CW complex structure on a non-compact space such as $\mathbb{R}^n$ or on an open n-disc $\mathring{D}^n$. One might ask if the converse also holds. Are there non-finite CW complex structures on compact spaces? That the answer is negative is a consequence of the following lemma.

Lemma 4.1.11:
Each compact subset K of a CW complex X intersects only finitely many cells of X.

Proof:
Let I_n be the set that indexes the n-cells of X and $c_i : D^n \to X_n$ and $f_i : S^{n-1} \to X_{n-1}$ for $i \in I_n$ the associated characteristic maps and attaching maps.

Choose for each cell σ with $\sigma \cap K \neq \emptyset$ a point $x_\sigma \in \sigma \cap K$. These points form a set $S \subset K$, which is in bijection with the set of cells in X that intersect K. To show that S is finite, it is sufficient to show that S is discrete and closed in X. As closed subsets of compacta are compact and discrete compacta are finite, this implies that S is finite. Hence, we show that $R \cap X_n$ is closed in X_n for all $n \in \mathbb{N}_0$ and subsets $R \subset S$.

As a subset of the discrete set X_0, the set $R \cap X_0$ is closed in X_0 for any $R \subset S$. Suppose we showed that $R \cap X_k$ is closed in X_k for all subsets $R \subset S$ and $k \leq n-1$. Then for each $i \in I_k$ with $k \leq n-1$ the set $c_i^{-1}(R) = c_i^{-1}(R \cap X_k) \subset D^k$ is closed in D^k as the preimage of the closed set $R \cap X_k$ under the continuous map $c_i : D^k \to X_k$. Likewise, for each $i \in I_n$ the set $f_i^{-1}(R) = f_i^{-1}(R \cap X_{n-1}) \subset S^{n-1} = \partial D^n$ is closed in S^{n-1} as the preimage of the closed set $R \cap X_{n-1}$ under the continuous map f_i.

For each $i \in I_n$ the set $c_i^{-1}(R) \subset D^n$ is either equal to $f_i^{-1}(R) \subset S^{n-1} = \partial D^n$, or it is obtained from $f_i^{-1}(R)$ by adding a point in $\mathring{D}^n$. As $f_i^{-1}(R)$ is closed in ∂D^n, it follows in both cases that $c_i^{-1}(D^n)$ is closed in D^n. By Remark 4.1.6, 2. the set $R \cap X_n$ is closed in X_n. □

Corollary 4.1.12: Every compact CW complex is finite.

Proof:
If X is a compact CW complex, we can consider the compact subset $K = X$. As it intersects all cells of X, it follows with Lemma 4.1.11 that X is finite. □

We will now look in more depth at the topological properties of CW complexes and relative CW complexes. We already addressed the question of compactness. It is also clear from the definition that we cannot say anything general about connectedness or path-connectedness of CW complexes - this will depend on the individual CW complex. However, we can address their separation properties. For this, we need to consider closed subsets of relative CW complexes and to construct open neighbourhoods of such subsets.

We can also investigate *local* homotopy theoretical properties of CW complexes, such as local contractibility or local path-connectedness. In this case, we want to show that certain neighbourhoods of a point that are contained in a given open neighbourhood deformation retract to the point. This requires the following technical lemma.

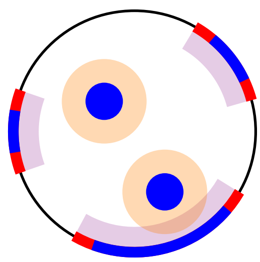
Lemma 4.1.13: Let (X, X_{-1}) be a relative CW complex, $A \subset X$ a closed subspace and $U \subset X$ open with $A \subset U$. Suppose there is an open neighbourhood $N(A)_{-1}$ of $A \cap X_{-1}$ in X_{-1} with $\overline{N(A)_{-1}} \subset U \cap X_{-1}$.

1. Then there is an open neighbourhood $N(A)$ of A with $\overline{N(A)} \subset U$.
2. If $A \cap \sigma = \emptyset$ for all n-cells σ, then $N(A) \cap X_n$ strongly deformation retracts to $N(A) \cap X_{n-1}$.

Proof:
1. We construct open neighbourhoods $N(A)_k$ of $A_k := A \cap X_k$ in X_k with $\overline{N(A)}_k \subset U \cap X_k =: U_k$ by induction. For $n = -1$ such a neighbourhood exists by assumption. Suppose we constructed such neighbourhoods for all $k \leq n-1$.

For $i \in I_n$ and $\epsilon_i \in (0,1)$ let $Y_i = V_i \cup W_i$ be the union of the open sets

$$V_i = \{x \in \mathring{D}^n \mid d(x, c_i^{-1}(A) \cap \mathring{D}^n) < \epsilon_i\} \subset \mathring{D}^n$$
$$W_i = \{\lambda v \mid v \in c_i^{-1}(N(A)_{n-1}),\ 1 - \epsilon_i < \lambda \leq 1\} \subset D^n.$$



Disc D^2 with the set $c_i^{-1}(A)$ in blue, $c_i^{-1}(N(A)_1)$ in red, V_i in orange and W_i in violet.

As A is closed and c_i continuous, the set $c_i^{-1}(A) \subset D^n$ is closed and hence compact. As U is open, the set $c_i^{-1}(U)$ is open in D^n, and as $A \subset U$ we have $\overline{c_i^{-1}(A)} \subset c_i^{-1}(U)$. By choosing $\epsilon_i > 0$ sufficiently small, we can achieve $\overline{V}_i \subset c_i^{-1}(U)$. Likewise, $c_i^{-1}(\overline{N(A)}_{n-1}) \subset S^{n-1}$ is closed and hence compact and contained in $c_i^{-1}(U)$ by assumption. Hence, there is an $\epsilon_i > 0$ such that $\overline{W}_i \subset c_i^{-1}(U)$. Thus, for $\epsilon_i > 0$ sufficiently small, we have $\overline{Y}_i \subset c_i^{-1}(U)$. We define

$$N(A)_n = N(A)_{n-1} \cup (\cup_{i \in I_n} c_i(Y_i)) = N(A)_{n-1} \cup \left(\cup_{i \in I_n} c_i(Y_i \cap \mathring{D}^n)\right).$$

We show that $N(A)_n$ is open in X_n. The set $N(A)_n \cap X_{n-1} = N(A)_{n-1}$ is open in X_{n-1} by assumption. For each $i \in I_n$, we have $c_i^{-1}(N(A)_n) = c_i^{-1}(c_i(Y_i)) = Y_i$, as $c_i|_{\mathring{D}^n}$ is injective and

$Y_i \cap \partial D^n = c_i^{-1}(N(A)_{n-1})$, which gives $c_i^{-1}(c_i(Y_i \cap \partial D^n)) = c_i^{-1}(N(A)_{n-1}) = Y_i \cap \partial D^n$. As Y_i is open in D^n by construction, Remark 4.1.6, 2. implies that $N(A)_n$ is open in X_n.

As $c_i|_{\mathring{D}^n} : \mathring{D}^n \to c_i(\mathring{D}^n)$ is a homeomorphism for all $i \in I_n$ and $c_i(\partial D^n) \subset X_{n-1}$ for $i \in I_n$ we have $\overline{\cup_{i\in I_n} c_n(Y_i)} \subset \cup_{i\in I} c_i(\overline{Y}_i) \cup \overline{N(A)}_{n-1}$. This yields

$$\begin{aligned} &\overline{N(A)}_n = \overline{N(A)}_{n-1} \cup (\cup_{i\in I_n} c_i(\overline{Y}_i)) \subset U_n = U \cap X_n \\ &A_n \subset A_{n-1} \cup (\cup_{i\in I_n} c_i(Y_i)) \subset N(A)_{n-1} \cup (\cup_{i\in I_n} c_i(Y_i)) = N(A)_n. \end{aligned}$$

Then the open set $N(A) = \cup_{n=-1}^{\infty} N_n(A)$ satisfies $N(A)_k = N(A) \cap X_k$ for all $k \geq -1$ as well as $A \subset N(A)$ and $\overline{N(A)} \subset U$.

2. If $A \cap c_i(\mathring{D}^n) = \emptyset$ for all $i \in I_n$, then $Y_i = W_i$ for all $i \in I_n$. Then $c_i^{-1}(N(A)_{n-1})$ is a strong deformation retract of W_i for all $i \in I_n$ with the retraction and homotopy

$$r_i : W_i \to c_i^{-1}(N(A)_{n-1}),\ v \mapsto v/||v|| \qquad h_i : [0,1] \times W_i \to W_i,\ (t,v) \mapsto (1-t)v + tv/||v||$$

Consequently, the set $N(A)_{n-1}$ is a strong deformation retract of $N(A)_n$ with retraction and homotopy induced by the continuous maps

$$r_i' : c_i(W_i) \to c_i(W_i) \cap N_{n-1}(A), \quad x \mapsto \begin{cases} x & x \in c_i(W_i) \cap N(A)_{n-1} \\ c_i \circ r_i \circ (c_i|_{\mathring{D}^n})^{-1}(x) & x \in c_i(W_i \cap \mathring{D}^n) \end{cases}$$

$$h_i' : [0,1] \times c_i(W_i) \to c_i(W_i), \quad (t,x) \mapsto \begin{cases} x & x \in c_i(W_i) \cap N(A)_{n-1} \\ c_i \circ h_i(t, (c_i|_{\mathring{D}^n})^{-1}(x)) & x \in c_i(W_i \cap \mathring{D}^n). \end{cases}$$

□

Corollary 4.1.14:

1. If X is a relative CW complex with X_{-1} normal, then X is normal.
2. Every CW complex is normal, in particular, Hausdorff.

Proof:
Let X be a relative CW complex such that X_{-1} is normal. We show that X is T_1 and T_4. The separation axioms T_0, T_2 and T_3 then follow.

1. We show that X is a T_1-space:
Let $x \in X$ and $k = \min\{n \geq -1 \mid x \in X_k\}$. If $k = -1$, then $\{x\}$ is closed in X_{-1}, as X_{-1} is normal. Otherwise, there is a unique $i \in I_k$ with $x \in c_i(\mathring{D}^k)$ and a unique $p \in \mathring{D}^k$ with $c_i(p) = x$. Then $\{p\} = c_i^{-1}(x)$ is closed in D^k, $c_j^{-1}(x) = \emptyset$ for all $j \in I_k \setminus \{i\}$ and $\{x\} \cap X_{k-1} = \emptyset$. This shows that $\{x\}$ is closed in X_k by Remark 4.1.6, 2.

That $\{x\}$ is closed in X_n for $n > k$ follows inductively. Suppose $\{x\} \subset X_m$ is closed in X_m for all $m \leq n-1$. Then for all $i \in I_n$ one has $c_i^{-1}(x) = f_i^{-1}(x) \subset S^{n-1}$ closed in D^n as the preimage of the closed set $\{x\} \subset X_{n-1}$ under the attaching map $f_i : S^{n-1} \to X_{n-1}$. With Remark 4.1.6, 2. it follows that $\{x\}$ is closed in X_n. This shows that $\{x\}$ is closed in X and X is T_1.

2. We show that X is a T_4-space:
Let $A, B \subset X$ closed with $A \cap B = \emptyset$. Then by Remark 4.1.6 $A_{-1} = A \cap X_{-1}$ and $B_{-1} = B \cap X_{-1}$ are closed and disjoint. As X_{-1} is normal, there are disjoint open subsets $O_A, O_B \subset X_{-1}$ with $A_{-1} \subset O_A$ and $B_{-1} \subset O_B$.

We set $U_A := X \setminus B$ and $N(A)_{-1} := O_A$. Then we have $A \subset U_A$ as well as

$$\overline{N(A)_{-1}} = \overline{O}_A \subset \overline{X_{-1} \setminus O_B} = X_{-1} \setminus O_B \subset X_{-1} \setminus B_{-1} = X_{-1} \cap U_A.$$

By Lemma 4.1.13 there is an open neighbourhood $N(A)$ of A in X such that $\overline{N(A)} \subset X \setminus B$. Then $N(A)$ and $X \setminus \overline{N(A)}$ are disjoint and open with $A \subset N(A)$ and $B \subset X \setminus \overline{N(A)}$.

This shows that X is T_4 and with 1. that X is normal. □

We have now proven that the cell decomposition $X = \dot\cup_{n\in\mathbb{N}_0}\dot\cup_{i\in I_n} c_i(\mathring{D}^n)$ for a CW complex X satisfies all the conditions in Remark 4.1.7:

- Lemma 4.1.14 implies that X is a Hausdorff space.
- Definition 4.1.5 yields continuous maps $c_i : D^n \to X$ for each $i \in I_n$, $n \in \mathbb{N}_0$ that induce homeomorphisms $c_i|_{\mathring{D}^n} : \mathring{D}^n \to c_i(\mathring{D}^n)$ by Remark 4.1.6, 6. and such that $c_i(\partial D^n)$ intersects only cells of dimension $\leq n-1$ for $i \in I_n$. This is the characteristic map axiom.
- By Remark 4.1.6, 2. a subset $U \subset X$ is open, if and only $c_i^{-1}(U)$ is open for all $i \in I_n$, $n \in \mathbb{N}_0$. This is equivalent to $U \cap c_i(D^n)$ is open for all $i \in I_n$ and $n \in \mathbb{N}_0$. As $c_i(D^n)$ is the closure of $c_i(\mathring{D}^n)$ by Exercise 45, this proves the weak topology axiom.
- By Lemma 4.1.11 the closure $c_i(D^n)$ of each cell $c_i(\mathring{D}^n)$ can intersect only finitely many cells, which proves the closure finiteness axiom.

We will now apply the second part of Lemma 4.1.13 to investigate the homotopy theoretic properties of a CW complex. More specifically, we prove that any CW complex is locally contractible and hence locally path-connected and locally connected. We also show that for each *relative* CW complex (X, X_{-1}) the subspace X_{-1} is a strong deformation retract of X. For this we need to combine the retractions and homotopies for subsequent skeleta from Lemma 4.1.13 into retractions and homotopies for the entire relative CW complex. This is achieved with the following technical lemma.

Lemma 4.1.15: Let $X_0 \subset X_1 \subset X_2 \subset \ldots$ a sequence of subspaces and $X = \cup_{n=0}^\infty X_n$ their union with the final topology induced by the inclusions $\iota_n : X_n \to X$. If X_{n-1} is a strong deformation retract of X_n for all $n \in \mathbb{N}$, then X_0 is a strong deformation retract of X.

Proof:
We denote by $\iota_{n-1,n} : X_{n-1} \to X_n$ the inclusions. Let $r_n : X_n \to X_{n-1}$ be a retraction and $h_n : [0,1] \times X_n \to X_n$ a homotopy from id_{X_n} to $\iota_{n-1,i} \circ r_n$ relative to X_{n-1}:

$$h_n(0,x) = x \;\; \forall x \in X_n, \quad h_n(1,x) = r_n(x) \;\; \forall x \in X_n, \quad h_n(s,x) = x \;\; \forall x \in X_{n-1}, s \in [0,1].$$

For $k, n \in \mathbb{N}_0$ we consider the continuous maps $r_n^k : X_k \to X_n$ given by

$$r_n^k = \begin{cases} r_{n+1} \circ \ldots \circ r_k & k > n \\ \mathrm{id}_{X_n} & k = n \\ \iota_{n-1,n} \circ \ldots \circ \iota_{k,k+1} & k < n. \end{cases}$$

As they satisfy $r_n^k(x) = r_n^{k-1}(x)$ for all $x \in X_{k-1}$ and $k, n \in \mathbb{N}_0$, they define a map

$$R_n : X \to X_n, \quad X_k \ni x \mapsto r_n^k(x) \in X_n.$$

The map R_n is continuous, because $R_n^{-1}(U) = \cup_{k=0}^{\infty}(r_n^k)^{-1}(U)$ is open for all open $U \subset X$ by continuity of the maps r_n^k. As $R_n(x) = x$ for all $x \in X_n$, it is a retraction from X to X_n.

A homotopy from $\iota_n \circ R_n$ to $\iota_{n-1} \circ R_{n-1}$ relative to X_{n-1} is given by

$$h'_n : [0,1] \times X \to X, \quad (t,x) \mapsto h_n(t, R_n(x)).$$

Composing the maps h'_n for $n \in \mathbb{N}_0$ yields a homotopy from id_X to $\iota_0 \circ R_0$ relative to X_0

$$h : [0,1] \times X \to X, \quad (t,x) \mapsto \begin{cases} h'_n(2^n(t-2^{-n}), x) & t \in [2^{-n}, 2^{-n+1}] \\ x & t = 0. \end{cases}$$

It satisfies $h(0,x) = x$ for all $x \in X$, $h(1,x) = h'_1(1,x) = h_1(1, R_1(x)) = r_1 \circ R_1(x) = R_0(x)$ for all $x \in X$ and $h(s,x) = x$ for all $x \in X_0$. It is continuous, because its restriction to $[0,1] \times X_k$ is continuous for all $k \in \mathbb{N}_0$

$$x \in X_k \quad \Rightarrow \quad h(t,x) = \begin{cases} x & t \in [0, 2^{-k}] \\ h_k(2^k(t-2^{-k}), x) & t \in [2^{-k}, 2^{-k+1}] \\ h_n(2^n(t-2^{-n}), R_n(x)) & t \in [2^{-n}, 2^{-n+1}],\ n < k. \end{cases}$$

This yields $h^{-1}(U) = \cup_{k=0}^{\infty} h^{-1}(U) \cap ([0,1] \times X_k) = \cup_{k=0}^{\infty}(h|_{[0,1]\times X_k})^{-1}(U)$ open for all open subsets $U \subset X$. □

Proposition 4.1.16:

1. Every CW complex X is locally contractible.
2. In every relative CW complex (X, X_{-1}) the space X_{-1} has an open neighbourhood $N(X_{-1}) \subset X$ that strongly deformation retracts to X_{-1}.

Proof:
1. Let X be a CW complex, $x \in X$ and U an open neighbourhood of x. We construct an open neighbourhood $x \in N(x) \subset U$ such that $\{x\}$ is a strong deformation retract of $N(x)$.

We apply Lemma 4.1.13 to the subset $A = \{x\}$, which is closed by Corollary 4.1.14, and to the open set U. This yields an open neighbourhood $N(x)$ of x such that $\overline{N(x)} \subset U$.

We show that $\{x\}$ is a strong deformation retract of $N(x)$. By Remark 4.1.6, 5. there is a unique $m \in \mathbb{N}_0$ and a unique $i \in I_m$ such that $x \in c_i(\mathring{D}^m)$. By construction of the neighbourhood $N(x)$ in Lemma 4.1.13 it follows that $N(x)_k = \emptyset$ for all $k < m$. For $k = m$ we have $N(x)_m = c_i(B_\epsilon(p))$, the image of an open ϵ-ball around $p = c_i^{-1}(x)$ under the characteristic map $c_i : D^m \to X_m$. For all k-cells σ with $k > m$ we have $\sigma \cap \{x\} = \emptyset$. By Lemma 4.1.13 this implies that $N(x)_k$ is a strong deformation retract of $N(x)_{k+1}$ for all $k \geq m$.

Because all subspaces $N(x)_k$ are open, the topology on $N(x) = \cup_{k=0}^{\infty} X_k$ is the final topology induced by the inclusions $\iota'_k : N(x)_k \to N(x)$ by Exercise 49. Applying Lemma 4.1.15 to the sequence of subspaces $N(x)_m \subset N(x)_{m+1} \subset N(x)_{m+2} \subset \ldots$ then shows that $N(x)_m = c_i(B_\epsilon(p))$ is a strong deformation retract of $N(x)$. As $\{x\}$ is a strong deformation retract of $N(x)_m$, it follows that $N(x) \subset U$ is contractible.

2. We apply Lemma 4.1.13 to the subspace $A = X_{-1}$, which is closed by Remark 4.1.6, 3. and $U = X$. This yields an open neighbourhood $N(X_{-1})$ of X_{-1} with $N(X_{-1})_{-1} = X_{-1}$. By Remark

4.1.6, 5. none of the n-cells for $n \geq 0$ intersect X_{-1}. With Lemma 4.1.13 this implies that $N(X_{-1})_{n-1}$ is a strong deformation retract of $N(X_{-1})_n$ for all $n \in \mathbb{N}_0$.

Because all subsets $N(X_{-1})_k$ are open in X, the topology on $N(X_{-1})$ is the final topology induced by the inclusions $\iota'_k : N(X_{-1})_k \to N(X_{-1})$ by Exercise 49. Applying Lemma 4.1.15 to the sequence of subspaces $N(X_{-1})_{-1} \subset N(X_{-1})_0 \subset \ldots$ shows that $X_{-1} = N(X_{-1})_{-1}$ is a strong deformation retract of $N(X_{-1})$. □

Our main applications of the second result are skeleta of CW complexes. As each skeleton X_n of a CW complex X defines a relative CW complex by Example 4.1.8, 7. it has an open neighbourhood $N(X_n)$ that strongly deformation retracts to X_n by Lemma 4.1.16. In other words, the pair (X, X_n) is a good pair in the sense of Definition 3.5.2. Their relative homologies are thus given by the homologies of their quotient by Proposition 3.5.3. For a pair (X_n, X_{n-1}) of subsequent skeleta, this quotient takes a particularly simple form.

Corollary 4.1.17: For every CW complex X the pairs (X, X_n) and (X_k, X_n) are relative CW complexes and hence good pairs for all $n \in \mathbb{N}_0$ and $k \geq n$.

Lemma 4.1.18: For every CW complex X and all $n \in \mathbb{N}$ we have homeomorphisms

$$X_n \setminus X_{n-1} \cong \amalg_{i\in I_n} \mathring{D}^n \qquad\qquad X_n/X_{n-1} \cong \vee_{i\in I_n} S^n.$$

Proof:
The first identity follows directly from the definition of a CW complex. The second identity follows from the following two commuting diagrams, whose outer rectangles coincide

$$\begin{tikzcd}
X_n/X_{n-1} & \bullet \arrow[l] & & X_n/X_{n-1} & \bullet \arrow[l] \\
X_n \arrow[u, "\pi_n"] & X_{n-1} \arrow[l, "i_{n-1,n}"'] \arrow[u] & & \amalg_{i\in I_n} D^n/S^{n-1} \arrow[u, "(c'_i)_{i\in I}"] & \amalg_{i\in I_n}\bullet \arrow[l] \arrow[u] \\
\amalg_{i\in I_n} D^n \arrow[u, "(c_i)_{i\in I_n}"] & \amalg_{i\in I_n} S^{n-1} \arrow[l, "(\iota)_{i\in I_n}"] \arrow[u, "(f_i)_{i\in I_n}"'] & & \amalg_{i\in I_n} D^n \arrow[u, "(\pi)_{i\in I_n}"] & \amalg_{i\in I_n} S^{n-1}. \arrow[l, "(\iota)_{i\in I_n}"] \arrow[u]
\end{tikzcd}$$

Here, the maps $c'_i : D^n/S^{n-1} \to X_n/X_{n-1}$ are the unique continuous maps with $c'_i \circ \pi = \pi_n \circ c_i$, induced by the characteristic maps $c_i : D^n \to X_n$ with $c_i(S^{n-1}) \subset X_{n-1}$ via the universal property of the quotient.

In the diagram on the left the lower square is a pushout by definition of a CW complex and the upper diagram is a pushout by Example 4.1.4, 3, as it describes the construction of the quotient. By Exercise 43, the outer diagram is a pushout as well.

In the diagram on the right the lower square is a pushout by Example 4.1.4, 3. and by Exercise 44, which states that pushouts are preserved by topological sums. As the outer rectangle is a pushout, it follows with Exercise 43 that the upper square is a pushout as well. The latter describes a wedge sum by Example 4.1.4, 2. and yields with $D^n/S^{n-1} = D^n/\partial D^n \cong S^n$

$$X_n/X_{n-1} \cong \left(\amalg_{i\in I_n} D^n/S^{n-1}\right) / \left(\amalg_{i\in I_n}\bullet\right) \cong \left(\amalg_{i\in I_n} S^n\right) / \left(\amalg_{i\in I_n}\bullet\right) \cong \vee_{i\in I} S^n.$$

□

4.2 Cellular homologies

In this section, we use the results on CW complexes to systematically compute their homologies. As a first step, we compute the relative and some absolute homologies of their skeleta. Relative homologies of subsequent skeleta can be computed with Proposition 3.5.3 and Lemma 4.1.18 by identifying them with a wedge sum of spheres. The absolute homologies are then obtained from the long exact homology sequence. As in Definition 4.1.5 we characterise a CW complex X by the following pushouts that describe the attaching of n-cells for $n \in \mathbb{N}_0$

$$\begin{tikzcd} X_n & X_{n-1} \arrow[l, "i_{n-1,n}"'] \\ \amalg_{i\in I_n} D^n \arrow[u, "(c_i)_{i\in I_n}"] & \amalg_{i\in I_n} S^{n-1}. \arrow[l, "(\iota)_{i\in I_n}"] \arrow[u, "(f_i)_{i\in I_n}"'] \end{tikzcd}$$

Proposition 4.2.1: Let X be a CW complex.

1. The relative homologies of its skeleta are given by

$$H_k(X_n, X_{n-1}) = \begin{cases} \langle I_n \rangle_{\mathbb{Z}} & k = n, \\ 0 & k \neq n. \end{cases}$$

2. The absolute homologies of its skeleta satisfy

$$H_k(X_n) = 0 \quad k > n \qquad\qquad H_k(X_n) \cong H_k(X) \quad k < n.$$

Proof:
1. The case $k = 0$, $n \in \mathbb{N}$ follows from Exercise 25, and for $k = n = 0$ we have

$$H_0(X_0, X_{-1}) = H_0(X_0, \emptyset) = H_0(X_0) = H_0(\amalg_{i\in I_0} \bullet) \cong \oplus_{i\in I_0} \mathbb{Z} = \langle I_0 \rangle_{\mathbb{Z}}.$$

As (X_n, X_{n-1}) is a good pair we have in the other cases

$$\begin{aligned} H_k(X_n, X_{n-1}) &\overset{3.5.3}{\cong} \tilde{H}_k(X_n/X_{n-1}) \overset{4.1.18}{\cong} \tilde{H}_k(\vee_{i\in I_n} S^n) \overset{3.5.9}{\cong} \oplus_{i\in I_n} \tilde{H}_k(S^n) \\ &\overset{3.5.6}{\cong} \begin{cases} \oplus_{i\in I_n}\mathbb{Z} & k = n \neq 0 \\ 0 & \text{else} \end{cases} \cong \begin{cases} \langle I_n \rangle_{\mathbb{Z}} & n = k \neq 0 \\ 0 & \text{else.} \end{cases} \end{aligned}$$

2.(a) We consider the long exact homology sequence for the pair (X_n, X_{n-1})

$$\ldots \to H_{k+1}(X_n, X_{n-1}) \xrightarrow{\partial_{k+1}} H_k(X_{n-1}) \xrightarrow{H_k(i_{n-1,n})} H_k(X_n) \to H_k(X_n, X_{n-1}) \xrightarrow{\partial_k} \ldots$$

- If $k + 1 < n$ or $k > n$ we have $H_k(X_n, X_{n-1}) = H_{k+1}(X_n, X_{n-1}) = 0$ by 1, and the map $H_k(i_{n-1,n})$ is an isomorphism.
- For $k = n - 1$, we still have $H_k(X_n, X_{n-1}) = 0$, and $H_{n-1}(i_{n-1,n})$ is surjective.

For $k > n$ this proves the first claim in 2: $H_k(X_n) \cong H_k(X_{n-1}) \cong \ldots \cong H_k(X_0) = 0$.

For $k < n$ this gives $H_k(X_n) \cong H_k(X_{n+1}) \cong H_k(X_{n+2}) \cong \ldots$. If X is a finite-dimensional CW complex, there is an $m \in \mathbb{N}_0$ with $X = X_m$. This implies $H_k(X_n) \cong H_k(X_m) = H_k(X)$ for all $k < n$ and proves the second claim in 2.

To prove the second claim in 2. in general, we show that $H_n(i_{n+1}) : H_n(X_{n+1}) \to H_n(X)$ is an isomorphism for all $n \in \mathbb{N}_0$. For this, note that the inclusions $i_{n,n+1} : X_n \to X_{n+1}$ and the inclusions $i_n : X_n \to X$ define the following commuting diagram in Top

$$\begin{tikzcd}
 & & & X & & & \\
\ldots \arrow[r] & X_{n-2} \arrow[urr, bend left, "i_{n-2}"] \arrow[r, "i_{n-2,n-1}"'] & X_{n-1} \arrow[ur, "i_{n-1}"] \arrow[r, "i_{n-1,n}"'] & X_n \arrow[u, "i_n"] \arrow[r, "i_{n,n+1}"'] & X_{n+1} \arrow[ul, "i_{n+1}"'] \arrow[r, "i_{n+1,n+2}"'] & X_{n+2} \arrow[ull, bend right, "i_{n+2}"'] \arrow[r] & \ldots
\end{tikzcd} \tag{32}$$

2.(b) We show that $H_n(i_{n+1}) : H_n(X_{n+1}) \to H_n(X)$ is surjective.

As the image of every singular n-simplex $\sigma : \Delta^n \to X$ is compact, it intersects only finitely many cells by Lemma 4.1.11 and hence is contained in some skeleton of X. Hence, for every singular n-cycle $z = \Sigma_{j=0}^l a_j \sigma_j \in Z_n(X)$ there is an $m \in \mathbb{N}_0$ with $\sigma_j(\Delta^n) \subset X_m$ for all $j = 0, \ldots, l$. Hence, the homology class $[z] \in H_n(X)$ is of the form $[z] = H_n(i_m)[z']$, where $z' \in Z_n(X_m)$ is the associated n-cycle in X_m. The map

$$H_n(i_{m-1,m} \circ \ldots \circ i_{n+1,n+2}) = H_n(i_{m-1,m}) \circ \ldots \circ H_n(i_{n+1,n+2}) : H_n(X_{n+1}) \to H_n(X_m)$$

is surjective, as $H_n(i_{n+x,n+x+1})$ is an isomorphism by 2.(a) for all $x \in \mathbb{N}$. Hence, there is a $[z''] \in H_n(X_{n+1})$ with $[z'] = H_n(i_{m-1,m} \circ \ldots \circ i_{n+1,n+2})[z'']$. With (32) we obtain

$$[z] = H_n(i_m)[z'] = H_n(i_m \circ i_{m-1,m} \circ \ldots \circ i_{n+1,n+2})[z''] = H_n(i_{n+1})[z''].$$

Thus, $H_n(i_{n+1}) : H_n(X_{n+1}) \to H_n(X)$ is surjective.

2.(c) We show that $H_n(i_{n+1}) : H_n(X_{n+1}) \to H_n(X)$ is injective.

Suppose that $z \in Z_n(X_{n+1})$ with $H_n(i_{n+1})[z] = 0$. Then there is an $(n+1)$-chain $x \in S_{n+1}(X)$ with $S_n(i_{n+1})z = d_{n+1}(x)$. As the images of all simplexes in x are compact, there is some $m \geq n+1$ such that the images of all simplexes in x are contained in X_m. This implies $x \in S_{n+1}(X_m)$ and $S_n(i_{m-1,m} \circ \ldots \circ i_{n+1,n+2})(z) = d_{n+1}(x)$ and $H_n(i_{m-1,m} \circ \ldots \circ i_{n+1,n+2})[z] = 0$. As $H_n(i_{k,k+1})$ is an isomorphism for $k > n$ by 2.(a), this implies $[z] = 0$. □

Remark 4.2.2: Diagram (32) that relates the skeleta X_n to the CW complex X and to their neighbours is an example of a categorical construction called **sequential colimit** or, confusingly, **direct limit** in the category Top. The arguments in the second part of the proof of Proposition 4.2.1 are typical for such colimits and are often called *colimit argument.*

Corollary 4.2.3: Let X, Y be CW complexes.

1. If $X_n \cong Y_n$, then $H_k(X) \cong H_k(Y)$ for all $k < n$.
2. If X has no n-cells, then $H_n(X) = 0$.

Proof:
1. By Proposition 4.2.1 we have $H_k(X) \cong H_k(X_n) \cong H_k(Y_n) \cong H_k(Y)$ for all $k < n$.

2. If X has no n-cells, then $X_n = X_{n-1}$, and Proposition 4.2.1, 2. gives $0 = H_n(X_{n-1}) = H_n(X_n)$.
□

The relative homologies of a CW complex X are not just useful for computations with the long exact homology sequences. They have a direct geometrical meaning - their generators represent the n-cells in X - and they can be used to define a simplified chain complex, the cellular complex. This complex consists of the relative homologies $H_n(X_n, X_{n-1})$ in degree n and a boundary operator that is defined by the connecting homomorphisms and the projection maps for the relative chain complex. As it has the same homologies as the singular chain complex, it gives a simple and efficient way to compute homologies of CW complexes.

Proposition 4.2.4: Let X be a CW complex. Then the abelian groups of **cellular chains** $C_n(X) = H_n(X_n, X_{n-1}) = \langle I_n \rangle_{\mathbb{Z}}$ and the **cellular boundary operators**

$$d_n = H_{n-1}(\pi_{n-1}) \circ \partial_n : H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2})$$

induced by the morphisms $\pi_n : (X_n, \emptyset) \to (X_n, X_{n-1})$ in Top(2) define a chain complex $C_\bullet(X)$, the **cellular chain complex** of X.

Proof:
We consider for $n \in \mathbb{N}_0$ the short exact sequences of chain complexes

$$0 \to S_\bullet(X_{n-1}) \xrightarrow{S_\bullet(i_n)} S_\bullet(X_n) \xrightarrow{S_\bullet(\pi_n)} S_\bullet(X_n)/S_\bullet(X_{n-1}) \to 0.$$

induced by the morphisms $\pi_n : (X_n, \emptyset) \to (X_n, X_{n-1})$ and $i_n : (X_{n-1}, \emptyset) \to (X_n, \emptyset)$. The associated long exact sequences of homologies define a commuting diagram with exact rows

$$\begin{tikzcd}
\ldots \arrow[r, "H_n(\pi_n)"] & H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[d, "\partial_n"] \arrow[dr, dashed, "d_n"] & H_{n-1}(X_{n-1}) \arrow[r, "H_{n-1}(i_n)"] \arrow[d, "H_{n-1}(\pi_{n-1})"] & H_{n-1}(X_n) \arrow[r] & \ldots \\
\ldots \arrow[r, "H_{n-1}(i_{n-1})"] & H_{n-1}(X_{n-1}) \arrow[r, "H_{n-1}(\pi_{n-1})"] & H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "\partial_{n-1}"] \arrow[d, "\partial_{n-1}"] \arrow[dr, dashed, "d_{n-1}"] & H_{n-2}(X_{n-2}) \arrow[r] \arrow[d, "H_{n-2}(\pi_{n-2})"] & \ldots \\
\ldots \arrow[r, "\partial_{n-2}"] & H_{n-2}(X_{n-3}) \arrow[r, "H_{n-2}(i_{n-2})"] & H_{n-2}(X_{n-2}) \arrow[r, "H_{n-2}(\pi_{n-2})"] & H_{n-2}(X_{n-2}, X_{n-3}) \arrow[r] & \ldots
\end{tikzcd}$$

As the squares in this diagram commute and the rows are exact, we have

$$d_{n-1} \circ d_n = H_{n-2}(\pi_{n-2}) \circ \underbrace{\partial_{n-1} \circ H_{n-1}(\pi_{n-1})}_{=0} \circ \partial_n = 0.$$

$\square$

Theorem 4.2.5: For every CW complex X there are isomorphisms

$$H_n C_\bullet(X) \cong H_n(X) \qquad \forall n \in \mathbb{N}_0.$$

Proof:
We consider the following commuting diagram in which the row and both columns are exact

$$\begin{tikzcd}
 & H_{n+1}(X_{n+1}, X_n) \arrow[d, "\partial_{n+1}"] \arrow[dr, "d_{n+1}"] & & H_{n-1}(X_{n-2}) \overset{4.2.1}{=} 0 \arrow[d, "H_{n-1}(i_{n-1})"] \\
0 \overset{4.2.1}{=} H_n(X_{n-1}) \arrow[r, "H_n(i_n)"] & H_n(X_n) \arrow[r, "H_n(\pi_n)"] \arrow[d, "H_n(i_{n+1})"] & H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[d, "H_{n-1}(\pi_{n-1})"] \\
 & H_n(X_{n+1}) \arrow[d, "H_n(\pi_{n+1})"] & & H_{n-1}(X_{n-1}, X_{n-2}) \\
 & H_n(X_{n+1}, X_n) \overset{4.2.1}{=} 0. & &
\end{tikzcd}$$

As the leftmost entry of the horizontal row is zero, we have that $H_n(\pi_n)$ is injective, which implies $H_n(X_n) \cong \mathrm{im}\, H_n(\pi_n)$ and $\mathrm{im}\, d_{n+1} = \mathrm{im}\,(H_n(\pi_n) \circ \partial_{n+1}) \cong \mathrm{im}\, \partial_{n+1}$. As the top entry of the right column is zero, $H_{n-1}(\pi_{n-1})$ is injective and $\ker d_n = \ker(H_{n-1}(\pi_{n-1}) \circ \partial_n) = \ker \partial_n$. As the bottom entry of the left column is zero, $H_n(i_{n+1})$ is surjective and $H_n(X_{n+1}) = H_n(X_n)/\mathrm{im}\, \partial_{n+1}$. Combining these statements yields with Proposition 4.2.1

$$H_n C_\bullet(X) = \frac{\ker d_n}{\mathrm{im}\, d_{n+1}} = \frac{\ker \partial_n}{\mathrm{im}\, d_{n+1}} = \frac{\mathrm{im}\, H_n(\pi_n)}{\mathrm{im}\, d_{n+1}} = \frac{H_n(X_n)}{\mathrm{im}\, \partial_{n+1}} \cong H_n(X_{n+1}) \overset{4.2.1}{\cong} H_n(X).$$

$\square$

In some cases, no information about the boundary operator of the cellular chain complex is required. Just counting the cells in each dimension of a CW complex is sufficient to compute its homologies. This applies to the homologies of the n-spheres, if they are given the CW structure from Example 4.1.8, 2. but also to complex projective space and its infinite counterpart.

Example 4.2.6:
For complex projective space $\mathbb{C}\mathrm{P}^n$ with the CW structure from Example 4.1.8, 5. we have

$$X_0 = X_1 = \{\bullet\}, \quad X_2 = X_3 = \mathbb{C}\mathrm{P}^1, \ldots, \quad X_{2n-2} = X_{2n-1} = \mathbb{C}\mathrm{P}^{n-1}, \quad X_{2n} = \mathbb{C}\mathrm{P}^n,$$

where X_{2k} is obtained from X_{2k-1} by attaching a $2k$-disc D^{2k} with the canonical surjection $\pi_k : S^{2k-1} \to \mathbb{C}\mathrm{P}^{k-1}$ for $1 \leq k \leq n$.

This implies $C_{2k}(\mathbb{C}\mathrm{P}^n) = H_{2k}(X_{2k}, X_{2k-1}) \cong \mathbb{Z}$ for $0 \leq k \leq n$ and $C_k(\mathbb{C}\mathrm{P}^n) \cong H_k(X_k, X_{k-1}) = 0$ for $k > 2n$ or k odd. The cellular complex is given by

$$0 \to \mathbb{Z} \xrightarrow{d_{2n}} 0 \xrightarrow{d_{2n-1}} \mathbb{Z} \xrightarrow{d_{2n-2}} 0 \to \ldots \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} 0 \xrightarrow{d_1} \mathbb{Z} \to 0,$$

and its homologies are

$$H_k(\mathbb{C}\mathrm{P}^n) = H_k C_\bullet(\mathbb{C}\mathrm{P}^n) = \frac{\ker d_k}{\mathrm{im}\, d_{k+1}} = \begin{cases} \mathbb{Z} & 0 \leq k \leq 2n,\ k \text{ even} \\ 0 & \text{else.} \end{cases}$$

Example 4.2.7: Consider infinite complex projective space $\mathbb{C}\mathrm{P}^\infty$ with the CW structure from Example 4.1.8, 6. We have $X_{2k} = X_{2k+1} = \mathbb{C}\mathrm{P}^k$ for all $k \in \mathbb{N}_0$ and X_{2k} is obtained from X_{2k-1} by attaching a $2k$-disc D^{2k} with the canonical surjection $\pi_k : S^{2k-1} \to \mathbb{C}\mathrm{P}^{k-1}$ for all $k \in \mathbb{N}$.

This yields $C_{2k}(\mathbb{C}\mathrm{P}^\infty) = H_{2k}(X_{2k}, X_{2k-1}) \cong \mathbb{Z}$ and $C_{2k+1}(\mathbb{C}\mathrm{P}^\infty) = H_{2k+1}(X_{2k+1}, X_{2k}) \cong 0$ for all $k \in \mathbb{N}_0$. The cellular complex reads

$$\ldots \to \mathbb{Z} \xrightarrow{d_{2n+2}} 0 \xrightarrow{d_{2n+1}} \mathbb{Z} \xrightarrow{d_{2n}} 0 \xrightarrow{d_{2n-1}} \mathbb{Z} \xrightarrow{d_{2n-2}} 0 \to \ldots \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} 0 \xrightarrow{d_1} \mathbb{Z} \to 0,$$

and its homologies are given by

$$H_k(\mathbb{C}\mathrm{P}^\infty) = H_k C_\bullet(\mathbb{C}\mathrm{P}^\infty) = \begin{cases} \mathbb{Z} & k \in \mathbb{N}_0 \text{ even} \\ 0 & \text{else.} \end{cases}$$

In general, computing the homologies of a CW complex also requires an understanding of the cellular boundary operator. However, its definition in Proposition 4.2.4 makes use of the relative singular homologies and is not adapted to the geometrical picture in terms of cells. The goal is to have a more explicit and geometrical description of this operator in terms of cells and their attaching maps.

We consider a CW complex X and denote by by $f_i : S^{n-1} \to X_n$ the attaching map and by $c_i : D^n \to X_n$ the characteristic map for the n-cell associated to $i \in I_n$

$$\begin{tikzcd} X_n & X_{n-1} \arrow[l, "i_{n-1,n}"'] \\ \amalg_{i\in I_n} D^n \arrow[u, "(c_i)_{i\in I_n}"] & \amalg_{i\in I_n} S^{n-1}. \arrow[l, "(\iota)_{i\in I_n}"] \arrow[u, "(f_i)_{i\in I_n}"'] \end{tikzcd} \tag{33}$$

The cellular boundary operator assigns to every n-cell $\sigma_i = c_i(\mathring{D}^n)$ a linear combination of $(n-1)$-cells $\tau_j = c_j(\mathring{D}^{n-1})$. We must express the coefficients of this linear combination in terms of cells. This is achieved by combining two maps that relate the cells σ_i and τ_j to the $(n-1)$-sphere S^{n-1}, composing them and computing their mapping degree.

The first is the attaching map $f_i : S^{n-1} \to X_{n-1}$ that describes how $\overline{\sigma}_i$ is attached to the $(n-1)$-skeleton X_{n-1}. The second is the collapsing map $\pi_j : X_{n-1} \to S^{n-1}$ for an $(n-1)$-cell τ_j, that collapses the complement of τ_j in X_{n-1} to a point. It is the composite

$$\pi_j : X_{n-1} \xrightarrow{\pi} X_{n-1}/X_{n-2} \cong \vee_{k\in I_{n-1}} S^{n-1} \xrightarrow{p_j} S^{n-1}$$

of the canonical surjection $\pi : X_{n-1} \to X_{n-1}/X_{n-2} \cong \vee_{k\in I_{n-1}} S^{n-1}$ from Lemma 4.1.18 and the map $p_j : \vee_{k\in I_{n-1}} S^{n-1} \to S^{n-1}$ from Remark 3.5.8 that selects the $(n-1)$-sphere for $j \in I_{n-1}$.

The composite $\pi_j \circ f_i : S^{n-1} \to X_{n-1} \to S^{n-1}$ describes the contribution of the boundary of the n-cell σ_i to the $(n-1)$-sphere obtained by collapsing the boundary of τ_j. As a map between $(n-1)$-spheres it has a mapping degree $d_{ij} = \deg(\pi_j \circ f_i) \in \mathbb{Z}$. This is the coefficient of τ_j in the expression for the cellular boundary $d_n(\sigma_i)$ as a linear combination of $(n-1)$-cells.

Note in particular that there can be only finitely many non-vanishing mapping degrees d_{ij} for a given n-cell σ_j. As $f_i(S^{n-1})$ is compact, it intersects only finitely many $(n-1)$-cells τ_j by Lemma 4.1.11. If it does not intersect an $(n-1)$-cell τ_j, the map $\pi_j \circ f_i$ is not surjective and hence the mapping degree vanishes by Lemma 3.6.6, 4.

Proposition 4.2.8 (cellular boundary formula):
The boundary operators of the cellular chain complex are given by

$$\begin{aligned} &d_1 = \partial_1 = d_1 : \langle I_1\rangle_{\mathbb{Z}} \to \langle I_0\rangle_{\mathbb{Z}}, \quad i \mapsto c_i(1) - c_i(-1) && n = 1 \\ &d_n : \langle I_n\rangle_{\mathbb{Z}} \to \langle I_{n-1}\rangle_{\mathbb{Z}}, \quad i \mapsto \Sigma_{j\in I_{n-1}} d_{ij} j && n \geq 2, \end{aligned}$$

where $d_{ij} = \deg(\pi_j \circ f_i)$, the map $f_i : S^{n-1} \to X_{n-1}$ is the attaching map for $c_i(\mathring{D}^n)$ and the map $\pi_j : X_{n-1} \to S^{n-1}$ collapses the complement of $c_j(\mathring{D}^{n-1})$ to a point.

Proof:
For $n = 1$ the cellular boundary operator is given by $d_1 = \partial_1 : \langle I_1\rangle_{\mathbb{Z}} \to Z_0(X_0)$, $i \mapsto c_i(1) - c_i(-1)$, as X_0 is a discrete set and every point $x \in X_0$ is its own path component.

Let $n \geq 2$. From the pairs (D^n, S^{n-1}) and (X_n, X_{n-1}) we obtain for all $i \in I_n$ a commuting diagram of chain complexes with exact rows

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet(S^{n-1}) \arrow[r, "S_\bullet(\iota)"] \arrow[d, "S_\bullet(f_i)"] & S_\bullet(D^n) \arrow[r, "S_\bullet(\pi)"] \arrow[d, "S_\bullet(c_i)"] & S_\bullet(D^n, S^{n-1}) \arrow[r] \arrow[d, "S_\bullet(c_i)"] & 0 \\
0 \arrow[r] & S_\bullet(X_{n-1}) \arrow[r, "S_\bullet(i_{n,n-1})"'] & S_\bullet(X_n) \arrow[r, "S_\bullet(\pi)"'] & S_\bullet(X_n, X_{n-1}) \arrow[r] & 0,
\end{tikzcd}$$

in which the left square commutes by (33), which implies $i_{n,n-1} \circ f_i = c_i \circ \iota$. By combining the associated long exact homology sequences with the definition of the cellular boundary operators in Proposition 4.2.4 we obtain for all $i \in I_n$ and $j \in I_{n-1}$ the following commuting diagram

$$\begin{tikzcd}
 & \mathbb{Z} \arrow[r, "\deg(\pi_j \circ f_i)"] \arrow[d, "\phi", "\cong"'] & \mathbb{Z} \arrow[d, "\phi", "\cong"'] \\
H_n(D^n, S^{n-1}) \arrow[r, "\partial_n", "\cong"'] \arrow[d, "H_n(c_i)"] & H_{n-1}(S^{n-1}) \arrow[r, "H_{n-1}(\pi_j \circ f_i)"] \arrow[d, "H_{n-1}(f_i)"] & H_{n-1}(S^{n-1}) \\
\langle I_n \rangle_{\mathbb{Z}} \cong H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & H_{n-1}(X_{n-1}) \arrow[r, "H_{n-1}(\pi)"] \arrow[ur, "H_{n-1}(\pi_j)"] \arrow[d, "H_{n-1}(\pi)"] & H_{n-1}(X_{n-1}/X_{n-2}) \cong H_{n-1}(\vee_{I_{n-1}} S^{n-1}) \arrow[u, "H_{n-1}(p_j)"'] \arrow[dl, "\cong"] \\
 & \langle I_{n-1} \rangle_{\mathbb{Z}} \cong H_{n-1}(X_{n-1}, X_{n-2}), &
\end{tikzcd}$$

in which

- the upper left square commutes by naturality of the connecting homomorphism,
- the lower left triangle by definition of the cellular boundary operator,
- the upper right square by Definition 3.6.5 of the mapping degree,
- the middle right square by by the identity $\pi_j = p_j \circ \pi$,
- the lower right triangle by Proposition 3.5.3.

Comparing the paths that go along the boundary of the diagram then shows that

$$H_{n-1}(p_j) \circ d_n \circ H_n(c_i) = \phi \circ (z \mapsto \deg(\pi_j \circ f_i) \cdot z) \circ \phi^{-1} \circ \partial_n.$$

□

Example 4.2.9: We consider real projective space $\mathbb{RP}^n = S^n / \sim$ with $x \sim -x$ for all $x \in S^n$ and the CW structure from Example 4.1.8, 4. The skeleta are given by $X_k = \mathbb{RP}^k$ for $0 \leq k \leq n$ with $\mathbb{RP}^0 = \bullet$. the k-skeleton $\mathbb{RP}^k$ is obtained from $\mathbb{RP}^{k-1}$ by attaching a single k-disc with the canonical surjection $f_k : S^{k-1} \to \mathbb{RP}^k$. The cellular chain complex is of the form

$$C_\bullet(\mathbb{RP}^n) = (0 \to \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \mathbb{Z} \xrightarrow{d_{n-2}} \ldots \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0).$$

As a group endomorphism of $\mathbb{Z}$, each cellular boundary operator is of the form $d_k : \mathbb{Z} \to \mathbb{Z}$, $z \mapsto n_k \cdot z$ with an integer $n_k \in \mathbb{Z}$.

To compute the integers n_k, we apply the cellular boundary formula in Proposition 4.2.8. As $X_0 = \bullet$, we have $d_1 = 0 : \mathbb{Z} \to \mathbb{Z}$. The other cellular boundary operators are given by the integers $n_k = \deg(\pi_{k-1} \circ f_k)$, where $\pi_{k-1} : \mathbb{RP}^{k-1} \to S^{k-1}$ collapses the complement of the $(k-1)$-cell in $\mathbb{RP}^{k-1}$ to a point.

This map is induced by the pinch map $P_{k-1} : S^{k-1} \to S^{k-2}/S^{k-1} \cong S^{k-1} \vee S^{k-1}$ that squeezes the equator of S^{k-1} to a point and by the fold map $F_{k-1} : S^{k-1} \vee S^{k-1} \to S^{k-1}$ from Definition 3.6.10. More specifically, as the antipodal map $a_{k-1} : S^{k-1} \to S^{k-1}$, $x \mapsto -x$ identifies the lower hemisphere with the upper hemisphere and can be adjusted with a rotation to preserve a selected point, we have the commuting diagram

$$\begin{tikzcd}
S^{k-1} \arrow[rr, "f_k"] \arrow[d, "P_{k-1}"'] & & \mathbb{R}\mathrm{P}^{k-1} = S^{k-1}/\sim \arrow[d, "\pi_{k-1}"] \\
S^{k-1}/S^{k-2} \cong S^{k-1} \vee S^{k-1} \arrow[r, "\mathrm{id}_{S^{k-1}} \vee a_{k-1}"'] & S^{k-1} \vee S^{k-1} \arrow[r, "F_{k-1}"'] & S^{k-1}.
\end{tikzcd}$$

As $\deg(\mathrm{id}_{S^{k-1}}) = 1$ by Lemma 3.6.6 and $\deg(a_{k-1}) = (-1)^k$ by Corollary 3.6.8, this implies

$$\begin{aligned} n_k &= \deg(\pi_{k-1} \circ f_k) = \deg(F_{k-1} \circ (\mathrm{id}_{S^{k-1}} \vee a_{k-1}) \circ P_{k-1}) \overset{3.6.11}{=} \deg(\mathrm{id}_{S^{k-1}}) + \deg(a_{k-1}) \\ &= 1 + (-1)^k. \end{aligned}$$

Thus, the cellular chain complex is

$$C_\bullet(\mathbb{R}\mathrm{P}^n) = (0 \to \mathbb{Z} \xrightarrow{d_n : z \mapsto (1+(-1)^n) \cdot z} \dots \xrightarrow{d_3 : z \mapsto 0} \mathbb{Z} \xrightarrow{d_2 : z \mapsto 2z} \mathbb{Z} \xrightarrow{d_1 : z \mapsto 0} \mathbb{Z} \to 0),$$

and its homologies are given by

$$H_k(\mathbb{R}\mathrm{P}^n) = H_k C_\bullet(\mathbb{R}\mathrm{P}^n) = \begin{cases} \mathbb{Z} & k = 0 \text{ or } k = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 1 \leq k < n \text{ odd} \\ 0 & k \text{ even or } k > n. \end{cases}$$

5 Homology with coefficients

5.1 Homology with coefficients

In Proposition 3.1.5 we defined the singular chain complex of a topological space X by considering the free abelian groups $S_n(X)$ generated by the set $\mathrm{Sing}_n(X) = \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, X)$ of its singular n-simplexes $\sigma : \Delta^n \to X$. Elements of $S_n(X)$ are finite linear combinations of singular n-simplexes with coefficients in $\mathbb{Z}$. The boundary operator is obtained by pre-composing each n-simplex with face maps and taking an alternating sum over all such composites. The result is an ordinary cohomology theory in the sense of Definition 3.7.1 with coefficient group $H_0(\bullet) = \mathbb{Z}$.

To construct homology theories with other coefficient groups, note that an analogous construction can be performed for any abelian group M instead of $\mathbb{Z}$, by replacing the free abelian group $S_n(X) = \oplus_{\mathrm{Sing}_n(X)}\mathbb{Z}$ by the direct sum $S_n(X;M) = \oplus_{\mathrm{Sing}_n(X)}M$. By Definition 1.1.8 elements of $S_n(X;M)$ are tuples $(m_\sigma)_{\sigma\in\mathrm{Sing}_n(X)}$, where $m_\sigma = 0$ for almost all $\sigma \in \mathrm{Sing}_n(X)$, or, equivalently, finite linear combinations $\Sigma_{i\in I} m_i\sigma_i$ of singular n-simplexes with coefficients $m_i \in M$. The boundary operator acts again by pre-composing each n-simplex with face maps and taking an alternating sum over all such composites.

Definition 5.1.1: Let M be an abelian group.
The **singular chain complex** $S_\bullet(X;M)$ with **coefficients in** M consists of the abelian groups

$$S_n(X;M) = \oplus_{\mathrm{Sing}_n(X)}M,$$

and the boundary operators $d_n : S_n(X;M) \to S_{n-1}(X;M)$, $\Sigma_{i\in I} m_i\sigma_i \mapsto \Sigma_{i\in I} m_i d_n(\sigma_i)$. The nth **singular homology with coefficients** in M is

$$H_n(X;M) := H_n S_\bullet(X;M).$$

Note that we have $S_n(X;\mathbb{Z}) = S_n(X)$ by definition. The identities $d_{n-1} \circ d_n = 0$ follow analogously to the corresponding identities for $M = \mathbb{Z}$. Note also that we have $S_n(\bullet;M) = M$ and $d_n = \mathrm{id}_M : S_n(\bullet;M) \to S_n(\bullet;M)$ for all $n \in \mathbb{N}_0$, which yields $H_n(\bullet;M) = 0$ for $n \neq 0$ and $H_0(\bullet;M) = M$. This shows in particular that any abelian group can be realised as the coefficient group of a homology theory.

In the same way, we can also generalise cellular homology of CW complexes from proposition 4.2.4. In this case, we replace the free abelian groups generated by the set I_n of n-cells by the direct sum $\oplus_{I_n}M$. The action of the boundary operator is given by its actions on n-cells.

Definition 5.1.2: Let M be an abelian group, X a CW complex and I_n its set of n-cells.
The **cellular chain complex** $C_\bullet(X;M)$ with **coefficients in** M is given by the abelian groups

$$C_n(X;M) = \oplus_{I_n}M$$

and the boundary operator $d_n : C_n(X;M) \to C_{n-1}(X;M)$, $\Sigma_{i\in I_n} m_i c_i \mapsto \Sigma_{i\in I} m_i d_n(c_i)$. Its nth homology $H_n C_\bullet(X;M)$ is called the nth **cellular homology with coefficients** in M.

One might wonder why it is necessary or interesting to extend the singular homologies of topological spaces from the ones with coefficients in $\mathbb{Z}$ to coefficients in an abelian group M.

One of the reasons is that there are many other approaches to homology and cohomology in specific settings. For instance, de Rham cohomology of smooth manifolds is formulated in terms of differential forms. This yields a cohomology theory with coefficients in the abelian group $(\mathbb{R}, +)$. In order to compare these approaches, one needs to handle this case in the framework of singular and cellular homology as well.

The other reason is that, depending on the choice of the abelian group M, these homology theories contain different information and yield different results. This is apparent when one considers the cellular homologies of $\mathbb{R}\mathrm{P}^n$.

Example 5.1.3: By Example 4.2.9 the cellular chain complex for $\mathbb{R}\mathrm{P}^n$ is given by

$$C_\bullet(\mathbb{R}\mathrm{P}^n) = (0 \to \mathbb{Z} \xrightarrow{d_n} \mathbb{Z} \xrightarrow{d_{n-1}} \mathbb{Z} \xrightarrow{d_{n-2}} \ldots \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0) \qquad d_k = 1 + (-1)^k.$$

with homologies

$$H_k(\mathbb{R}\mathrm{P}^n) = H_k C_\bullet(\mathbb{R}\mathrm{P}^n) = \begin{cases} \mathbb{Z} & k = 0 \text{ or } k = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 1 \leq k < n \text{ odd} \\ 0 & k \text{ even or } k > n. \end{cases}$$

• If $M = \mathbb{Z}/2\mathbb{Z}$, we have $d_k = 1 + (-1)^k = 2 = 0$, and this yields

$$C_\bullet(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2\mathbb{Z}) = (0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{d_n=0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{d_{n-1}=0} \ldots \xrightarrow{d_2=0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{d_1=0} \mathbb{Z}/2\mathbb{Z} \to 0)$$

and homologies

$$H_k C_\bullet(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & 0 \leq k \leq n \\ 0 & \text{else.} \end{cases}$$

• If $M = (\mathbb{F}, +)$ is the additive group of a field $\mathbb{F}$ of characteristic $\mathrm{char}\,\mathbb{F} \neq 2$, then

$$C_\bullet(\mathbb{R}\mathrm{P}^n, \mathbb{F}) = (0 \to \mathbb{F} \xrightarrow{d_n} \mathbb{F} \xrightarrow{d_{n-1}} \ldots \mathbb{F} \xrightarrow{d_3=0} \mathbb{F} \xrightarrow{d_2=2} \mathbb{F} \xrightarrow{d_1=0} \mathbb{F} \to 0) \qquad d_k = 1 + (-1)^k.$$

As $2\mathbb{F} \cong \mathbb{F}$, the homologies are then given by

$$H_k C_\bullet(\mathbb{R}\mathrm{P}^n, \mathbb{F}) = \begin{cases} \mathbb{F} & k = 0 \text{ or } k = n \text{ odd} \\ 0 & \text{else.} \end{cases}$$

5.2 Tensor product of abelian groups

In the following sections, we will compare the homology groups with coefficients in different abelian groups systematically and finally show that they are all determined by the homologies with coefficients in $\mathbb{Z}$, together with information that only depends on the abelian groups.

In this section, we introduce the algebraic background for this comparison, namely tensor products of abelian groups. This is a special case of the tensor product of modules over rings. Hence, it can be viewed as the abelian group counterpart of the tensor product of vector spaces, but it behaves differently, especially for abelian groups with torsion.

Definition 5.2.1: The **tensor product of abelian groups** A, B is the abelian group $A \otimes B$ with generating set $A \times B$ and relations

$$(a+a', b) - (a, b) - (a', b) \qquad (a, b+b') - (a, b) - (a, b') \qquad \forall a, a' \in A,\ b, b' \in B.$$

We denote by $\tau = \pi \circ \iota : A \times B \to A \otimes B$ the composite of the inclusion $\iota : A \times B \to \langle A \times B \rangle_{\mathbb{Z}}$ and the canonical surjection $\pi : \langle A \times B \rangle_{\mathbb{Z}} \to A \otimes B$ and set $a \otimes b = \tau(a, b)$ for all $a \in A$, $b \in B$.

Remark 5.2.2:

1. The relations of the tensor product imply for all $n \in \mathbb{Z}$, $a, a' \in A$, $b, b' \in B$

   $$(a+a') \otimes b = a \otimes b + a' \otimes b, \qquad a \otimes (b+b') = a \otimes b + a \otimes b', \qquad n(a \otimes b) = (na) \otimes b = a \otimes (nb).$$

2. The elements $a \otimes b$ for $a \in A$ and $b \in B$ generate the abelian group $A \otimes B$:
   Any element of $A \otimes B$ is a finite sum $\Sigma_{i=1}^m n_i\, a_i \otimes b_i$ with $m \in \mathbb{N}_0$, $n_i \in \mathbb{Z}$, $a_i \in A$, $b_i \in B_i$.

   This follows, because the elements of the free group $\langle A \times B \rangle_{\mathbb{Z}} = \oplus_{A \times B} \mathbb{Z}$ are finite linear combinations of elements in $A \times B$ and $\pi : \langle A \times B \rangle_{\mathbb{Z}} \to A \otimes B$ is surjective.

3. Due to 2. we often denote a group homomorphism $f : A \otimes B \to C$ into an abelian group C by $f : A \otimes B \to C$, $a \otimes b \mapsto f(a \otimes b)$. As the elements $a \otimes b$ generate $A \otimes B$, the values $f(a \otimes b)$ determine f completely.

The tensor product of abelian groups has a universal property that is analogous to the one of tensor products of vector spaces. The only difference is that vector spaces are replaced by abelian groups and instead of (bi)linear maps over a field, one considers $\mathbb{Z}$-(bi)linear maps. It arises from the universal property of the free group, cf. Remark 1.1.12, and the universal property of the factor group.

Lemma 5.2.3: The tensor product has the following **universal property**:
The map $\tau = \pi \circ \iota : A \times B \to A \otimes B$ is $\mathbb{Z}$-bilinear:

$$\tau(a+a', b) = \tau(a, b) + \tau(a', b) \qquad \tau(a, b+b') = \tau(a, b) + \tau(a, b') \quad \forall a, a' \in A,\ b, b' \in B.$$

For every $\mathbb{Z}$-bilinear map $f : A \times B \to C$ into an abelian group C, there is a unique group homomorphism $f' : A \otimes B \to C$ with $f' \circ \tau = f$

$$\begin{tikzcd} A \times B \arrow[rr, "f"] \arrow[dr, "\tau"'] & & C \\ & A \otimes B \arrow[ur, dashed, "\exists ! f'"'] & \end{tikzcd}$$

Proof:
The $\mathbb{Z}$-bilinearity of τ holds by definition, due to the relations. To show uniqueness, suppose $g, h : A \otimes B \to C$ are group homomorphisms with $g \circ \tau = f = h \circ \tau$. Then we have $g \circ \pi \circ \iota = h \circ \pi \circ \iota$ and $g \circ \pi = h \circ \pi$ by the universal property of the free group. By surjectivity of π this implies $g = h$. This shows uniqueness.

To show existence, note that by the universal property of the free group, there is a unique group homomorphism $f'' : \langle A \times B \rangle_{\mathbb{Z}} \to C$ with $f'' \circ \iota = f$. Due to the $\mathbb{Z}$-bilinearity of f and because f'' is a group homomorphism, we have

$$f''((a+a', b) - (a, b) - (a', b)) = f(a+a', b) - f(a, b) - f(a', b) = 0$$
$$f''((a, b+b') - (a, b) - (a, b')) = f(a, b+b') - f(a, b) - f(a, b') = 0.$$

Hence, the set $U \subset \langle A \times B \rangle_{\mathbb{Z}}$ of relations satisfies $U \subset \ker f''$. By the universal property of the factor group there is a unique group homomorphism $f' : A \otimes B \to C$ with $f' \circ \pi = f''$. This implies $f' \circ \tau = f' \circ \pi \circ \iota = f'' \circ \iota = f$. □

Clearly, the universal property of the tensor product is useful to construct group homomorphisms $f' : A \otimes B \to C$ *from* the tensor product of abelian groups A, B into an abelian group C. To construct them, it is sufficient to specify a $\mathbb{Z}$-bilinear map from $A \times B$ to C. An applicatoin is the group isomorphism in the following example, which is left as an exercise.

Example 5.2.4: (Exercise 54)
For $m, n \in \mathbb{N}$, $m, n \geq 2$ the tensor product of the abelian groups $\mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ is

$$\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}.$$

The universal property of the tensor product also allows one to extend the tensor product from abelian groups to group homomorphisms between them. As this is compatible with the identity maps and the composition of group homomorphisms, it defines a functor $-\otimes- : \mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}$. Fixing one of its arguments to an abelian group A yields functors $A \otimes -, -\otimes A : \mathrm{Ab} \to \mathrm{Ab}$. Any group homomorphism defines natural transformations between such functors.

Proposition 5.2.5: The tensor product defines

- a functor $-\otimes- : \mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}$,
- for each abelian group A functors $A \otimes - : \mathrm{Ab} \to \mathrm{Ab}$ and $-\otimes A : \mathrm{Ab} \to \mathrm{Ab}$,
- for each group homomorphism $f : A \to A'$ natural transformations $f \otimes - : A \otimes - \Rightarrow A' \otimes -$ and $-\otimes f : -\otimes A \Rightarrow -\otimes A'$.

Proof:
1. For all group homomorphisms $f : A \to A'$ and $g : B \to B'$ between abelian groups, the map $\tau' \circ (f, g) : A \times B \to A' \otimes B'$ is $\mathbb{Z}$-bilinear

$$\begin{aligned}
\tau' \circ (f,g)(a + a', b) &= \tau'(f(a + a'), g(b)) = \tau'(f(a) + f(a'), g(b)) = \tau'(f(a), g(b)) + \tau'(f(a'), g(b)) \\
&= \tau' \circ (f,g)(a,b) + \tau' \circ (f,g)(a', b) \\
\tau' \circ (f,g)(a, b + b') &= \tau'(f(a), g(b + b')) = \tau'(f(a), g(b) + g(b')) = \tau'(f(a), g(b)) + \tau'(f(a), g(b')) \\
&= \tau' \circ (f,g)(a,b) + \tau' \circ (f,g)(a, b').
\end{aligned}$$

By the universal property of the tensor product, there exists a unique group homomorphism $f \otimes g : A \otimes B \to A' \otimes B'$ with $(f \otimes g) \circ \tau = \tau' \circ (f, g)$. This is equivalent to the condition $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$ for all $a \in A$, $b \in B$.

$$\begin{tikzcd}
A \times B \arrow[r, "(f,g)"] \arrow[d, "\tau"] & A' \times B' \arrow[d, "\tau'"] \\
A \otimes B \arrow[r, "f \otimes g"'] & A' \otimes B'.
\end{tikzcd} \tag{34}$$

The functor $-\otimes- : \mathrm{Ab} \times \mathrm{Ab} \to \mathrm{Ab}$ assigns

- to a pair (A, B) of abelian groups the tensor product $A \otimes B$,
- to a pair (f, g) of group homomorphisms $f : A \to A'$ and $g : B \to B'$ of abelian groups the group homomorphism $f \otimes g : A \otimes B \to A' \otimes B'$, $a \otimes b \mapsto f(a) \otimes g(b)$.

That it is a functor follows, because the following diagrams commute

$$\begin{tikzcd}
A\times B \arrow[r, "(\mathrm{id}_A,\mathrm{id}_B)"] \arrow[d, "\tau"] & A\times B \arrow[d, "\tau"] \\
A\otimes B \arrow[r, "\mathrm{id}_{A\otimes B}"'] & A\otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
A\times B \arrow[r, "(f,g)"'] \arrow[rr, bend left, "(f'\circ f,g'\circ g)"] \arrow[d, "\tau"] & A'\times B' \arrow[r, "(f',g')"'] \arrow[d, "\tau'"] & A''\times B'' \arrow[d, "\tau''"] \\
A\otimes B \arrow[r, "f\otimes g"'] & A'\otimes B' \arrow[r, "f'\otimes g'"'] & A''\otimes B''.
\end{tikzcd}$$

The functor $-\otimes A : \mathrm{Ab} \to \mathrm{Ab}$ for an abelian group A assigns

- to an abelian group B the abelian group $B\otimes A$,
- to a group homomorphism $f : B \to B'$ the group homomorphism $f\otimes \mathrm{id}_A : B\otimes A \to B'\otimes A$,

and the functor $A\otimes - : \mathrm{Ab} \to \mathrm{Ab}$ is defined analogously. Claims 2. and 3. follow from 1. and the properties of products of groups (Exercise). $\square$

We now look in more depth at the properties of tensor products, in particular, its interaction with direct sums of abelian groups and the role of the abelian group $\mathbb{Z}$. The following lemma shows that we can see abelian groups as an analogue of elements of a commutative unital ring, with the direct sum replacing the ring addition and the tensor product replacing the ring multiplication. The neutral element of the direct sum is the trivial group, and the neutral element of the tensor product the group $\mathbb{Z}$. There is also an associativity law for the tensor product and a distributive law between tensor products and direct sums. Both hold up to canonical isomorphisms.

Lemma 5.2.6 (properties of the tensor product):
The tensor product of abelian groups has the following properties:

1. $0\otimes A = A\otimes 0 = 0$ for all abelian groups A,
2. $\mathbb{Z}\otimes A \cong A\otimes\mathbb{Z} \cong A$ for all abelian groups A,
3. $A\otimes B \cong B\otimes A$ for all abelian groups A, B,
4. $(A\otimes B)\otimes C \cong A\otimes(B\otimes C)$ for all abelian groups A, B, C,
5. $A\otimes(\oplus_{i\in I} B_i) \cong \oplus_{i\in I}(A\otimes B_i)$ and $(\oplus_{i\in I} B_i)\otimes A \cong \oplus_{i\in I}(B_i\otimes A)$ for all abelian groups A and families $(B_i)_{i\in I}$ of abelian groups,
6. $f\otimes 0 = 0\otimes g = 0 : A\otimes B \to A'\otimes B'$ for all group homomorphisms $f : A \to A'$, $g : B \to B'$,
7. For all group homomorphisms $f, f' : A \to A'$ and $g, g' : B \to B'$

$$(f+f')\otimes g = f\otimes g + f'\otimes g \qquad f\otimes(g+g') = f\otimes g + f\otimes g'.$$

Proof:
1. We have $0\otimes a = (0\cdot 0)\otimes a = 0\cdot(0\otimes a) = 0$ for all $a \in A$. As the elements $0\otimes a$ generate $0\otimes A$, this proves the claim. The proof for $A\otimes 0$ is analogous.

2. The map $f : \mathbb{Z}\otimes A \to A$, $\lambda\otimes a \mapsto \lambda a$ is a group isomorphism with inverse $f^{-1} : A \to \mathbb{Z}\otimes A$, $a \mapsto 1\otimes a$. One has $f\circ f^{-1}(a) = a$ and $f^{-1}\circ f(z\otimes a) = z(1\otimes a) = z\otimes a$ for all $a \in A$ and $z \in \mathbb{Z}$.

3. The group isomorphism is given by $f : A\otimes B \to B\otimes A$, $a\otimes b \mapsto b\otimes a$.

4. The isomorphism is given by $f : (A\otimes B)\otimes C \to A\otimes(B\otimes C)$, $(a\otimes b)\otimes c \mapsto a\otimes(b\otimes c)$.

5. The map $f : A\otimes(\oplus_{i\in I}B_i) \to \oplus_{i\in I}(A\otimes B_i)$, $a\otimes(b_i)_{i\in I} \to (a\otimes b_i)_{i\in I}$ is a group homomorphism with inverse $f^{-1} : \oplus_{i\in I}(A\otimes B_i) \to A\otimes(\oplus_{i\in I}B_i)$, $(a\otimes b_i)_{i\in I} \mapsto a\otimes(b_i)_{i\in I}$.

6.-7. Exercise. $\square$

Note that there is no counterpart of claim 5. for direct products of abelian groups. For an abelian group A and a family $(B_i)_{i\in I}$ indexed by an infinite index set I, the abelian group $A\otimes(\Pi_{i\in I}B_i)$ is in general not isomorphic to $\Pi_{i\in I}(A\otimes B_i)$ (Exercise 56).

Claims 2. and 5. in Lemma 5.2.6 relate tensor products $\langle Y\rangle_{\mathbb{Z}}\otimes M$ of a free abelian group $\langle Y\rangle_{\mathbb{Z}}$ with an abelian group M to direct sums of M with itself over the index set Y:

$$\oplus_{y\in Y}M \overset{2.}{\cong} \oplus_{y\in Y}(\mathbb{Z}\otimes M) \overset{5.}{\cong} (\oplus_{y\in Y}\mathbb{Z})\otimes M = \langle Y\rangle_{\mathbb{Z}}\otimes M.$$

This allows us to rewrite the groups of n-chains of singular chain complex $S_\bullet(X;M)$ in Definition 5.1.1 as the tensor products $S_n(X;M) = S_n(X)\otimes M$. Their boundary operators are then given by $d_n\otimes \mathrm{id}_M : S_n(X)\otimes M \to S_{n-1}(X)\otimes M$, $\Sigma_{i\in I}\sigma_i\otimes m_i \mapsto \Sigma_{i\in I}d_n(\sigma_i)\otimes m_i$.

The benefit of this viewpoint is that it allows us to express the singular chain complexes with coefficients in an abelian group M as a functor, analogously to the singular chain complex functor $S_\bullet : \mathrm{Top} \to \mathrm{Ch}_{\mathrm{Ab}}$ from Proposition 3.1.5.

To achieve this, we extend the functor $-\otimes M : \mathrm{Ab} \to \mathrm{Ab}$ from Proposition 5.2.5 to a functor $-\otimes M : \mathrm{Ch}_{\mathrm{Ab}} \to \mathrm{Ch}_{\mathrm{Ab}}$ by tensoring each group of n-chains with M and each boundary operator with the identity id_M. Composing it with the singular homology functor $S_\bullet : \mathrm{Top} \to \mathrm{Ch}_{\mathrm{Ab}}$ then yields singular homology with coefficients in M. Likewise, composing it with the relative chain complex functor $S_\bullet : \mathrm{Top}(2) \to \mathrm{Ch}_{\mathrm{Ab}}$ from Proposition 3.3.4 yields relative chain complexes with coefficients in M.

Corollary 5.2.7: Let M be an abelian group. The tensor product of abelian groups defines a functor $-\otimes M : \mathrm{Ch}_{\mathrm{Ab}} \to \mathrm{Ch}_{\mathrm{Ab}}$ that assigns

- to a chain complex $X_\bullet$ the chain complex $X_\bullet\otimes M$ with $(X_\bullet\otimes M)_n = X_n\otimes M$ and boundary operators $d_n = d_n\otimes \mathrm{id}_M : X_n\otimes M \to X_{n-1}\otimes M$,
- to a chain map $f_\bullet : X_\bullet \to X'_\bullet$ the chain map $f_\bullet\otimes \mathrm{id}_M : X_\bullet\otimes M \to X'_\bullet\otimes M$ with components $f_n\otimes \mathrm{id}_M : X_n\otimes M \to X'_n\otimes M$.

Proof:
That $X_\bullet\otimes M$ is indeed a chain complex follows from the functoriality of $-\otimes M : \mathrm{Ab} \to \mathrm{Ab}$, together with item 6. in Lemma 5.2.6, which give

$$(d_{n-1}\otimes \mathrm{id}_M)\circ(d_n\otimes \mathrm{id}_M) = (d_{n-1}\circ d_n)\otimes \mathrm{id}_M = 0\otimes \mathrm{id}_M \overset{6.}{=} 0 : X_n\otimes M \to X_{n-2}\otimes M.$$

That the group homomorphisms $f_n\otimes \mathrm{id}_M : X_n\otimes M \to X'_n\otimes M$ define a chain map, and that these assignments are compatible with the composition of morphism follows again from the functoriality of $-\otimes M : \mathrm{Ab} \to \mathrm{Ab}$

$$(f_{n-1}\otimes \mathrm{id}_M)\circ(d_n\otimes \mathrm{id}_M) = (f_{n-1}\circ d_n)\otimes \mathrm{id}_M = (d'_n\circ f_n)\otimes \mathrm{id}_M = (d'_n\otimes \mathrm{id}_M)\circ(f_n\otimes \mathrm{id}_M)$$
$$(g_n\otimes \mathrm{id}_M)\circ(f_n\otimes \mathrm{id}_M) = (g_n\circ f_n)\otimes \mathrm{id}_M \qquad (\mathrm{id}_{X_n}\otimes \mathrm{id}_M) = \mathrm{id}_{X_n\otimes M}$$

$\square$

Corollary 5.2.8: Let M be an abelian group. Then the singular chain complex with coefficients in M is given by the functor $S_\bullet(-;M) = (-\otimes M)S_\bullet : \text{Top} \to \text{Ch}_{\text{Ab}}$ that assigns

- to a topological space X the chain complex $S_\bullet(X;M) = S_\bullet(X)\otimes M$,
- to a continuous map $f : X \to Y$ the chain map $S_\bullet(f)\otimes\text{id}_M : S_\bullet(X;M) \to S_\bullet(Y;M)$.

There is a functor $S_\bullet(-;M) : \text{Top}(2) \to \text{Ch}_{\text{Ab}}$, the **relative chain complex functor with coefficients** in M, that assigns

- to a pair of spaces (X,A) the chain complex $S_\bullet(X,A;M) = S_\bullet(X,A)\otimes M$,
- to a morphism of pairs $f : (X,A) \to (Y,B)$ the chain map

$$S_\bullet(f)\otimes\text{id}_M : S_\bullet(X,A;M) \to S_\bullet(Y,B;M).$$

5.3 The torsion functor

Corollary 5.2.8 establishes that singular chain complexes with coefficients in an abelian group A are obtained by post-composing the singular homology functor $S_\bullet : \text{Top} \to \text{Ch}_{\text{Ab}}$ with the functor $-\otimes A : \text{Ch}_{\text{Ab}} \to \text{Ch}_{\text{Ab}}$. The aim is now to understand how this affects their homologies. In other words, we need to understand how tensoring with an abelian group A affects the kernels and images of the boundary operators and their quotients.

By Remark 2.2.2 we can characterise this in terms of short exact sequences. The homologies of a chain complex $X_\bullet$ are given as the quotients $H_n(X) = \ker d_n/\text{im}\, d_{n+1}$ with respect to the subgroup $\text{im}\, d_{n+1} \subset \ker d_n$. Remark 2.2.2 states that this is equivalent to a short exact sequence

$$0 \to B_n(X) = \text{im}\, d_{n+1} \xrightarrow{\iota_n} Z_n(X) = \ker d_n \xrightarrow{\pi_n} H_n(X) = \ker d_n/\text{im}\, d_{n+1} \to 0. \tag{35}$$

We therefore need to investigate how tensoring a short exact sequence of abelian groups with a fixed abelian group affects its exactness. Unfortunately, it turns out that tensoring with an abelian group A does in general not preserve exactness.

Example 5.3.1: Take the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\iota: z\mapsto n\cdot z} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$ for $2 \leq n \in \mathbb{N}$.

Then tensoring with the abelian group $\mathbb{Z}/n\mathbb{Z}$ yields the sequence

$$0 \to \mathbb{Z}\otimes\mathbb{Z}/n\mathbb{Z} \xrightarrow{\iota\otimes\text{id}} \mathbb{Z}\otimes\mathbb{Z}/n\mathbb{Z} \xrightarrow{\pi\otimes\text{id}} \mathbb{Z}/n\mathbb{Z}\otimes\mathbb{Z}/n\mathbb{Z} \to 0.$$

The map $\iota\otimes\text{id} = 0 : \mathbb{Z}\otimes\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}\otimes\mathbb{Z}/n\mathbb{Z}$ is trivial, as we have for all $\bar{k} \in \mathbb{Z}/n\mathbb{Z}$ and $z \in \mathbb{Z}$

$$(\iota\otimes\text{id})(z\otimes\bar{k}) = (nz)\otimes\bar{k} = n(z\otimes\bar{k}) = z\otimes(n\bar{k}) = z\otimes\bar{0} = 0.$$

Hence, $\iota\otimes\text{id}$ is not injective. We still have $\pi\otimes\text{id}$ surjective and also $\text{im}\,(\iota\otimes\text{id}) = 0 = \ker\,(\pi\otimes\text{id})$, as $(\pi\otimes\text{id})(z\otimes\bar{k}) = \bar{z}\otimes\bar{k} = 0$ implies either $\bar{k} = 0$ and hence $z\otimes\bar{k} = 0$ or $\bar{z} = 0$ and hence $z = nw$ with some $w \in \mathbb{Z}$, which also gives $z\otimes\bar{k} = (nw)\otimes\bar{k} = w\otimes(n\bar{k}) = w\otimes 0 = 0$.

In this example, exactness of the sequence is partially preserved, namely on the right of the sequence, but not on the left. We will see that this is not specific to the chosen short exact sequence or the abelian group it is tensored with, but holds in full generality.

In contrast, the example shows that exactness need not be preserved on the *left* of a sequence: the image $\iota\otimes\text{id}_M : A\otimes M \to B\otimes M$ of an injective group homomorphism $\iota : A \to B$ the under the functor $-\otimes M : \mathrm{Ab} \to \mathrm{Ab}$ need no longer be injective. So tensoring with an abelian group does not necessarily preserve kernels. Exactness on the left of the sequence depends on the short exact sequence and on the abelian group tensored to this sequence.

The main examples of short exact sequences, where tensoring with *any* abelian group yields a short exact sequence are *split exact sequences*. Recall from Exercise 8 that a short exact sequence $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ is called *split exact*, if there is a group isomorphism $\phi : B \to A \oplus C$ such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\iota"] \arrow[dr, "\iota_1 : a \mapsto (a,0)"'] & B \arrow[r, "\pi"] \arrow[d, "\phi", "\cong"'] & C \arrow[r] & 0 \\ & & A \oplus C. \arrow[ur, "\pi_2 : (a,c) \mapsto c"'] & & \end{tikzcd} \tag{36}$$

Lemma 5.3.2: Let M be an abelian group.

1. The functors $M\otimes-$ and $-\otimes M$ are **right exact**:
   for any exact sequence $A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ the following sequences are exact
   $$M\otimes A \xrightarrow{\mathrm{id}\otimes\iota} M\otimes B \xrightarrow{\mathrm{id}\otimes\pi} M\otimes C \to 0 \qquad A\otimes M \xrightarrow{\iota\otimes\mathrm{id}} B\otimes M \xrightarrow{\pi\otimes\mathrm{id}} C\otimes M \to 0.$$
2. If $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ is a **split exact sequence**, then the sequences
   $$0 \to M\otimes A \xrightarrow{\mathrm{id}\otimes\iota} M\otimes B \xrightarrow{\mathrm{id}\otimes\pi} M\otimes C \to 0 \qquad 0 \to A\otimes M \xrightarrow{\iota\otimes\mathrm{id}} B\otimes M \xrightarrow{\pi\otimes\mathrm{id}} C\otimes M \to 0$$
   are split exact. In particular, this holds, whenever C is a free group.

Proof:
1. We prove this for the second sequence. As $A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ is a chain complex, we have $\pi \circ \iota = 0$, and it follows that $(\pi\otimes\mathrm{id}) \circ (\iota\otimes\mathrm{id}) = (\pi \circ \iota)\otimes\mathrm{id} = 0\otimes\mathrm{id} = 0$.

Exactness of the sequence $A\otimes M \xrightarrow{\iota\otimes\mathrm{id}} B\otimes M \xrightarrow{\pi\otimes\mathrm{id}} C\otimes M \to 0$ is then equivalent to the statement that $\pi\otimes\mathrm{id}$ is surjective and $\ker(\pi\otimes\mathrm{id}) = \mathrm{im}\,(\iota\otimes\mathrm{id})$.

To show that $\pi\otimes\mathrm{id}$ is surjective, let $x = \Sigma_{i=1}^k n_i(c_i\otimes m_i) \in C\otimes M$ with $n_i \in \mathbb{Z}$. By surjectivity of π, there are elements $b_i \in B$ with $\pi(b_i) = c_i$, and this implies $(\pi\otimes\mathrm{id})(\Sigma_{i=1}^k n_i(b_i\otimes m_i)) = x$, thus $\pi\otimes\mathrm{id}$ is surjective. As $(\pi\otimes\mathrm{id}) \circ (\iota\otimes\mathrm{id}) = 0$, we also have $\mathrm{im}\,(\iota\otimes\mathrm{id}) \subset \ker(\pi\otimes\mathrm{id})$.

To show that $\mathrm{im}\,(\iota\otimes\mathrm{id}) = \ker(\pi\otimes\mathrm{id})$, let $p : B\otimes M \to (B\otimes M)/\mathrm{im}\,(\iota\otimes\mathrm{id})$ be the canonical surjection. The $\mathbb{Z}$-bilinear map $\phi = p \circ \tau : B \times M \to (B\otimes M)/\mathrm{im}\,(\iota\otimes\mathrm{id})$ satisfies

$$\phi \circ (\iota, \mathrm{id}) = p \circ \tau(\iota, \mathrm{id}) = p \circ (\iota\otimes\mathrm{id}) \circ \tau = 0.$$

We obtain a $\mathbb{Z}$-bilinear map $\psi : C \times M \to (B\otimes M)/\mathrm{im}\,(\iota\otimes\mathrm{id})$, $(\pi(b), m) \mapsto \phi(b, m)$, which is defined, because π is surjective and because $\pi(b) = \pi(b')$ implies $b' - b \in \ker\pi = \mathrm{im}\,\iota$ and $\phi(b', m) = \phi(b, m) + \phi(b' - b, m) = \phi(b, m)$. By construction, we have $\psi \circ (\pi, \mathrm{id}) = \phi$.

By the universal property of the tensor product, there exists a unique group homomorphism $\psi' : C\otimes M \to (B\otimes M)/\mathrm{im}\,(\iota\otimes\mathrm{id})$ with $\psi' \circ \tau = \psi$. This implies

$$\psi' \circ (\pi\otimes\mathrm{id}) \circ \tau = \psi' \circ \tau \circ (\pi, \mathrm{id}) = \psi \circ (\pi, \mathrm{id}) = \phi = p \circ \tau$$

and by the universal property of the tensor product $\psi' \circ (\pi \otimes \mathrm{id}) = p$. Hence, we have $\ker(\pi \otimes \mathrm{id}) \subset \ker p$ and $\ker(\pi \otimes \mathrm{id}) = \mathrm{im}\,(\iota \otimes \mathrm{id})$.

2. If the sequence is split exact, then tensoring diagram (36) with an abelian group M and using the group isomorphism $(A \oplus C) \otimes M \cong (A \otimes M) \oplus (C \otimes M)$ yields the commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & A\otimes M \arrow[rr, "\iota\otimes\mathrm{id}"] \arrow[drr, "\iota_1\otimes\mathrm{id}: a\otimes m \mapsto (a\otimes m, 0)"'] & & B\otimes M \arrow[rr, "\pi\otimes\mathrm{id}"] \arrow[d, "\phi\otimes\mathrm{id}"', "\cong"] & & C\otimes M \arrow[r] & 0 \\ & & & (A\otimes M)\oplus(C\otimes M). \arrow[urr, "\pi_2\otimes\mathrm{id}: (a\otimes m, c\otimes m)\mapsto c\otimes m"'] & & & \end{tikzcd}$$

This shows that the sequence is exact. If $C = \langle X \rangle_{\mathbb{Z}}$ is a free group, we can choose for every $x \in X$ an element $h(x) \in B$ with $\pi(h(x)) = x$ by surjectivity of π. This defines a map $h : X \to B$, and by the universal property of the free group a unique group homomorphism $h : C \to B$ with $\pi \circ h = \mathrm{id}_C$. By Exercise 8 the sequence splits. □

Remark 5.3.3: Lemma 5.3.2 can also be phrased as the statement that tensoring with an abelian group M preserves cokernels. The **cokernel** of a group homomorphism $\iota : A \to B$ is the factor group $\mathrm{coker}(\iota) = B/\mathrm{im}\,\iota$, together with the canonical surjection $p : B \to B/\mathrm{im}\,\iota$. Exactness of the sequence $A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ is equivalent to $C = \mathrm{im}\,\pi \cong B/\ker\pi = B/\mathrm{im}\,\iota$. Thus, Lemma 5.3.2 states that for any such exact sequence

$$(B\otimes M)/\mathrm{im}\,(\iota\otimes\mathrm{id}) = \mathrm{coker}(\iota\otimes\mathrm{id}) = \mathrm{coker}(\iota)\otimes M = (B/\mathrm{im}\,\iota)\otimes M.$$

Lemma 5.3.2 shows that for any short exact sequence that ends in a *free* group C on the right, tensoring with any abelian group M yields again again a short exact sequence. It follows that whenever the nth homology $H_n(X)$ of a topological space X is a *free* group, the image of the short exact sequence (35) is again a short exact sequence

$$0 \to B_n(X)\otimes M \xrightarrow{\iota_n\otimes\mathrm{id}} Z_n(X)\otimes M \xrightarrow{\pi_n\otimes\mathrm{id}} H_n(X)\otimes M \to 0.$$

We will see in the following that this implies $H_n(X; M) = H_n(X)\otimes M$. However, to treat the general case, we require more background. In general, the abelian group C on the right need not be a free group, but by Remark 1.1.12 it can be *presented*, that is, realised as a quotient of a free group. Any presentation $C = \langle M \mid U \rangle$ yields a description of C as a quotient $C = \langle M \rangle_{\mathbb{Z}}/\langle U \rangle$, where $\langle M \rangle_{\mathbb{Z}}$ is the free abelian group generated by M and $\langle U \rangle \subset \langle M \rangle_{\mathbb{Z}}$ is the subgroup generated by U. This corresponds to a short exact sequence

$$0 \to \langle U \rangle \xrightarrow{\iota} \langle M \rangle_{\mathbb{Z}} \xrightarrow{\pi} C \to 0,$$

where $\iota : \langle U \rangle \to \langle M \rangle_{\mathbb{Z}}$ is the inclusion and $\pi : \langle M \rangle_{\mathbb{Z}} \to C$ the canonical surjection. Such a short exact sequence is also called a *free resolution* of C.

Definition 5.3.4: Let A be an abelian group. A **free resolution** of A is a short exact sequence $0 \to K \xrightarrow{\iota} F \xrightarrow{\pi} A \to 0$ of abelian groups, where F is a free abelian group.

Remark 5.3.5:

1. Every abelian group A has a resolution with $F = \langle A \rangle_{\mathbb{Z}} = \oplus_A \mathbb{Z}$. It is given by the group homomorphism $\pi : \langle A \rangle_{\mathbb{Z}} \to A$, $a \mapsto a$ and the inclusion $\iota : \ker \pi \to \langle A \rangle_{\mathbb{Z}}$ and called the **standard resolution** of A.

2. As subgroups of free abelian groups are free and $K \cong \iota(K) \subset F$, the group K in a free resolution is also a free abelian group.

Example 5.3.6: The sequence $0 \to \mathbb{Z} \xrightarrow{\iota: z \mapsto n \cdot z} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}/n\mathbb{Z}$.

We will now use presentations of abelian groups or, equivalently, free resolutions, to describe the non-exactness of a functor $-\otimes M : \mathrm{Ch}_{\mathrm{Ab}} \to \mathrm{Ch}_{\mathrm{Ab}}$ and to relate homologies with coefficients in M to homologies with coefficients in $\mathbb{Z}$. More specifically, we will use the quantity $\ker(\iota \otimes \mathrm{id}) \subset K \otimes M$ to characterise the non-exactness of the functor $-\otimes M$. However, the choice of a presentation and hence of a free resolution is highly non-unique. The first step is to address this non-uniqueness, by understanding how group homomorphisms between abelian groups interact with their free resolutions.

Proposition 5.3.7: Let A, A' be abelian groups, $A_\bullet = 0 \to K \xrightarrow{\iota} F \xrightarrow{\pi} A \to 0$ a free resolution of A and $A'_\bullet = 0 \to K' \xrightarrow{\iota} F' \xrightarrow{\pi} A' \to 0$ a short exact sequence.

1. For any group homomorphism $f : A \to A'$ there are group homomorphisms $g : F \to F'$ and $h : K \to K'$ that form a chain map $f_\bullet = (h, g, f) : A_\bullet \to A'_\bullet$

$$
\begin{tikzcd}
A_\bullet = \arrow[d, "f_\bullet"] & 0 \arrow[r] & K \arrow[r, "\iota"] \arrow[d, "h"] & F \arrow[r, "\pi"] \arrow[d, "g"] & A \arrow[r] \arrow[d, "f"] & 0 \\
A'_\bullet = & 0 \arrow[r] & K' \arrow[r, "\iota'"] & F' \arrow[r, "\pi'"] & A' \arrow[r] & 0.
\end{tikzcd}
\tag{37}
$$

2. The chain map $f_\bullet : A_\bullet \to A'_\bullet$ is unique up to chain homotopy.

Proof:
1. Suppose that $F = \langle M \rangle_{\mathbb{Z}} = \oplus_{m \in M} \mathbb{Z}$ for some set M. To define g, choose for every $m \in M$ an element $f'_m \in F'$ with $\pi'(f'_m) = f \circ \pi(m)$, which exists by surjectivity of π'. This defines a map $g : M \to F'$, $m \mapsto f'_m$ and by the universal property of the free abelian group a group homomorphism $g : F \to F'$ with $\pi' \circ g = f \circ \pi$.

The group homomorphism h is defined analogously. As K is free, there is a set N such that $K = \langle N \rangle_{\mathbb{Z}} = \oplus_{n \in N} \mathbb{Z}$. For every $n \in N$ we have $\pi' \circ g \circ \iota(n) = f \circ \pi \circ \iota(n) = f(0) = 0$ and thus $g \circ \iota(n) \in \ker \pi' = \operatorname{im} \iota'$. Hence, there is a $k'_n \in K'$ with $\iota'(k'_n) = g \circ \iota(n)$, and k'_n is unique by injectivity of ι'. This defines a map $h : N \to K'$, $n \mapsto k'_n$, and by the universal property of the free group a group homomorphism $h : K \to K'$ with $\iota' \circ h = g \circ \iota$.

2. Suppose $g' : F \to F'$ and $h' : K \to K'$ are group homomorphisms that define another chain map $f'_\bullet : A_\bullet \to A'_\bullet$. We define a chain homotopy $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ by setting $h_0 = 0 : A \to F'$. Then we have $\pi' \circ h_0 = 0 = f - f$.

To define $h_1 : F \to K'$, note that we must have $\iota' \circ h_1 + h_0 \circ \pi = \iota' \circ h_1 = g' - g$. As we have $\pi' \circ g = \pi' \circ g' = f \circ \pi$, the map $g' - g : F \to F'$ takes values in $\ker \pi' = \operatorname{im} \iota'$. Thus, for every $m \in M$ there is a unique $k_m \in K'$ such that $g'(m) - g(m) = \iota'(k_m)$. This defines a map $h_1 : M \to K'$, $m \mapsto k'_m$ and by the universal property of the free group a group homomorphism $h_1 : F \to K'$ with $\iota' \circ h_1 = g' - g$.

As we have short exact sequences, we must set $h_2 = 0 : K \to 0$. The condition on a chain homotopy then becomes $h_1 \circ \iota = h' - h$. By 1. we have $\iota' \circ (h' - h) = (g' - g) \circ \iota = \iota' \circ h_1 \circ \iota$, and by injectivity of ι' this yields $h' - h = h_1 \circ \iota$. □

Corollary 5.3.8: All free resolutions of an abelian group A are chain homotopy equivalent.

Proof:
Let $A_\bullet$ and $A'_\bullet$ be free resolutions of A. By Proposition 5.3.7 the identity map $\mathrm{id}_A : A \to A$ induces chain maps $\phi_\bullet : A_\bullet \to A'_\bullet$ and $\psi_\bullet : A'_\bullet \to A_\bullet$. The composites $\psi_\bullet \circ \phi_\bullet : A_\bullet \to A_\bullet$ and $\phi_\bullet \circ \psi_\bullet : A'_\bullet \to A'_\bullet$ are again chain maps that extend the identity id_A. As $\mathrm{id}_{A_\bullet} : A_\bullet \to A_\bullet$ and $\mathrm{id}_{A'_\bullet} : A'_\bullet \to A'_\bullet$ also extend the identity maps, we have $\psi_\bullet \circ \phi_\bullet \sim \mathrm{id}_{A_\bullet}$ and $\phi_\bullet \circ \psi_\bullet \sim \mathrm{id}_{A'_\bullet}$. $\square$

Corollary 5.3.8 shows that the choice of a free resolution does not affect its homologies, and Proposition 5.3.7 allows us to extend group homomorphisms between abelian groups to chain maps between their free resolutions. We can thus characterise the non-exactness of the functor $-\otimes M : \mathrm{Ch}_{\mathrm{Ab}} \to \mathrm{Ch}_{\mathrm{Ab}}$ for an abelian group M by choosing for any abelian group A a free resolution and considering the homology $\ker(\iota \otimes \mathrm{id})$ of its image. As this is compatible with group homomorphisms, this defines a functor, the *torsion functor*.

Proposition 5.3.9:
Any abelian group M defines a functor $\mathrm{Tor}(-, M) : \mathrm{Ab} \to \mathrm{Ab}$ that assigns

- to an abelian group A the group $\mathrm{Tor}(A, M) = \ker(\iota \otimes \mathrm{id})$ for any free resolution $A_\bullet = 0 \to K \xrightarrow{\iota} F \xrightarrow{\pi} A \to 0$,
- to a group homomorphism $f : A \to A'$ the group homomorphism

$$\mathrm{Tor}(f, M) = h \otimes \mathrm{id} : \ker(\iota \otimes \mathrm{id}_M) \to \ker(\iota' \otimes \mathrm{id}_M)$$

  for any chain map $f_\bullet = (h, g, f) : A_\bullet \to A'_\bullet$ between free resolutions that extends f.

Proof:
1. Given free resolutions $A_\bullet = 0 \to K \xrightarrow{\iota} F \xrightarrow{\pi} A \to 0$ and $A'_\bullet = 0 \to K' \xrightarrow{\iota'} F' \xrightarrow{\pi'} A' \to 0$ of abelian groups A and A' and a chain map $f_\bullet = (h, g, f) : A_\bullet \to A'_\bullet$ that extends a group homomorphism $f : A \to A'$, we can apply the functor $-\otimes M : \mathrm{Ab} \to \mathrm{Ab}$ for an abelian group M to diagram (37). This yields the commuting diagram

$$\begin{tikzcd} A_\bullet \otimes M = \arrow[d, "f_\bullet \otimes \mathrm{id}"] & 0 \arrow[r] & K \otimes M \arrow[r, "\iota \otimes \mathrm{id}"] \arrow[d, "h \otimes \mathrm{id}"] & F \otimes M \arrow[r, "\pi \otimes \mathrm{id}"] \arrow[d, "g \otimes \mathrm{id}"] & A \otimes M \arrow[r] \arrow[d, "f \otimes \mathrm{id}"] & 0 \\ A'_\bullet \otimes M = & 0 \arrow[r] & K' \otimes M \arrow[r, "\iota' \otimes \mathrm{id}"'] & F' \otimes M \arrow[r, "\pi' \otimes \mathrm{id}"'] & A' \otimes M \arrow[r] & 0. \end{tikzcd} \tag{38}$$

in which the rows are exact in the last two entries. By replacing the last non-trivial terms on the right by zeros, we obtain a chain map

$$\begin{tikzcd} (A_\bullet \otimes M)^{red} = \arrow[d, "f_\bullet^{red}"] & 0 \arrow[r] & K \otimes M \arrow[r, "\iota \otimes \mathrm{id}"] \arrow[d, "h \otimes \mathrm{id}"] & F \otimes M \arrow[r] \arrow[d, "g \otimes \mathrm{id}"] & 0 \\ (A'_\bullet \otimes M)^{red} = & 0 \arrow[r] & K' \otimes M \arrow[r, "\iota' \otimes \mathrm{id}"'] & F' \otimes M \arrow[r] & 0. \end{tikzcd} \tag{39}$$

The homologies of the reduced chain complexes are given by $H_0((A_\bullet \otimes M)^{red}) = A \otimes M$ and $H_1((A_\bullet \otimes M)^{red}) = \ker(\iota \otimes \mathrm{id})$, and the induced map between the first homologies by

$$\mathrm{Tor}(f, M) = H_1(f_\bullet^{red}) : \ker(\iota \otimes \mathrm{id}) \to \ker(\iota' \otimes \mathrm{id}), \; x \mapsto h(x).$$

As any two resolutions $A^1_\bullet$ and $A^2_\bullet$ of A are chain homotopy equivalent by Corollary 5.3.8, this also holds for the chain complexes $A^1_\bullet \otimes M$ and $A^2_\bullet \otimes M$ in (38) and for the resulting chain complexes $(A^1_\bullet \otimes M)^{red}$ and $(A^2_\bullet \otimes M)^{red}$ in (39). Thus, their homologies do not depend on the choice of the resolutions. This shows that $\mathrm{Tor}(A, M)$ is defined.

By Proposition 5.3.7, any two extensions $f^1_\bullet$ and $f^2_\bullet$ of $f : A \to A'$ are chain homotopic. This also holds for the induced chain maps $f^1_\bullet \otimes \mathrm{id}$ and $f^2_\bullet \otimes \mathrm{id}$ in (38) and the induced chain maps $f^{1red}_\bullet$ and $f^{2red}_\bullet$ in (39). Hence, their associated maps $H_1(f^{1red}_\bullet) = H_1(f^{2red}_\bullet)$ agree. This shows that $\mathrm{Tor}(f, M) : \mathrm{Tor}(A, M) \to \mathrm{Tor}(A', M)$ is defined.

2. We show that $\mathrm{Tor}(-, M)$ is a functor. For the identity map $\mathrm{id}_A : A \to A$ we can choose free resolutions $A_\bullet = A'_\bullet$ and $(\mathrm{id}_A)_\bullet = (\mathrm{id}_K, \mathrm{id}_F, \mathrm{id}_A) : A_\bullet \to A_\bullet$. This yields

$$\mathrm{Tor}(\mathrm{id}_A, M) = H_1((\mathrm{id}_A)^{red}_\bullet) = \mathrm{id} : \mathrm{Tor}(A, M) \to \mathrm{Tor}(A, M).$$

Given group homomorphisms $f : A \to A'$ and $f' : A' \to A''$ with associated chain maps $f_\bullet = (h, g, f) : A_\bullet \to A'_\bullet$ and $f'_\bullet = (h', g', f') : A'_\bullet \to A''_\bullet$ that extend f and f', we obtain the chain map $f'_\bullet \circ f_\bullet = (h' \circ h, g' \circ g, f' \circ f) : A_\bullet \to A''_\bullet$ that extends $f' \circ f : A \to A''$. This gives

$$\mathrm{Tor}(f' \circ f, M) = H_1((f' \circ f)^{red}_\bullet) = H_1(f'^{red}_\bullet \circ f^{red}_\bullet) = H^1(f'^{red}_\bullet) \circ H_1(f^{red}_\bullet) = \mathrm{Tor}(f', M) \circ \mathrm{Tor}(f, M).$$

□

Example 5.3.10:

1. For any $n \in \mathbb{N}$ and any abelian group M we have $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, M) = \{m \in M \mid nm = 0\}$. This follows by choosing the free resolution $0 \to \mathbb{Z} \xrightarrow{\iota : z \mapsto n \cdot z} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$. We then have

$$\begin{tikzcd} \mathbb{Z} \otimes M \arrow[r, "\iota \otimes \mathrm{id} : z \otimes m \mapsto nz \otimes m"] \arrow[d, "z \otimes m \mapsto zm"', "\cong"] & \mathbb{Z} \otimes M \arrow[d, "\cong"', "z \otimes m \mapsto zm"] \\ M \arrow[r, "\iota' : m \mapsto n \cdot m"'] & M, \end{tikzcd}$$

and this shows that $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, M) = \ker(\iota') = \{m \in M \mid nm = 0\}$.

2. Example 1. implies in particular $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z}$ and $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{F}) = 0$ for any field $\mathbb{F}$ whose characteristic does not divide n.

3. For all families $(A_i)_{i \in I}$ of abelian groups the compatibility between tensor products and direct sums and the compatibiklity between homologies and direct sums imply (Exercise)

$$\mathrm{Tor}(\oplus_{i \in I} A_i, B) \cong \oplus_{i \in I} \mathrm{Tor}(A_i, B) \qquad \mathrm{Tor}(B, \oplus_{i \in I} A_i) \cong \oplus_{i \in I} \mathrm{Tor}(B, A_i).$$

4. For any free group F and abelian group A, we have $\mathrm{Tor}(A, F) = \mathrm{Tor}(F, A) = 0$.

   The free resolution $0 \to 0 \to F \xrightarrow{\mathrm{id}} F \to 0$ yields $0 \to 0 \to F \otimes A \xrightarrow{\mathrm{id}} F \otimes A \to 0$, which is a short exact sequence, and hence $\mathrm{Tor}(F, A) = 0$.

   To show that $\mathrm{Tor}(A, F) = 0$, note that $\mathrm{Tor}(A, \mathbb{Z}) = 0$, as tensoring a short exact sequence with $\mathbb{Z}$ yields a sequence chain isomorphic to the original sequence. Applying 3. then proves the claim for any free group F.

The identity $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, M) = \{m \in M \mid nm = 0\}$ in Example 5.3.10, 1. is the reason why the functor $\mathrm{Tor}(-, M) : \mathrm{Ab} \to \mathrm{Ab}$ is called *torsion* functor. By comparing with Definition 1.1.6 one sees that $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, M)$ computes the n-torsion subgroup of the abelian group M.

One can also show that the functor Tor is symmetric in both arguments: $\mathrm{Tor}(A, M) = \mathrm{Tor}(M, A)$ for all abelian groups A, M, but this requires more theory, see for instance [Me, Section 4.5], in particular [Me, Theorem 4.5.6].

5.4 Tensor products of chain complexes and the Künneth formula

In this section we relate singular homology with coefficients in an abelian group M to singular homology with coefficients in $\mathbb{Z}$. They key ingredient is the (algebraic) Künneth formula[1]. It also plays an important role in the computation of homologies of product spaces. The first step is to generalise the concept of a tensor product from tensor products of abelian groups to tensor products of chain complexes.

Definition 5.4.1: Let $X_\bullet$ and $Y_\bullet$ chain complexes. The **tensor product** of X and Y is the chain complex $X_\bullet \otimes Y_\bullet$ with abelian groups $(X_\bullet \otimes Y_\bullet)_n = \oplus_{p+q=n} X_p \otimes Y_q$ and boundary operators

$$d_n : \oplus_{p+q=n} X_p \otimes Y_q \to \oplus_{p+q=n-1} X_p \otimes Y_q, \quad X_p \otimes Y_q \ni x \otimes y \mapsto d_p(x) \otimes y + (-1)^p x \otimes d_q(y)$$

The sign in the formula for the boundary operator ensures that the boundary operator satisfies $d_{n-1} \circ d_n = 0$ and is indeed a boundary operator.

The tensor product of chain complexes also sheds some light on the definition of a chain homotopy in Definition 2.1.9. In Remark 2.1.11 we showed that a chain homotopy $h_\bullet : f_\bullet \Rightarrow g_\bullet$ between chain maps $f_\bullet, g_\bullet : X_\bullet \to X'_\bullet$ can be viewed as a chain map $k_\bullet : X''_\bullet \to X_\bullet$, where $X''_\bullet$ is a chain complex constructed from $X_\bullet$, which remained somewhat mysterious. With the tensor product of chain complexes, we can understand this chain complex as the tensor product $I_\bullet \otimes X_\bullet$, where $I_\bullet$ is a chain complex that generalises the unit interval.

Example 5.4.2: Let $I_\bullet$ be the chain complex with $I_0 = \mathbb{Z} \oplus \mathbb{Z}$, $I_1 = \mathbb{Z}$ and $I_k = 0$ for all $k \neq 0, 1$ with boundary operator $d_1 : I_1 \to I_0$, $z \mapsto (z, -z)$

$$I_\bullet = 0 \to \mathbb{Z} \xrightarrow{z \mapsto (z,-z)} \mathbb{Z} \oplus \mathbb{Z} \to 0.$$

Then the tensor product $I_\bullet \otimes X_\bullet$ is given by $(I_\bullet \otimes X_\bullet)_n = I_0 \otimes X_n + I_1 \otimes X_{n-1} \cong X_n \oplus X_n \oplus X_{n-1}$ with boundary operator

$$d_n : (I_\bullet \otimes X_\bullet)_n \to (X_\bullet \otimes I_\bullet)_{n-1}, \quad (x, x', x'') \mapsto (d_n(x) + x'', d_n(x) - x'', -d_{n-1}(x)).$$

This is the chain complex Remark 2.1.11.

Lemma 5.4.3:
The tensor product of chain complexes defines a functor $\otimes : \mathrm{Ch}_{\mathrm{Ab}} \times \mathrm{Ch}_{\mathrm{Ab}} \to \mathrm{Ch}_{\mathrm{Ab}}$ that assigns

- to a pair $(X_\bullet, Y_\bullet)$ of chain complexes the chain complex $X_\bullet \otimes Y_\bullet$,
- to a pair $(f_\bullet, g_\bullet)$ of chain maps $f_\bullet : X_\bullet \to X'_\bullet$ and $g_\bullet : Y_\bullet \to Y'_\bullet$ the chain map

$$f_\bullet \otimes g_\bullet : X_\bullet \otimes Y_\bullet \to X'_\bullet \otimes Y'_\bullet, \quad X_p \otimes Y_q \ni x \otimes y \mapsto f_p(x) \otimes g_q(y) \in X'_p \otimes Y'_q.$$

It satisfies $X_\bullet \otimes Y_\bullet \cong Y_\bullet \otimes X_\bullet$ for all chain complexes $X_\bullet \otimes Y_\bullet$.

Proof:
It is clear that these assignments are compatible with the composition of morphisms and with identity morphisms. The isomorphism $\phi_\bullet : X_\bullet \otimes Y_\bullet \to Y_\bullet \otimes X_\bullet$ has components

$$\phi_n : \oplus_{p+q=n} X_p \otimes Y_q \to \oplus_{p+q=n} Y_q \otimes X_p, \quad X_p \otimes Y_q \ni x \otimes y \mapsto (-1)^{pq} y \otimes x \in Y_q \otimes X_p.$$

[1]This formula is due to Dr. Hermann Lorenz Künneth, a high school teacher in Erlangen who became an adjunct professor at the Friedrich-Alexander-Universität after his retirement.

A direct computation shows that this is indeed a chain map. For all $x \in X_p$ and $y \in Y_q$ one has

$$\begin{aligned}&d_n \circ \phi_n(x\otimes y) = (-1)^{pq} d_n(y\otimes x) = (-1)^{pq}(d_q(y)\otimes x + (-1)^q y\otimes d_p(x))\\&= (-1)^{pq} d_q(y)\otimes x + (-1)^{(p-1)q} y\otimes d_p(x) = \phi_{n-1}(d_p(x)\otimes y + (-1)^p x\otimes d_q(y)) = \phi_{n-1}\circ d_n(x\otimes y).\end{aligned}$$

□

To relate the singular homologies of a topological space X to its singular homologies with coefficients in an abelian group M we use the fact that every abelian group $S_n(X)$ in the chain complex $S_\bullet(X)$ is a free abelian group.

Such chain complexes are called free chain complexes. We observe first that if a free chain complex $F_\bullet$ has a trivial boundary operator, the homologies of the chain complex $F_\bullet\otimes X_\bullet$ are obtained by tensoring the homologies of $X_\bullet$ with the abelian groups in $F_\bullet$.

Definition 5.4.4:
A chain complex $X_\bullet$ in Ab is called **free** if all groups X_n for $n \in \mathbb{Z}$ are free abelian groups.

Lemma 5.4.5: Let $F_\bullet$ a free chain complex with trivial differential $d_n = 0 : F_n \to F_{n-1}$. Then for any chain complex $X_\bullet$ the following maps are isomorphisms natural in $X_\bullet$ and $F_\bullet$

$$\phi_n^{F_\bullet,X_\bullet} : \oplus_{p+q=n} F_p\otimes H_q(X) = \oplus_{p+q=n} H_p(F_\bullet)\otimes H_q(X_\bullet) \to H_n(F_\bullet\otimes X_\bullet), \quad f\otimes[x] \mapsto [f\otimes x].$$

Proof:
As the boundary operators in $F_\bullet$ are trivial, we have $Z_n(F_\bullet) = F_n$, $B_n(F_\bullet) = 0$ and $H_n(F_\bullet) = F_n$. The boundary operators of $F_\bullet\otimes X_\bullet$ are given by

$$d_n : \oplus_{p+q=n} F_p\otimes X_q \to \oplus_{p+q=n-1} F_p\otimes X_q, \quad F_p\otimes X_q \ni f\otimes x \mapsto (-1)^p f\otimes d_q(x) \in F_p\otimes X_{q-1}.$$

For every $n \in \mathbb{Z}$ we have a short exact sequence $0 \to B_q(X_\bullet) \xrightarrow{\iota_q} Z_q(X_\bullet) \xrightarrow{\pi_q} H_q(X_\bullet) \to 0$. As tensoring with a free abelian group preserves short exact sequences by Example 5.3.10, 4., tensoring with F_p yields a short exact sequence

$$0 \to F_p\otimes B_q(X_\bullet) \xrightarrow{\mathrm{id}\otimes\iota_q} F_p\otimes Z_q(X_\bullet) \xrightarrow{\mathrm{id}\otimes\pi_q} F_p\otimes H_q(X_\bullet) \to 0.$$

Taking direct sums over all p, q with $p + q = n$ yields a short exact sequence

$$0 \to \underbrace{\oplus_{p+q=n} F_p\otimes B_q(X_\bullet)}_{=B_n(F_\bullet\otimes X_\bullet)} \xrightarrow{\mathrm{id}\otimes\iota_q} \underbrace{\oplus_{p+q=n} F_p\otimes Z_q(X_\bullet)}_{=Z_n(F_\bullet\otimes X_\bullet)} \xrightarrow{\mathrm{id}\otimes\pi_q} \oplus_{p+q=n} F_p\otimes H_q(X_\bullet) \to 0,$$

which shows that $H_n(F_\bullet\otimes X_\bullet) \cong \oplus_{p+q=n} F_p\otimes H_q(X_\bullet)$ and the maps $\phi_n^{F_\bullet,X_\bullet}$ are isomorphisms. To prove their naturality consider chain maps $g_\bullet : F_\bullet \to F'_\bullet$ and $k_\bullet : X_\bullet \to X'_\bullet$ and $f \in F_p$, $x \in Z_q(X_\bullet)$ with $p + q = n$. Then we have

$$\begin{aligned}\phi_n^{F'_\bullet,X'_\bullet} \circ (H_p(g_\bullet)\otimes H_q(k_\bullet))(f\otimes[x]) &= \phi_n^{F'_\bullet,X'_\bullet}(g_p(f)\otimes[k_q(x)]) = [g_p(f)\otimes k_q(x)]\\&= H_n(g_\bullet\otimes k_\bullet)[f\otimes x] = H_n(g_\bullet\otimes k_\bullet)\circ\phi_n^{F_\bullet,X_\bullet}(f\otimes[x]).\end{aligned}$$

□

With this preparations we can now prove a general version of the Künneth formula, which is also sometimes called the *algebraic* Künneth formula.

Theorem 5.4.6 (Künneth formula):
Let $F_\bullet$ be a free chain complex and $X_\bullet$ a chain complex in Ab. Then for every $n \in \mathbb{Z}$ the following sequence is exact

$$0 \to \oplus_{p+q=n} H_p(F_\bullet)\otimes H_q(X_\bullet) \xrightarrow{\phi_n:[f]\otimes[x]\mapsto[f\otimes x]} H_n(F_\bullet\otimes X_\bullet) \to \oplus_{p+q=n-1}\mathrm{Tor}(H_p(F_\bullet), H_q(X_\bullet)) \to 0.$$

This sequence splits, but the splitting is not canonical

$$H_n(F_\bullet\otimes X_\bullet) \cong \big(\oplus_{p+q=n} H_p(F_\bullet)\otimes H_q(X_\bullet)\big) \oplus \big(\oplus_{p+q=n-1} \mathrm{Tor}(H_p(F_\bullet), H_q(X_\bullet))\big).$$

Proof:
1. For each $n \in \mathbb{Z}$ we have a short exact sequence

$$0 \to B_n(F_\bullet) \xrightarrow{i_n} Z_n(F_\bullet) \xrightarrow{\pi_n} H_n(F_\bullet) \to 0. \tag{40}$$

As F_n is free, the subgroups $Z_n(F_\bullet), B_n(F_\bullet) \subset F_n$ are free as well, and this short exact sequence is a free resolution of $H_n(F_\bullet)$. As we have commuting diagrams

$$\begin{tikzcd}
0 \arrow[r] & Z_n(F_\bullet) \arrow[r, "\iota_n"] \arrow[d, "0"] & F_n \arrow[r, "d_n"] \arrow[d, "d_n"] & B_{n-1}(F_\bullet) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1}(F_\bullet) \arrow[r, "\iota_{n-1}"'] & F_{n-1} \arrow[r, "d_{n-1}"'] & B_{n-2}(F_\bullet) \arrow[r] & 0
\end{tikzcd}$$

we can promote the groups $Z_n(F_\bullet)$ and $B_{n-1}(F_\bullet)$ to chain complexes by equipping them with the trivial differential and obtain a short exact sequence of chain complexes

$$0 \to Z_\bullet(F_\bullet) \xrightarrow{\iota_\bullet} F_\bullet \xrightarrow{d_\bullet} B^{-1}_\bullet(F_\bullet) \to 0, \tag{41}$$

where $B^{-1}_n(F_\bullet) = B_{n-1}(F_\bullet)$, $\iota_n : Z_n(F_\bullet) \to F_n$ is the inclusion and $d_n : F_n \to B_{n-1}(F_\bullet)$ the corestriction of the differential. As $B_{n-1}(F_\bullet) \subset F_{n-1}$ is free as a subgroup of a free abelian group, this sequence splits in each degree n by Lemma 5.3.2.

Tensoring with a chain complex $X_\bullet$ yields a short exact sequence of chain complexes

$$0 \to Z_\bullet(F_\bullet)\otimes X_\bullet \xrightarrow{\iota_\bullet\otimes\mathrm{id}} F_\bullet\otimes X_\bullet \xrightarrow{d_\bullet\otimes\mathrm{id}} B^{-1}_\bullet(F_\bullet)\otimes X_\bullet \to 0. \tag{42}$$

As F_p is a free abelian group, we have a splitting $F_p \cong \ker d_p \oplus \mathrm{im}\, d_p = Z_p(F_\bullet) \oplus B_{p-1}(F_\bullet)$. Because the boundary operators of $Z_\bullet(F_\bullet)$ and $B^{-1}_\bullet(X_\bullet)$ are trivial, this yields a splitting of the short exact sequence (42), given by the following isomorphisms, which define a chain map

$$(F_\bullet\otimes X_\bullet)_n = \oplus_{p+q=n} F_p\otimes X_q \cong (\oplus_{p+q=n} Z_p(F_\bullet)\otimes X_q) \oplus (\oplus_{p+q=n} B_{p-1}(F_\bullet)\otimes X_q)\,.$$

As $Z_\bullet(F_\bullet)$ and $B^{-1}_\bullet(F_\bullet)$ are free chain complexes with trivial boundary operators, by Lemma 5.4.5 the long exact homology sequence for (42) takes the form

$$\ldots \xrightarrow{\partial_{n+1}} (Z_\bullet(F_\bullet)\otimes H_\bullet(X_\bullet))_n \xrightarrow{H_n(\iota_\bullet\otimes\mathrm{id})} H_n(F_\bullet\otimes X_\bullet) \xrightarrow{H_n(d_\bullet\otimes\mathrm{id})} (B^{-1}_\bullet(F_\bullet)\otimes H_\bullet(X_\bullet))_n \xrightarrow{\partial_n} \ldots \tag{43}$$

The exactness of this sequence yields for all $n \in \mathbb{Z}$ short exact sequences

$$0 \to \mathrm{coker}\,\partial_{n+1} \to H_n(F_\bullet\otimes X_\bullet) \to \ker\partial_n \to 0. \tag{44}$$

By definition, the connecting homomorphism ∂_n assigns to $x \in \oplus_{p+q=n} B_{p-1}(F_\bullet)\otimes H_q(X_\bullet)$ the unique $y \in \oplus_{p+q=n} Z_{p-1}(F_\bullet)\otimes H_q(X_\bullet)$ with $H_{n-1}(\iota_\bullet\otimes\mathrm{id})(y) = d_n(w)$, where $w \in H_n(F_\bullet\otimes X_\bullet)$ satisfies $H_n(d_\bullet\otimes\mathrm{id})(w) = x$. If $x = \Sigma_{i=1}^m d_{p_i}(f_i)\otimes[x_i]$, we can take $w = \Sigma_{i=1}^m [f_i\otimes x_i]$, which implies $y = \Sigma_{i=1}^m d_p(f_i)\otimes[x_i]$. Thus, the connecting homomorphism is given by

$$\partial_n = (i_\bullet\otimes\mathrm{id})_n : \oplus_{p+q=n} B_{n-1}(F_\bullet)\otimes H_q(X_\bullet) \to \oplus_{p+q=n} Z_{n-1}(F_\bullet)\otimes H_q(X_\bullet),$$

where $i_\bullet : B_\bullet^{-1}(F_\bullet) \to Z_\bullet(F_\bullet)$ is the inclusion from (40). As (40) is a free resolution, this yields

$$\ker \partial_n = \oplus_{p+q=n} \mathrm{Tor}(H_{p-1}(F_\bullet), H_q(F_\bullet))$$
$$\operatorname{coker} \partial_{n+1} = \oplus_{p+q=n} H_p(F_\bullet)\otimes H_q(X_\bullet).$$

Inserting this back into (44) proves the claim.

2. We prove that the sequence splits only for free chain complexes $X_\bullet$, as this is the only case needed in the following. If $X_\bullet$ is free, the short exact sequence of chain complexes

$$0 \to Z_\bullet(X_\bullet) \xrightarrow{\iota'_\bullet} X_\bullet \xrightarrow{d'_\bullet} B_\bullet^{-1}(X_\bullet) \to 0$$

splits in each degree, analogously to (41), because $B_{n-1}(X_\bullet) \subset X_{n-1}$ is free as a subgroup of a free group. By Exercise 8 we can choose retractions $r_n : F_n \to Z_n(F_\bullet)$ and $r'_n : X_n \to Z_n(X_\bullet)$ with $r_n \circ \iota_n = \mathrm{id}_{Z_n(F_\bullet)}$ and $r'_n \circ \iota'_n = \mathrm{id}_{Z_n(X_\bullet)}$. The composite maps

$$s_n = \pi_n \circ r_n : F_n \to H_n(F_\bullet), f \mapsto [r_n(f)] \qquad s'_n = \pi'_n \circ r'_n : X_n \to H_n(X_\bullet), x \mapsto [r'_n(x)]$$

are surjective as composites of two surjective maps, because retractions are surjective. As $r_{n-1}|_{Z_{n-1}(F_\bullet)} = \mathrm{id}$ and $r'_{n-1}|_{Z_{n-1}(X_\bullet)} = \mathrm{id}$, we have for all $f \in F_n$ and $x \in X_n$

$$s_{n-1} \circ d_n(f) = \pi_{n-1} \circ r_{n-1} \circ d_n(f) = [d_n(f)] = 0 \quad s'_{n-1} \circ d'_n(x) = \pi'_{n-1} \circ r'_{n-1} \circ d'_n(x) = [d'_n(x)] = 0.$$

Hence, the maps $s_n : F_n \to H_n(F_\bullet)$ and $s'_n : X_n \to H_n(X_\bullet)$ define a chain map

$$\psi_\bullet : F_\bullet\otimes X_\bullet \to H_\bullet(F_\bullet)\otimes H_\bullet(X_\bullet), \quad F_p\otimes X_q \ni f\otimes x \mapsto [r_p(f)]\otimes[r'_q(x)] \in H_p(F_\bullet)\otimes H_q(X_\bullet),$$

where the chain complexes $H_\bullet(F_\bullet)$ and $H_\bullet(X_\bullet)$ are equipped with the trivial differential. The induced maps on the homologies

$$H_n(\psi_\bullet) : H_n(F_\bullet\otimes X_\bullet) \to \oplus_{p+q=n} H_p(F_\bullet)\otimes H_q(X_\bullet), \quad [f\otimes x] \mapsto [r_p(f)]\otimes[r'_q(x)]$$

are left inverses to the maps ϕ_n and split the short exact sequence in the Künneth formula. □

With the algebraic Künneth formula, we can now relate singular homology with coefficients in an abelian group M to singular homology with coefficients in $\mathbb{Z}$. For this, we consider the singular chain complex $S_\bullet(X)$ and interpret the coefficient group M as a chain complex with a single non-trivial entry. The algebraic Künneth formula then yields the following corollary.

Corollary 5.4.7 (universal coefficient theorem):
For every topological space X and abelian group M there is a short exact sequence

$$0 \to H_n(X)\otimes M \xrightarrow{[x]\otimes m \mapsto [x\otimes m]} H_n(X; M) \to \mathrm{Tor}(H_{n-1}(X), M) \to 0.$$

The sequence splits, but not canonically:

$$H_n(X; M) \cong \big(H_n(X)\otimes M\big) \oplus \mathrm{Tor}(H_{n-1}(X), M).$$

Proof:
We consider the singular chain complex $F_\bullet = S_\bullet(X)$, which is free, and a chain complex $X_\bullet$ with $X_0 = M$ and $X_k = 0$ for $k \neq 0$. As this implies $H_0(X_\bullet) = M$ and $H_q(X_\bullet) = 0$ for $q \neq 0$, this yields the short exact sequence in the corollary. Its splitting is given by the maps

$$\psi_n : H_n(X; M) \to H_n(X) \otimes M, \quad [x \otimes m] \mapsto [r_n(x)] \otimes m,$$

where $r_n : S_n(X) \to Z_n(X)$ is a retraction of $\iota_n : Z_n(X) \to S_n(X)$, as in step 2. of the proof of Theorem 5.4.6. □

The universal coefficient theorem in Corollary 5.4.7 shows that singular homologies with coefficients in an abelian group are determined by the ones with coefficients in $\mathbb{Z}$. Hence, the additional information contained in them is purely algebraic. However, they are nevertheless useful in many applications. For instance, homologies with coefficients in certain fields can contain less information, but may be easier to compute in some cases. In the setting of manifolds, it is sometimes more natural to work with homologies with real coefficients rather than integers.

Example 5.4.8:

1. If X is a topological space such that $H_n(X)$ is free for all $n \in \mathbb{N}_0$, then we have $\mathrm{Tor}(H_{n-1}(X), M) = 0$ for all $n \in \mathbb{N}$ and abelian groups M. The homology groups with coefficients in M are then given by $H_n(X; M) = H_n(X) \otimes M$.

2. In particular, we have for all abelian groups M

$$H_n(S^k; M) = \begin{cases} M \oplus M & n = k = 0 \\ M & n = 0, k \in \mathbb{N} \text{ or } n = k \in \mathbb{N} \\ 0 & \text{else.} \end{cases}$$

3. If X is a CW complex of finite type and M torsion free then $H_n(X; M) = H_n(X) \otimes M$ for all $n \in \mathbb{N}_0$ (Exercise 65).

4. By Example 5.1.3, real projective space $\mathbb{RP}^n$ has the homology groups

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k = 0 \text{ or } k = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 1 \leq k < n \text{ odd} \\ 0 & k \text{ even or } k > n. \end{cases}$$

The universal coefficient theorem gives

$$H_k(\mathbb{RP}^n; M) = \begin{cases} M & k = 0 \text{ or } k = n \text{ odd} \\ M/2M & 1 \leq k < n \text{ odd} \\ \mathrm{Tor}_2(M) & 1 \leq k \leq n \text{ even} \\ 0 & k \geq n+1, \end{cases}$$

where we used $\mathbb{Z}/2\mathbb{Z} \otimes M = M/2M$ and $\mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, M) = \mathrm{Tor}_2(M) = \{m \in M \mid 2m = 0\}$ from Example 5.3.10, 1.

5.5 Homologies of product spaces

An important application of the Künneth formula in Theorem 5.4.6 are homologies of product spaces. Together with the Eilenberg-Zilber Theorem, which we will prove in this section, the Künneth formula yields a short exact sequence that allows one to compute homologies of product spaces $X \times Y$ in terms of the homologies of X and Y.

For this we need to relate the singular chain complex $S_\bullet(X \times Y)$ for the product of two topological spaces X, Y to the tensor product $S_\bullet(X) \otimes S_\bullet(Y)$ of their singular chain complexes. This amounts to relating the functors

$$S_\bullet \times : \mathrm{Top} \times \mathrm{Top} \xrightarrow{\times} \mathrm{Top} \xrightarrow{S_\bullet} \mathrm{Ch}_{\mathrm{Ab}} \qquad \otimes(S_\bullet \times S_\bullet) : \mathrm{Top} \times \mathrm{Top} \xrightarrow{S_\bullet \times S_\bullet} \mathrm{Ch}_{\mathrm{Ab}} \times \mathrm{Ch}_{\mathrm{Ab}} \xrightarrow{\otimes} \mathrm{Ch}_{\mathrm{Ab}}.$$

The appropriate mathematical structure to relate them are natural transformations

$$f : S_\bullet \times \Rightarrow \otimes(S_\bullet \times S_\bullet) \qquad g : \otimes(S_\bullet \times S_\bullet) \Rightarrow S_\bullet \times,$$

whose components for a pair (X, Y) of topological spaces are chain maps

$$f_\bullet^{X,Y} : S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y) \qquad g_\bullet^{X,Y} : S_\bullet(X) \otimes S_\bullet(Y) \to S_\bullet(X \times Y).$$

Their naturality states that for all continuous maps $\alpha : X \to X'$ and $\beta : Y \to Y'$

$$f_\bullet^{X',Y'} \circ S_\bullet(\alpha \times \beta) = (S_\bullet(\alpha) \otimes S_\bullet(\beta)) \circ f_\bullet^{X,Y} \quad g_\bullet^{X',Y'} \circ (S_\bullet(\alpha) \otimes S_\bullet(\beta)) = S_\bullet(\alpha \times \beta) \circ g_\bullet^{X,Y}. \tag{45}$$

This naturality condition is very strong and determines these chain maps almost completely.

The basis elements of $S_p(X)$ and $S_q(Y)$ are singular n-simplexes $\nu : \Delta^p \to X$ and $\rho : \Delta^q \to Y$, which we can also interpret as continuous maps $\alpha = \nu : \Delta^p \to X$ and $\beta = \rho : \Delta^q \to Y$. This gives $\nu \otimes \rho = (S_p(\nu) \otimes S_q(\rho))(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q})$. The naturality condition (45) then allows us to express $g_{p+q}^{X,Y}(\nu \otimes \rho)$ in terms of $g_{p+q}^{\Delta^p,\Delta^q}(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q})$.

Likewise, a basis of $S_n(X \times Y)$ are singular n-simplexes $\chi : \Delta^n \to X \times Y$ or, equivalently, pairs (σ, τ) of singular n-simplexes $\sigma : \Delta^n \to X$ and $\tau : \Delta^n \to Y$, obtained by projecting on X, Y. In this case, we have $(\sigma, \tau) = S_n(\sigma, \tau)(D_n)$ with the diagonal map $D_n : \Delta^n \to \Delta^n \times \Delta^n$, $x \mapsto (x, x)$. The naturality condition (45) then allows us to express $f_n^{X,Y}$ in terms of $f_n^{\Delta^n,\Delta^n}(D_n)$.

The advantage of this is that the chain complexes $S_\bullet(\Delta^n)$ have trivial homologies $H_k(\Delta^n) = 0$ for all $k \neq 0$, as Δ^n is convex and hence contractible. More generally, a chain complex $X_\bullet$ with $H_k(X_\bullet) = 0$ for $k \neq 0$ is called **acyclic**. For this reason, the constructions in this section are called the method of **acyclic models**.

The first step is to look at the properties of the chain complexes $S_\bullet(\Delta^p \times \Delta^q)$ and $S_\bullet(\Delta^p) \otimes S_\bullet(\Delta^q)$ with boundary operators $d_1^\times$ and $d_1^\otimes$, respectively.

Lemma 5.5.1: Let $p, q \in \mathbb{N}_0$.

1. The chain complexes $S_\bullet(\Delta^p \times \Delta^q)$ and $S_\bullet(\Delta^p) \otimes S_\bullet(\Delta^q)$ are free and acyclic.
2. The group isomorphism $\phi : S_0(\Delta^p \times \Delta^q) \to S_0(\Delta^p) \otimes S_0(\Delta^q)$, $(x, y) \mapsto x \otimes y$ satisfies

$$\phi \circ d_1^\times S_1(\Delta^p \times \Delta^q) \subset d_1^\otimes(S_\bullet(\Delta^p) \otimes S_\bullet(\Delta^q))_1$$
$$\phi^{-1} \circ d_1(S_1(\Delta^p) \otimes S_0(\Delta^q)) \subset d_1^\times S_1(\Delta^p, \Delta^q) \qquad \phi^{-1} \circ d_1(S_0(\Delta^p) \otimes S_1(\Delta^q)) \subset d_1^\times S_1(\Delta^p, \Delta^q).$$

Proof:
1. As all abelian groups $S_n(\Delta^p)$ and $S_n(\Delta^p \times \Delta^q)$ are free by definition, their tensor products are free by Lemma 5.2.6, and so are their direct sums. As the standard n-simplexes Δ^n are convex and hence contractible, so are their products $\Delta^p \times \Delta^q$. This implies $H_n(\Delta^p) = 0$ and $H_n(\Delta^p \times \Delta^q) = H_n S_\bullet(\Delta^p \times \Delta^q) = 0$ for all $n > 0$ and $p, q \in \mathbb{N}_0$. As all abelian groups $S_n(\Delta^p)$ for $n \in \mathbb{Z}$ are free with $H_0(\Delta^p) = \mathbb{Z}$ the Künneth formula from Theorem 5.4.6 implies for $n \neq 0$

$$H_n(S_\bullet(\Delta^p) \otimes S_\bullet(\Delta^q)) \cong \big(\oplus_{j+k=n} H_j(\Delta^p) \otimes H_k(\Delta^q) \big) \oplus \big(\oplus_{j+k=n-1} \mathrm{Tor}(H_j(\Delta^p), H_k(\Delta^q)) \big) = 0.$$

2. The universal property of the product identifies singular simplexes $\chi : \Delta^n \to \Delta^p \times \Delta^q$ with pairs (ν, ρ) of singular simplexes $\nu = \pi_1 \circ \chi : \Delta^n \to \Delta^p$ and $\rho = \pi_2 \circ \chi : \Delta^n \to \Delta^q$. We then have for all 1-simplexes $\nu : \Delta^1 \to \Delta^p$ and $\rho : \Delta^n \to \Delta^q$ and $x \in \Delta^p$, $y \in \Delta^q$

$$\begin{aligned}
\phi \circ d_1^\times(\nu, \rho) &= \nu(1) \otimes \rho(1) - \nu(0) \otimes \rho(0) = (\nu(1) - \nu(0)) \otimes \rho(1) + \nu(0) \otimes (\rho(1) - \rho(0)) \\
&= d_1(\nu) \otimes \rho(1) + \nu(0) \otimes d_1(\rho) = d_1(\nu \otimes \rho(1) + \nu(0) \otimes \rho) \in d_1^\otimes (S_\bullet(\Delta^p) \otimes S_\bullet(\Delta^q))_1 \\
\phi^{-1} \circ d_1^\otimes(\nu \otimes y) &= (d_1(\nu), y) = d_1(\nu, y) \in d_1^\times S_1(\Delta^p \times \Delta^q) \\
\phi^{-1} \circ d_1^\otimes(x \otimes \rho) &= (x, d_1(\rho)) = d_1(x, \rho) \in d_1^\times S_1(\Delta^p \times \Delta^q).
\end{aligned}$$

□

We will now use the fact that the chain complexes $S_\bullet(\Delta^p \times \Delta^q)$ and $S_\bullet(\Delta^p) \otimes S_\bullet(\Delta^q)$ are acyclic to construct natural transformations $f : S_\bullet \times \Rightarrow \otimes(S_\bullet \times S_\bullet)$ and $g : \otimes(S_\bullet \times S_\bullet) \Rightarrow S_\bullet \times$, as outlined above. Naively, one could expect that these should be natural isomorphisms, but this is neither true nor required. In order to induce isomorphisms on the homologies, it is sufficient that their components define chain homotopy equivalences between $S_\bullet(X \times Y)$ and $S_\bullet(X) \otimes S_\bullet(Y)$.

The existence of such natural transformations follows with the Theorem of Eilenberg and Zilber. To keep notation simple, we identify for all topological spaces X and Y the free abelian groups $S_0(X \times Y) = \langle X \times Y \rangle_\mathbb{Z} \cong \langle X \rangle_\mathbb{Z} \otimes \langle Y \rangle_\mathbb{Z} = S_0(X) \otimes S_0(Y)$.

Theorem 5.5.2: (**Eilenberg-Zilber**)

1. There are natural transformations $f : S_\bullet \times \Rightarrow \otimes(S_\bullet \times S_\bullet)$ and $g : \otimes(S_\bullet \times S_\bullet) \Rightarrow S_\bullet \times$ whose components are chain maps with $f_0^{X,Y} = g_0^{X,Y} = \mathrm{id}$.
2. Their components $f_\bullet^{X,Y} : S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$, $g_\bullet^{X,Y} : S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$ are unique up to chain homotopies natural in X, Y.
3. Their components $f_\bullet^{X,Y}$ and $g_\bullet^{X,Y}$ form chain homotopy equivalences with chain homotopies that are natural in X, Y.

Any such pair of natural transformations f and g is called an **Eilenberg-Zilber map**.

Proof:
1. We construct $f_n^{X,Y}$ and $g_n^{X,Y}$ inductively. For $n = 0$ we set $f_0^{X,Y} = g_0^{X,Y} = \mathrm{id}$.

Suppose we constructed for all $0 \leq k \leq n-1$ group homomorphisms $f_k^{X,Y}$ and $g_k^{X,Y}$ that satisfy for all $k < n$ and continuous maps $\alpha : X \to X'$ and $\beta : Y \to Y'$

$$d_k^\otimes \circ f_k^{X,Y} = f_{k-1}^{X,Y} \circ d_k^\times \qquad d_k^\times \circ g_k^{X,Y} = g_{k-1}^{X,Y} \circ d_k^\otimes \tag{46}$$

$$(S_\bullet(\alpha) \otimes S_\bullet(\beta))_k \circ f_k^{X,Y} = f_k^{X',Y'} \circ S_k(\alpha, \beta) \quad S_k(\alpha, \beta) \circ g_k^{X,Y} = g_k^{X',Y'} \circ (S_\bullet(\alpha) \otimes S_\bullet(\beta))_k. \tag{47}$$

We denote by $D_n : \Delta^n \to \Delta^n \times \Delta^n$, $x \mapsto (x,x)$ the diagonal map.

If $n = 1$, the second claim in Lemma 5.5.1 implies that there is a $z \in (S_\bullet(\Delta^1) \times S_\bullet(\Delta^1))_1$ and for $p+q=1$ a $w \in S_1(\Delta^p \times \Delta^q)$ with

$$d_1^{\otimes}(z) = f_0^{\Delta^1,\Delta^1} \circ d_1^{\times}(D_1) = d_1^{\times}(D_1) \qquad d_1^{\times}(w) = g_0^{\Delta^p,\Delta^q} \circ d_1^{\otimes}(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) = d_1^{\otimes}(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}).$$

If $n > 1$, then (46) implies for $p+q=n$

$$\begin{aligned} d_{n-1}^{\otimes}(f_{n-1}^{\Delta^n,\Delta^n} \circ d_n^{\times}(D_n)) &= f_{n-2}^{\Delta^n,\Delta^n} \circ d_{n-1}^{\times} \circ d_n^{\times}(D_n) = 0 \\ d_{n-1}^{\times}(g_{n-1}^{\Delta^p,\Delta^q} \circ d_n^{\otimes}(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q})) &= g_{n-2}^{\Delta^p,\Delta^q} \circ d_{n-1}^{\otimes} \circ d_n^{\otimes}(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) = 0. \end{aligned}$$

This shows that the terms in the brackets are $(n-1)$-cycles and hence $(n-1)$-boundaries, as $H_{n-1}(\Delta^n \times \Delta^n) = H_{n-1}(S_\bullet(\Delta^p) \otimes S_\bullet(\Delta^q)) = 0$ for $n > 1$.

We define for all singular simplexes $\sigma : \Delta^n \to X$, $\tau : \Delta^n \to Y$, $\nu : \Delta^p \to X$ and $\rho : \Delta^q \to Y$

$$\begin{aligned} f_n^{X,Y}(\sigma,\tau) &= (S_\bullet(\sigma) \otimes S_\bullet(\tau))_n(z) & d_n^{\otimes}(z) &= f_{n-1}^{\Delta^n,\Delta^n} \circ d_n^{\times}(D_n) \\ g_n^{X,Y}(\nu \otimes \rho) &= S_n(\nu,\rho)(w) & d_n^{\times}(w) &= g_{n-1}^{\Delta^p,\Delta^q} \circ d_n^{\otimes}(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}). \end{aligned} \tag{48}$$

We now show that (46) and (47) hold for $k = n$. For (46), we compute

$$\begin{aligned} d_n^{\otimes} \circ f_n^{X,Y}(\sigma,\tau) &\overset{(48)}{=} d_n^{\otimes} \circ (S_\bullet(\sigma) \otimes S_\bullet(\tau))_n(z) = (S_\bullet(\sigma) \otimes S_\bullet(\tau))_{n-1} \circ d_n^{\otimes}(z) \\ &\overset{(48)}{=} (S_\bullet(\sigma) \otimes S_\bullet(\tau))_{n-1} \circ f_{n-1}^{\Delta^n,\Delta^n} \circ d_n^{\times}(D_n) \overset{(47)}{=} f_{n-1}^{X,Y} \circ S_{n-1}(\sigma,\tau) \circ d_n^{\times}(D_n) \\ &= f_{n-1}^{X,Y} \circ d_n^{\times} \circ S_n(\sigma,\tau)(D_n) = f_{n-1}^{X,Y} \circ d_n^{\times}(\sigma,\tau) \\ d_n^{\times} \circ g_n^{X,Y}(\nu \otimes \rho) &\overset{(48)}{=} d_n^{\times} \circ S_n(\nu,\rho)(w) = S_{n-1}(\nu,\rho) \circ d_n^{\times}(w) \overset{(48)}{=} S_{n-1}(\nu,\rho) \circ g_{n-1}^{\Delta^p,\Delta^q} \circ d_n^{\otimes}(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) \\ &\overset{(47)}{=} g_{n-1}^{X,Y} \circ (S_\bullet(\nu) \otimes S_\bullet(\rho))_{n-1} \circ d_n^{\otimes}(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) \\ &= g_{n-1}^{X,Y} \circ d_n^{\otimes} \circ (S_\bullet(\nu) \otimes S_\bullet(\rho))_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) = g_{n-1}^{X,Y} \circ d_n^{\otimes}(\nu \otimes \rho). \end{aligned}$$

For (47) we consider continuous maps $\alpha : X \to X'$ and $\beta : Y \to Y'$ and obtain

$$\begin{aligned} (S_\bullet(\alpha) \otimes S_\bullet(\beta))_n \circ f_n^{X,Y}(\sigma,\tau) &\overset{(48)}{=} (S_\bullet(\alpha \circ \sigma) \otimes S_\bullet(\beta \circ \tau))_n(z) \overset{(48)}{=} f_n^{X',Y'}(\alpha \circ \sigma, \beta \circ \tau) \\ &= f_n^{X',Y'} \circ S_n(\alpha,\beta)(\sigma,\tau) \\ S_n(\alpha,\beta) \circ g_n^{X,Y}(\nu,\rho) &\overset{(48)}{=} S_n(\alpha \circ \nu, \beta \circ \rho)(w) \overset{(48)}{=} g_n^{X',Y'}(\alpha \circ \nu \otimes \beta \circ \rho) \\ &= g_n^{X',Y'} \circ (S_\bullet(\alpha) \otimes S_\bullet(\beta))_n(\nu \otimes \rho). \end{aligned}$$

2. We show that natural transformations $f, f' : S_\bullet \times \Rightarrow \otimes(S_\bullet \times S_\bullet)$ and $g, g' : \otimes(S_\bullet \times S_\bullet) \Rightarrow S_\bullet \times$ with $f_0'^{X,Y} = f_0^{X,Y}$ and $g_0'^{X,Y} = g_0^{X,Y}$ are chain homotopic with a chain homotopy that is natural in X and Y.

We construct chain homotopies $h_\bullet^{X,Y} : f_\bullet^{X,Y} \Rightarrow f_\bullet'^{X,Y}$ and $k_\bullet^{X,Y} : g_\bullet^{X,Y} \Rightarrow g_\bullet'^{X,Y}$ inductively from

$$h_0^{X,Y} = 0 : S_0(X \times Y) \to (S_\bullet(X) \otimes S_\bullet(Y))_1 \qquad k_0^{X,Y} = 0 : (S_\bullet(X) \otimes S_\bullet(Y))_0 \to S_1(X \times Y).$$

Suppose we constructed $h_j^{X,Y}$ and $k_j^{X,Y}$ for all topological spaces X, Y and all $0 \leq j \leq n-1$ such that for all $0 \leq j \leq n-1$ and continuous maps $\alpha : X \to X'$ and $\beta : Y \to Y'$

$$d_{j+1}^{\otimes} \circ h_j^{X,Y} + h_{j-1}^{X,Y} \circ d_j^{\times} = f_j'^{X,Y} - f_j^{X,Y} \qquad d_{j+1}^{\times} \circ h_j^{X,Y} + h_{j-1}^{X,Y} \circ d_j^{\otimes} = g_j'^{X,Y} - g_j^{X,Y} \tag{49}$$

$$(S_\bullet(\alpha) \otimes S_\bullet(\beta))_{j+1} \circ h_j^{X,Y} = h_j^{X',Y'} \circ S_j(\alpha,\beta) \quad S_{j+1}(\alpha,\beta) \circ k_j^{X,Y} = k_j^{X',Y'} \circ (S_\bullet(\alpha) \otimes S_\bullet(\beta))_j. \tag{50}$$

Then we have from (49) for $j = n - 1$

$$
\begin{aligned}
&d_n^{\otimes}((f_n'^{\Delta^n,\Delta^n} - f_n^{\Delta^n,\Delta^n} - h_{n-1}^{\Delta^n,\Delta^n} \circ d_n^{\times})(D_n)) = h_{n-2}^{\Delta^n,\Delta^n} \circ d_{n-1}^{\times} \circ d_n^{\times}(D_n) = 0 \\
&d_n^{\times}((g_n'^{\Delta^n,\Delta^n} - g_n^{\Delta^n,\Delta^n} - k_{n-1}^{\Delta^n,\Delta^n} \circ d_n^{\otimes})(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q})) = k_{n-2}^{\Delta^n,\Delta^n} \circ d_{n-1}^{\otimes} \circ d_n^{\otimes}(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) = 0
\end{aligned}
$$

This shows that the terms in the brackets are n-cycles and hence n-boundaries, because we have $H_n(\Delta^n \times \Delta^n) = 0$ and $H_n(\Delta^p \times \Delta^q) = 0$ for $n > 0$. We define for all singular n-simplexes $\sigma : \Delta^n \to X$, $\tau : \Delta^n \to Y$, $\nu : \Delta^p \to X$ and $\rho : \Delta^q \to Y$

$$
\begin{aligned}
&h_n^{X,Y}(\sigma, \tau) = (S_\bullet(\sigma) \otimes S_\bullet(\tau))_{n+1}(z) && d_{n+1}^{\otimes}(z) = (f_n'^{\Delta^n,\Delta^n} - f_n^{\Delta^n,\Delta^n} - h_{n-1}^{\Delta^n,\Delta^n} \circ d_n^{\times})(D_n) \qquad (51) \\
&k_n^{X,Y}(\nu \otimes \rho) = S_{n+1}(\nu, \rho)(w) && d_{n+1}^{\times}(w) = (g_n'^{\Delta^n,\Delta^n} - g_n^{\Delta^n,\Delta^n} - k_{n-1}^{\Delta^n,\Delta^n} \circ d_n^{\otimes})(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}).
\end{aligned}
$$

A direct computation shows that with this definition the identities (49) and (50) hold for $j = n$

$$
\begin{aligned}
d_{n+1}^{\otimes} \circ h_n^{X,Y}(\sigma, \tau) &\overset{(51)}{=} d_{n+1}^{\otimes} \circ (S_\bullet(\sigma) \otimes S_\bullet(\tau))_{n+1}(z) = (S_\bullet(\sigma) \otimes S_\bullet(\tau))_n \circ d_{n+1}^{\otimes}(z) \\
&\overset{(51)}{=} (S_\bullet(\sigma) \otimes S_\bullet(\tau))_n \circ (f_n'^{\Delta^n,\Delta^n} - f_n^{\Delta^n,\Delta^n} - h_{n-1}^{\Delta^n,\Delta^n} \circ d_n^{\times})(D_n) \\
&\overset{\text{nat } f,(50)}{=} (f_n'^{X,Y} - f_n^{X,Y}) \circ S_n(\sigma, \tau)(D_n) - h_{n-1}^{X,Y} \circ S_{n-1}(\sigma, \tau) \circ d_n^{\times}(D_n) \\
&= (f_n'^{X,Y} - f_n^{X,Y} - h_{n-1}^{X,Y} \circ d_n^{\times}) \circ S_n(\sigma, \tau)(D_n) \\
&= (f_n'^{X,Y} - f_n^{X,Y} - h_{n-1}^{X,Y} \circ d_n^{\times})(\sigma, \tau)
\end{aligned}
$$

$$
\begin{aligned}
d_{n+1}^{\times} \circ k_n^{X,Y}(\nu \otimes \rho) &\overset{(51)}{=} d_{n+1}^{\times} \circ S_n(\nu, \rho)(w) = S_n(\nu, \rho) \circ d_{n+1}^{\times}(w) \\
&\overset{(51)}{=} S_n(\nu, \rho) \circ (g_n'^{\Delta^p,\Delta^q} - g_n^{\Delta^p,\Delta^q} - k_{n-1}^{\Delta^p,\Delta^q} \circ d_n^{\otimes})(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) \\
&\overset{\text{nat } g,(50)}{=} (g_n'^{X,Y} - g_n^{X,Y}) \circ (S_\bullet(\nu) \otimes S_\bullet(\rho))_{n-1}(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) \\
&\quad - k_{n-1}^{X,Y} \circ (S_\bullet(\nu) \otimes S_\bullet(\rho))_{n-1} \circ d_n^{\otimes}(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) \\
&= (g_n'^{X,Y} - g_n^{X,Y} - k_{n-1}^{X,Y} \circ d_n^{\otimes}) \circ (S_\bullet(\nu) \otimes S_\bullet(\rho))_n(\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}) \\
&= (g_n'^{X,Y} - g_n^{X,Y} - k_{n-1}^{X,Y} \circ d_n^{\otimes})(\nu \otimes \rho)
\end{aligned}
$$

$$
\begin{aligned}
(S_\bullet(\alpha) \otimes S_\bullet(\beta))_{n+1} \circ h_n^{X,Y}(\sigma, \tau) &\overset{(51)}{=} (S_\bullet(\alpha \circ \sigma) \otimes S_\bullet(\beta \circ \tau))_{n+1}(z) \\
&= h_{n+1}^{X',Y'}(\alpha \circ \sigma, \beta \circ \tau) = h_n^{X',Y'} \circ S_n(\alpha, \beta)(\sigma, \tau) \\
S_{n+1}(\alpha, \beta) \circ k_n^{X,Y}(\nu \otimes \rho) &\overset{(51)}{=} S_{n+1}(\alpha \circ \nu, \beta \circ \rho)(w) = k_n^{X',Y'}(\alpha \circ \nu \otimes \beta \circ \rho) \\
&= k_n^{X',Y'} \circ (S_\bullet(\alpha) \otimes S_\bullet(\beta))_n(\nu \otimes \rho)
\end{aligned}
$$

3. The proof of this statement is analogous to the proof of 2. One considers natural transformations $F : S_\bullet \times \Rightarrow S_\bullet \times$ and $G : \otimes(S_\bullet \times S_\bullet) \Rightarrow \otimes(S_\bullet \times S_\bullet)$ with $F_0^{X,Y} = \mathrm{id}$ and $G_0^{X,Y} = \mathrm{id}$ for all topological spaces X, Y. One inductively constructs chain homotopies $h_\bullet^{X,Y} : \mathrm{id}_{S_\bullet(X \times Y)} \Rightarrow F_\bullet^{X,Y}$ and $k_\bullet^{X,Y} : \mathrm{id}_{S_\bullet(X) \otimes S_\bullet(Y)} \Rightarrow G_\bullet^{X,Y}$ that are natural in X and Y with $h_0^{X,Y} = 0$ and $k_0^{X,Y} = 0$. Setting $F = g \circ f$ and $G = f \circ g$ then proves the claim. (Exercise 66) □

The construction in the Eilenberg-Zilber Theorem is implicit - it just proves existence of the natural transformations $f : S_\bullet \times \Rightarrow \otimes(S_\bullet \times S_\bullet)$ and $g : \otimes(S_\bullet \times S_\bullet) \Rightarrow S_\bullet \times$ and the uniqueness of their components up to natural chain homotopy. It is possible to use the inductive construction to give concrete formulas for the components of the natural transformations f and g, but these can be complicated, and it is often not necessary to work with concrete formulas.

We now combine the Eilenberg-Zilber theorem with the Künneth formula in Theorem 5.4.6 to obtain a formula that allows us to compute the homologies of product spaces.

Corollary 5.5.3 (topological Künneth formula):
For all topological spaces X, Y there is a short exact sequence that is natural in X, Y

$$0 \to \bigoplus_{p+q=n} H_p(X)\otimes H_q(X) \to H_n(X \times Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0.$$

The sequence splits, but not canonically.

Proof:
For topological spaces X, Y the Künneth formula gives a short exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(X)\otimes H_q(X) \to H_n(S_\bullet(X)\otimes S_\bullet(Y)) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0.$$

As the homologies and Tor are functors, this is natural in X and Y. As $S_\bullet(X)$ and $S_\bullet(Y)$ are free chain complexes, this sequence splits, but not canonically. The Eilenberg-Zilber Theorem yields a chain homotopy equivalence $f_\bullet^{X,Y} : S_\bullet(X \times Y) \to S_\bullet(X)\otimes S_\bullet(Y)$ that is natural in X, Y. It induces an isomorphism on the homologies

$$H_n(S_\bullet(X)\otimes S_\bullet(Y)) \cong H_n S_\bullet(X \times Y) = H_n(X \times Y).$$

$\square$

Example 5.5.4:

1. If X is a topological space such that $H_n(X)$ is a free abelian group for all $n \in \mathbb{N}_0$, then $H_n(X \times Y) = \oplus_{p+q=n} H_p(X)\otimes H_q(Y)$ for all topological spaces Y.
2. In particular, if X is a contractible topological space, then $H_n(X \times Y) = H_n(Y)$ for all $n \in \mathbb{N}_0$, as $H_0(X) = \mathbb{Z}$ and $H_k(X) = 0$ for all $k > 0$.
3. The n-torus $T^n = (S^1)^{\times n} = S^1 \times \ldots \times S^1$ has the homologies $H_k(T^n) = \mathbb{Z}^{\binom{n}{k}}$. (Exercise 67).

6 Singular cohomology

6.1 The singular cochain complex and singular cohomology

As explained already in Section 2.1, see in particular Definition 2.1.4, there is a notion of *cochain complex* that is dual to the usual notion of chain complex. A cochain complex $X^\bullet = (X^n)_{n\in\mathbb{Z}}$ consists of a family of abelian groups X^n and coboundary operators $d^n : X^n \to X^{n+1}$ that raise the degree by one. Although every cochain complex $X^\bullet$ defines a chain complex $X_\bullet$ with $X_n = X^{-n}$ and $d_n = d^{-n} : X_n \to X_{n-1}$ and vice versa, it is sometimes preferable in algebraic topology to work with cochain complexes. This leads to the concept of *singular cohomology.* Just as singular homology, singular cohomology can be considered for the abelian group $\mathbb{Z}$ or with coefficients in a general abelian group M.

Definition 6.1.1: Let X be a topological space and M an abelian group.

The **singular cochain complex** $S^\bullet(X;M)$ **with coefficients** in M is the cochain complex with $S^n(X;M) = \mathrm{Hom}_{\mathrm{Ab}}(S_n(X),M)$ for $n\in\mathbb{Z}$ and coboundary operator

$$d^n = \mathrm{Hom}_{\mathrm{Ab}}(d_{n+1},M) : S^n(X;M) \to S^{n+1}(X;M), \quad \phi \mapsto \phi\circ d_{n+1}$$

- elements of $S^n(X;M)$ are called **singular cochains**,
- elements of $Z^n(X;M) = \ker d^n \subset S^n(X;M)$ are called **singular cocycles**,
- elements of $B^n(X;M) = \mathrm{im}\, d^{n-1} \subset Z^n(X;M)$ are called **singular coboundaries**.

The nth **singular cohomology** group with coefficients in M is

$$H^n(X;M) = \frac{Z^n(X;M)}{B^n(X;M)}.$$

For $M=\mathbb{Z}$ one writes $S^\bullet(X) := S^\bullet(X;\mathbb{Z})$ and $H^n(X) := H^n(X;\mathbb{Z})$.

The singular *cohomologies* of a topological space X with coefficients in an abelian group M are not independent from its singular homologies with coefficients in $\mathbb{Z}$. Just like singular *homologies* with coefficients in M, they can be reduced to the singular homologies with coefficients in $\mathbb{Z}$ and algebraic quantities given by the group M. This is analogous to the universal coefficient theorem for singular homology in Corollary 5.4.7. The only difference is that the role of the functor $-\otimes M : \mathrm{Ab}\to\mathrm{Ab}$ in Corollary 5.4.7 is taken by the functor $\mathrm{Hom}(-,M) : \mathrm{Ab}^{op}\to\mathrm{Ab}$.

To obtain this result, we proceed as in Section 5. The first step is to analyse the behaviour of the functor $\mathrm{Hom}(-,M) : \mathrm{Ab}^{op}\to\mathrm{Ab}$ on short exact sequences and to derive a counterpart of Lemma 5.3.2 for the functor $-\otimes M : \mathrm{Ab}\to\mathrm{Ab}$.

Lemma 6.1.2: Let M be an abelian group.

1. The functor $\mathrm{Hom}(-,M) : \mathrm{Ab}^{op}\to\mathrm{Ab}$ is **left exact**:
for any exact sequence $A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ the following sequence is exact
$$0 \to \mathrm{Hom}_{\mathrm{Ab}}(C,M) \xrightarrow{\pi^*:\phi\mapsto\phi\circ\pi} \mathrm{Hom}_{\mathrm{Ab}}(B,M) \xrightarrow{\iota^*:\phi\mapsto\phi\circ\iota} \mathrm{Hom}(A,M)$$
2. If $0\to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ is a split exact, then one has a split exact sequence sequence
$$0 \to \mathrm{Hom}_{\mathrm{Ab}}(C,M) \xrightarrow{\pi^*:\phi\mapsto\phi\circ\pi} \mathrm{Hom}_{\mathrm{Ab}}(B,M) \xrightarrow{\iota^*:\phi\mapsto\phi\circ\iota} \mathrm{Hom}(A,M) \to 0.$$
In particular, this holds, whenever C is a free group.

Proof:
1. We consider the group homomorphisms $\pi^* = \mathrm{Hom}(\pi, M) : \mathrm{Hom}_{\mathrm{Ab}}(C, M) \to \mathrm{Hom}_{\mathrm{Ab}}(B, M)$ and $\iota^* = \mathrm{Hom}(\pi, M) : \mathrm{Hom}_{\mathrm{Ab}}(B, M) \to \mathrm{Hom}_{\mathrm{Ab}}(A, M)$ and show that π^* is injective with $\mathrm{im}\,\pi^* = \ker \iota^*$. If $\pi^*(\phi) = \pi^*(\psi)$ for group homomorphisms $\phi, \psi : C \to M$ then $\phi \circ \pi = \psi \circ \pi$ and by surjectivity of π this implies $\phi = \psi$. We also have $\iota^* \circ \pi^*(\phi) = \phi \circ \pi \circ \iota = \phi \circ 0 = 0$ for all group homomorphisms $\phi : C \to M$, which implies $\mathrm{im}\,\pi^* \subset \ker \iota^*$.

If a group homomorphism $\chi : B \to M$ satisfies $\chi \in \ker \iota^*$, then we have $\iota^*(\chi) = \chi \circ \iota = 0$ and $\mathrm{im}\,\iota = \ker \pi \subset \ker \chi$. By surjectivity of π, it induces a group homomorphism $\chi' : C = \mathrm{im}\,\pi \cong B/\ker \pi \to M$ with $\pi^*(\chi') = \chi' \circ \pi = \chi$. This shows that $\chi \in \mathrm{im}\,\pi^*$ and $\mathrm{im}\,\pi^* = \ker \iota^*$.

2. The proof of the second claim is analogous to the one of Lemma 5.3.2. □

Example 6.1.3: We apply the functor $\mathrm{Hom}(-, \mathbb{Z}) : \mathrm{Ab}^{op} \to \mathrm{Ab}$ to the short exact sequence $0 \to \mathbb{Z} \xrightarrow{z \mapsto nz} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$. This yields the chain complex

$$0 \to \underbrace{\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})}_{\cong 0} \xrightarrow{\pi^* : f \mapsto f \circ \pi} \underbrace{\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, \mathbb{Z})}_{\cong \mathbb{Z}} \xrightarrow{\iota^* : f \mapsto f(n \cdot -)} \underbrace{\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, \mathbb{Z})}_{\cong \mathbb{Z}} \to 0$$

Then the map $\pi^* : 0 = \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$, $f \mapsto f \circ \pi$ is injective. As any group homomorphism $f : \mathbb{Z} \to \mathbb{Z}$ is of the form $f : \mathbb{Z} \to M$, $z \mapsto zf(1)$, we have $\mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}$ and $\ker(\iota^*) = \mathrm{Tor}_n(\mathbb{Z}) = \mathrm{im}\,\pi^* = 0$. However, $\iota^* : \mathbb{Z} \to \mathbb{Z}$, $z \mapsto nz$ is not surjective for $n \neq \pm 1$.

In Proposition 5.3.9 we used free resolutions to characterise the non-exactness of the functor $-\otimes M : \mathrm{Ab} \to \mathrm{Ab}$ in terms of the torsion functor $\mathrm{Tor}(-, M) : \mathrm{Ab} \to \mathrm{Ab}$. For this we applied the functor $-\otimes M$ to a free resolution of an abelian group A and removed the exact part of the resulting chain complex. On the group homomorphisms, the functor $\mathrm{Tor}(-, M) : \mathrm{Ab} \to \mathrm{Ab}$ was obtained by lifting group homomorphisms $f : A \to A'$ to chain maps between the associated free resolutions. Lemma 6.1.2 allows us to apply an analogous procedure to the functor $\mathrm{Hom}(-, M) : \mathrm{Ab}^{op} \to \mathrm{Ab}$. This yields the functor $\mathrm{Ext}(-, M) : \mathrm{Ab}^{op} \to \mathrm{Ab}$, whose name stems from the fact that $\mathrm{Ext}(A, M)$ classifies extensions of M by A.

Proposition 6.1.4:
Any abelian group M defines a functor $\mathrm{Ext}(-, M) : \mathrm{Ab}^{op} \to \mathrm{Ab}$ that assigns

- to an abelian group A the group $\mathrm{Ext}(A, M) = \mathrm{coker}\,\iota^* = \mathrm{Hom}(K, M)/\mathrm{im}\,\iota^*$ for any free resolution $A_\bullet = 0 \to K \xrightarrow{\iota} F \xrightarrow{\pi} A \to 0$,
- to a group homomorphism $f : A \to A'$ the group homomorphism

$$\mathrm{Ext}(f, M) = h^* : \mathrm{coker}(\iota'^*) \to \mathrm{coker}(\iota^*), \quad \phi + \mathrm{im}\,\iota'^* \to \phi \circ h + \mathrm{im}\,\iota^*$$

  for any chain map $f_\bullet = (h, g, f) : A_\bullet \to A'_\bullet$ between free resolutions that extends f.

Proof:
1. Given free resolutions $A_\bullet = 0 \to K \xrightarrow{\iota} F \xrightarrow{\pi} A \to 0$ and $A'_\bullet = 0 \to K' \xrightarrow{\iota'} F' \xrightarrow{\pi'} A' \to 0$ of abelian groups A and A' and a chain map $f_\bullet = (h, g, f) : A_\bullet \to A'_\bullet$ that extends a group homomorphism $f : A \to A'$, we can apply the functor $\mathrm{Hom}(-, M) : \mathrm{Ab}^{op} \to \mathrm{Ab}$ for an abelian

group M to diagram (37). This yields the commuting diagram

$$\begin{tikzcd}
\mathrm{Hom}(A_\bullet, M) = & 0 \arrow[r] & \mathrm{Hom}(A,M) \arrow[r, "\pi^*"] & \mathrm{Hom}(F,M) \arrow[r, "\iota^*"] & \mathrm{Hom}(K,M) \arrow[r] & 0 \\
\mathrm{Hom}(A'_\bullet, M) = \arrow[u, "\mathrm{Hom}(f_\bullet,M)"] & 0 \arrow[r] & \mathrm{Hom}(A',M) \arrow[r, "\pi'^*"'] \arrow[u, "f^*"] & \mathrm{Hom}(F',M) \arrow[r, "\iota'^*"'] \arrow[u, "g^*"] & \mathrm{Hom}(K',M) \arrow[r] \arrow[u, "h^*"] & 0.
\end{tikzcd} \tag{52}$$

in which the rows are exact in the first two entries. The cokernels of ι^* and ι'^* are the first cohomologies of the chain complexes in the diagram

$$\begin{tikzcd}
\mathrm{Hom}(A_\bullet, M)^{red} = & 0 \arrow[r] & \mathrm{Hom}(F,M) \arrow[r, "\iota^*"] & \mathrm{Hom}(K,M) \arrow[r] & 0 \\
\mathrm{Hom}(A'_\bullet, M)^{red} = \arrow[u, "\mathrm{Hom}(f_\bullet,M)^{red}"] & 0 \arrow[r] & \mathrm{Hom}(F',M) \arrow[r, "\iota'^*"'] \arrow[u, "g^*"] & \mathrm{Hom}(K',M) \arrow[r] \arrow[u, "h^*"] & 0,
\end{tikzcd} \tag{53}$$

and the induced map $\mathrm{Ext}(f,M) = h^* : \mathrm{coker}(\iota'^*) \to \mathrm{coker}(\iota^*)$ is given by

$$\mathrm{Ext}(f,M) = H^1(\mathrm{Hom}(f_\bullet), M)^{red} : H^1\mathrm{Hom}(A'_\bullet, M) \to H^1\mathrm{Hom}(A'_\bullet, M)^{red}$$

As all free resolutions $A^1_\bullet$ and $A^2_\bullet$ of A are chain homotopy equivalent by Corollary 5.3.8, the associated cochain complexes $\mathrm{Hom}(A^1_\bullet, M)$ and $\mathrm{Hom}(A^2_\bullet, M)$ in (52) are cochain homotopy equivalent, and so are the chain complexes $\mathrm{Hom}(A^1_\bullet, M)^{red}$ and $\mathrm{Hom}(A^2_\bullet, M)^{red}$ in (53). Thus, their cohomologies do not depend on the resolutions. This shows that $\mathrm{Ext}(A,M)$ is defined.

By Proposition 5.3.7, any two extensions $f^1_\bullet$ and $f^2_\bullet$ of $f: A \to A'$ are chain homotopic. This also holds for the induced cochain maps $\mathrm{Hom}(f^1_\bullet, M)$ and $\mathrm{Hom}(f^2_\bullet, M)$ in (52) and the induced chain maps $\mathrm{Hom}(f^1_\bullet, M)^{red}$ and $\mathrm{Hom}(f^2_\bullet, M)^{red}$ in (53). Hence, they induce the same homomorphisms between the cohomologies. This shows that $\mathrm{Ext}(f,M) : \mathrm{Ext}(A',M) \to \mathrm{Ext}(A,M)$ is defined.

2. That $\mathrm{Ext}(-,M) : \mathrm{Ab}^{op} \to \mathrm{Ab}$ is a functor follows as in the proof of Proposition 5.3.9 (Exercise). □

Example 6.1.5:

1. For all abelian groups M one has $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, M) = M/nM$. This follows by choosing the free resolution $0 \to \mathbb{Z} \xrightarrow{\iota: z \mapsto nz} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \to 0$. This yields a commuting diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{Z},M) \arrow[r, "\iota^*:\phi\mapsto n\phi"] \arrow[d, "\cong"', "\phi\mapsto\phi(1)"] & \mathrm{Hom}(\mathbb{Z},M) \arrow[d, "\cong"', "\psi\mapsto\psi(1)"] \\
M \arrow[r, "\iota':m\mapsto nm"'] & M
\end{tikzcd}$$

and gives $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, M) = \mathrm{coker}(\iota^*) \cong \mathrm{coker}(\iota') = M/\mathrm{im}\,\iota' = M/nM$.

2. If A is a free group, we can choose the free resolution $0 \to 0 \xrightarrow{\iota} A \xrightarrow{\pi=\mathrm{id}} A \to 0$, which gives $\mathrm{Ext}(A,M) = \mathrm{coker}(\iota^*) = \mathrm{Hom}(0,M)/\mathrm{im}\,\iota^* = 0$.

3. An abelian group M is called **divisible**, if for all $m \in M$ and $n \in \mathbb{Z}\setminus\{0\}$ there is an $m' \in M$ with $m = nm'$. One can show that for divisible M one has $\mathrm{Ext}(A,M) = 0$ for all abelian groups A. For finitely generated abelian groups, this follows directly from 1.

4. For all abelian groups M and families $(A_i)_{i\in I}$ of abelian groups one has

$$\mathrm{Ext}(\oplus_{i\in I}A_i, M) = \Pi_{i\in I}\mathrm{Ext}(A_i, M) \qquad \mathrm{Ext}(M, \Pi_{i\in I}A_i) \cong \Pi_{i\in I}\mathrm{Ext}(M, A_i)$$

   This follows, because the functors $\mathrm{Hom}(-, M) : \mathrm{Ab}^{op} \to \mathrm{Ab}$ and $\mathrm{Hom}(M, -) : \mathrm{Ab} \to \mathrm{Ab}$ satisfy $\mathrm{Hom}(\oplus_{i\in I}A_i, M) \cong \Pi_{i\in I}\mathrm{Hom}(A_i, M)$ and $\mathrm{Hom}(M, \Pi_{i\in I}A_i) \cong \Pi_{i\in I}\mathrm{Hom}(M, A_i)$ by Exercise 1 and by choosing appropriate resolutions (Exercise 68).

By making use of the functor $\mathrm{Ext} : \mathrm{Ab}^{op} \to \mathrm{Ab}$, we can now derive a counterpart of the universal coefficient theorem for singular homologies. For this, we first derive a simplified counterpart of the Künneth formula in Theorem 5.4.6, where we choose one of the relevant chain complexes to be of the form $0 \to M \to 0$. This is also known as the universal coefficient theorem for cochain complexes. Its proof is a direct counterpart of the proof of Theorem 5.4.6, up to some simplifications due to the choice of the chain complex $0 \to M \to 0$. By choosing for the free chain complex $F_\bullet$ in Theorem 6.1.6 the singular chain complex $S_\bullet(X)$ for a topological space X, we then obtain the universal coefficient theorem for singular cohomology.

Theorem 6.1.6 (universal coefficients for cochain complexes):
Let M be an abelian group. For every free chain complex $F_\bullet$ and $M^\bullet = \mathrm{Hom}(F_\bullet, M)$ and all $n \in \mathbb{Z}$ one has an exact sequence that splits

$$0 \to \mathrm{Ext}(H_{n-1}(F_\bullet), M) \to H^n(M^\bullet) \to \mathrm{Hom}(H_n(F_\bullet), M) \to 0.$$

Proof:
We consider the short exact sequence of chain complexes

$$0 \to B_n(F_\bullet) \xrightarrow{i_n} Z_n(F_\bullet) \xrightarrow{\pi_n} H_n(F_\bullet) \to 0, \tag{54}$$

where $B_n(F_\bullet) \subset Z_n(F_\bullet) \subset F_n$ are free as subgroups of free abelian groups. We also consider the short exact sequence of chain complexes

$$0 \to Z_\bullet(F_\bullet) \xrightarrow{\iota_\bullet} F_\bullet \xrightarrow{d_\bullet} B^{-1}_\bullet(F_\bullet) \to 0$$

where $B^{-1}_n(F_\bullet) = B_{n-1}(F_\bullet)$, $Z_\bullet(F_\bullet)$ and $B^{-1}_\bullet(F_\bullet)$ are equipped with the trivial differential, $\iota_n : Z_n(F_\bullet) \to F_n$ is the inclusion and $d_n : F_n \to B_{n-1}(F_\bullet)$ is the corestriction of the differential. As $B^{-1}_n(F_\bullet)$ is free for all $n \in \mathbb{Z}$, this sequence splits. Applying $\mathrm{Hom}(-, M) : \mathrm{Ab}^{op} \to \mathrm{Ab}$ yields a short exact sequence of chain complexes

$$0 \to \mathrm{Hom}(B^{-1}_\bullet(F_\bullet), M) \xrightarrow{d^*_\bullet} \mathrm{Hom}(F_\bullet, M) \xrightarrow{\iota^*_\bullet} \mathrm{Hom}(Z_\bullet(F_\bullet), M) \to 0.$$

As $B^{-1}_\bullet(F_\bullet)$ and $Z_\bullet(F_\bullet)$ have trivial boundary operators, the cohomologies of the first and last cochain complex are given by $H^n\mathrm{Hom}(B^{-1}_\bullet(F_\bullet), M) = \mathrm{Hom}(B_{n-1}(F_\bullet), M)$ and $H^n\mathrm{Hom}(Z_\bullet(F_\bullet), M) = \mathrm{Hom}(Z_n(F_\bullet), M)$. The long exact cohomology sequence takes the form

$$\ldots \xrightarrow{\partial^{n-1}} \mathrm{Hom}(B_{n-1}(F_\bullet), M) \xrightarrow{H^n(d^*_\bullet)} H^n(M^\bullet) \xrightarrow{H^n(\iota^*_\bullet)} \mathrm{Hom}(Z_n(F_\bullet), M) \xrightarrow{\partial^n} \ldots$$

This yields for each $n \in \mathbb{Z}$ a short exact sequence

$$0 \to \mathrm{coker}\, \partial^{n-1} \to H^n(M^\bullet) \to \ker \partial^n \to 0. \tag{55}$$

The connecting homomorphism is given by

$$\partial^n = i_n^* : \mathrm{Hom}(Z_n(F_\bullet), M) \to \mathrm{Hom}(B_n(F_\bullet), M), \quad \phi \mapsto \phi \circ i_n.$$

Because (54) is a free resolution of $H_n(F_\bullet)$, we have $\ker \partial^n = \ker i_n^* = \mathrm{im}\, \pi_n^* = \mathrm{Hom}(H_n(F_\bullet), M)$ and $\mathrm{coker}\, \partial^n = \mathrm{Ext}(H_{n-1}(F_\bullet), M)$. Inserting this into (55) gives the exact sequence in the theorem. That this short exact sequence splits follows as for the Künneth formula in Theorem 5.4.6, but we omit the proof. □

Corollary 6.1.7 (**universal coefficients for singular cohomology**):
For every topological space X, abelian group M and $n \in \mathbb{Z}$ one has a split exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(X), M) \to H^n(X; M) \to \mathrm{Hom}(H_n(X), M) \to 0.$$

Example 6.1.8: We compute the singular cohomologies of $\mathbb{R}\mathrm{P}^n$. By Corollary 6.1.7 we have a short split exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(\mathbb{R}\mathrm{P}^n), M) \to H^n(X; M) \to \mathrm{Hom}(H_n(X), M) \to 0.$$

From Example 4.2.9 we have

$$H_k(\mathbb{R}\mathrm{P}^n) = \begin{cases} \mathbb{Z} & k = 0 \text{ or } k = n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 1 \leq k < n \text{ odd} \\ 0 & k \text{ even or } k > n. \end{cases}$$

This yields $\mathrm{Ext}(H_{k-1}(\mathbb{R}\mathrm{P}^n), M) = M/2M$ if $2 \leq k \leq n$ even and $\mathrm{Ext}(H_{k-1}(\mathbb{R}\mathrm{P}^n), M) = 0$ else.

We also have $\mathrm{Hom}(H_k(\mathbb{R}\mathrm{P}^n), M) \cong M$ if $k = 0$ or $k = n$ odd, $\mathrm{Hom}(H_k(\mathbb{R}\mathrm{P}^n), M) = \mathrm{Tor}_2(M)$ if $1 \leq k < n$ odd and $\mathrm{Hom}(H_k(\mathbb{R}\mathrm{P}^n), M) = 0$ else. This gives

$$H^k(\mathbb{R}\mathrm{P}^n; M) \cong \mathrm{Ext}(H_{k-1}(\mathbb{R}\mathrm{P}^n), M) \oplus \mathrm{Hom}(H_k(\mathbb{R}\mathrm{P}^n), M) = \begin{cases} M & k = 0 \vee k = n \text{ odd} \\ \mathrm{Tor}_2(M) & 1 \leq k < n \text{ odd} \\ M/2M & 1 \leq k \leq n \text{ even} \\ 0 & k < n. \end{cases}$$

Corollary 6.1.7 shows that the singular cohomologies of a topological space X with coefficients in an abelian group M do not contain more information than the singular homologies with coefficients in $\mathbb{Z}$. They can be computed from the coefficient group M and the singular homologies with coefficients in $\mathbb{Z}$.

More generally, cohomology theories behave analogously to homology theories. Just as homology theories, they can be characterised by Eilenberg-Steenrod axioms, which are analogous to the Eilenberg-Steenrod axioms for homology theories from Definition 3.7.1. Singular cohomology with coefficients in an abelian group M satisfies these Eilenberg-Steenrod axioms and thus defines a cohomology theory with coefficients in M.

Recall from Definition 3.7.1 that a homology theory is characterised by a family of functors $H_n : \mathrm{Top}(2) \to \mathrm{Ab}$ from the category Top(2) of pairs of topological spaces and morphisms of pairs and by a family of natural transformations $\partial_n : H_n^3 \Rightarrow H_{n-1}^1$, where the functors

$H_n^1 : \mathrm{Top}(2) \to \mathrm{Ab}$ and $H_n^3 : \mathrm{Top}(2) \to \mathrm{Ab}$ assign to a pair (X, A) the homology $H_n(A; M)$ and the relative homology $H_n(X, A; M)$, respectively. These natural transformations encode the connecting homomorphisms. Likewise, a cohomology theory is characterised by a family of functors $H^n : \mathrm{Top}(2)^{op} \to \mathrm{Ab}$ and a family of natural transformations $\partial^n : H_1^n \Rightarrow H_3^{n+1}$, where $H_1^n, H_2^n, H_3^n : \mathrm{Top}(2)^{op} \to \mathrm{Ab}$ assign to a pair (X, A) the cohomologies $H^n(A; M)$, $H^n(X; M)$ and $H^n(X, A; M)$, respectively.

Theorem 6.1.9: Singular cohomology with coefficients in an abelian group M defines a

- collection of functors $H^n : \mathrm{Top}(2)^{op} \to \mathrm{Ab}$ for each $n \in \mathbb{Z}$,
- collection of natural transformations $\partial^n : H_1^n \Rightarrow H_3^{n+1}$ for all $n \in \mathbb{Z}$,

that satisfy the **Eilenberg-Steenrod axioms** for cohomology:

1. **Long exact sequence:** for every pair (X, A) there is a long exact sequence
$$\ldots \xleftarrow{\partial^n} H^n(A; M) \xleftarrow{H^n(i)} H^n(X; M) \xleftarrow{H^n(\pi)} H^n(X, A; M) \xleftarrow{\partial^{n-1}} H^{n-1}(A; M) \xleftarrow{H^{n-1}(i)} \ldots$$
2. **Homotopy invariance:** If $f, g : (X, A) \to (Y, B)$ are homotopic, then for all $n \in \mathbb{Z}$
$$H^n(f) = H^n(g) : H^n(Y, B; M) \to H^n(X, A; M).$$
3. **Excision:** For every pair (X, A) and every open subset $U \subset A$ with $\overline{U} \subset \mathring{A}$ the inclusions $\iota : (X \setminus U, A \setminus U) \to (X, A)$ induce isomorphisms
$$H^n(\iota) : H^n(X, A; M) \xrightarrow{\sim} H^n(X \setminus U, A \setminus U; M)$$
4. **Additivity:** For any family $(X_i, A_i)_{i \in I}$ of pairs of topological spaces and all $n \in \mathbb{Z}$
$$H^n(\amalg_{i \in I} X_i, \amalg_{i \in I} A_i; M) \cong \Pi_{i \in I} H^n(X_i, A_i; M).$$
5. **Dimension axiom:** $H^n(\bullet; M) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$.

The abelian group $H^0(\bullet) = M$ is called the **coefficient group** of the cohomology theory.

Proof:
The functor $\mathrm{Hom}(-, M) : \mathrm{Ab}^{op} \to \mathrm{Ab}$ defines a functor $\mathrm{Hom}(-, M) : \mathrm{Ch}_{\mathrm{Ab}}^{op} \to \mathrm{Ch}^{\mathrm{Ab}}$ that

- sends a chain complex $X_\bullet$ to the cochain complex $\mathrm{Hom}_{\mathrm{Ab}}(X_\bullet, M)$ with boundary operators
$$d^n : \mathrm{Hom}_{\mathrm{Ab}}(X_n, M) \to \mathrm{Hom}_{\mathrm{Ab}}(X_{n+1}, M), \quad \phi \mapsto \phi \circ d_{n+1},$$
- a chain map $f_\bullet : X_\bullet \to Y_\bullet$ to the cochain map $\mathrm{Hom}(f_\bullet, M) : \mathrm{Hom}(Y_\bullet, M) \to \mathrm{Hom}(X_\bullet, M)$
$$\mathrm{Hom}(f_n, M) : \mathrm{Hom}_{\mathrm{Ab}}(Y_n, M) \to \mathrm{Hom}_{\mathrm{Ab}}(X_n, M), \quad \phi \mapsto \phi \circ f_n.$$

That $\mathrm{Hom}_{\mathrm{Ab}}(X_\bullet, M)$ is indeed a cochain complex and $\mathrm{Hom}(f_\bullet, M)$ a cochain map follows, because one has for all group homomorphisms $\phi : X_n \to M$ and $\psi : Y_n \to M$
$$\begin{aligned} &d^{n+1} \circ d^n(\phi) = \phi \circ d_{n+1} \circ d_{n+2} = \phi \circ 0 = 0 : X_{n+2} \to M \\ &d^n \circ f_n(\psi) = \psi \circ f_n \circ d_{n+1} = \psi \circ d_{n+1} \circ f_{n+1} = f^{n+1} \circ d^n(\psi). \end{aligned}$$

The functors $H^n : \mathrm{Top}(2)^{op} \to \mathrm{Ab}$ are the composite functors
$$H^n : \mathrm{Top}(2)^{op} \xrightarrow{S_\bullet} \mathrm{Ch}_{\mathrm{Ab}}^{op} \xrightarrow{\mathrm{Hom}(-,M)} \mathrm{Ch}^{\mathrm{Ab}} \xrightarrow{H^n} \mathrm{Ab}.$$

1. To prove the long exact sequence axiom, recall that a pair (X, A) of topological spaces gives rise to a short exact sequence of chain complexes

$$0 \to S_\bullet(A) \xrightarrow{\iota_\bullet} S_\bullet(X) \xrightarrow{\pi_\bullet} S_\bullet(X, A) \to 0. \tag{56}$$

For each $n \in \mathbb{N}_0$ there is a retraction map $r_n : S_n(X) \to S_n(A)$ with $r_n(\sigma) = \sigma$ for each singular n-simplex $\sigma : \Delta^n \to X$ with $\sigma(\Delta^n) \subset A$ and $r_n(\sigma) = 0$ for $\sigma(\Delta^n) \not\subset A$. By Exercise 8 the short exact sequence (56) splits in each degree. Hence applying the functor $\mathrm{Hom}(-, M)$ yields a short exact sequence of cochain complexes

$$0 \to \mathrm{Hom}(S_\bullet(X, A), M) \xrightarrow{\pi^*_\bullet} \mathrm{Hom}(S_\bullet(X), M) \xrightarrow{\iota^*_\bullet} \mathrm{Hom}(S_\bullet(A), M) \to 0.$$

with an associated long exact cohomology sequence that defines the connecting homomorphisms and hence the natural transformations $\partial^n : H^n_1 \Rightarrow H^{n+1}_3$

$$\ldots \xrightarrow{\partial^{n-1}} H^n(X, A; M) \xrightarrow{H^n(\pi^*_\bullet)} H^n(X; M) \xrightarrow{H^n(\iota^*_\bullet)} H^n(A; M) \xrightarrow{\partial^n} H^{n+1}(X, A; M) \to \ldots$$

2. To show homotopy invariance, we show that the functor $\mathrm{Hom}(-; M) : \mathrm{Ch}^{op}_{\mathrm{Ab}} \to \mathrm{Ch}^{\mathrm{Ab}}$ sends chain homotopies to cochain homotopies. Any chain homotopy $h_\bullet : f_\bullet \Rightarrow g_\bullet$ between chain maps $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$ yields a cochain homotopy $k^\bullet = \mathrm{Hom}(h_\bullet, M) : \mathrm{Hom}(f_\bullet, M) \Rightarrow \mathrm{Hom}(g_\bullet, M)$

$$k^n = \mathrm{Hom}(h_n, M) : \mathrm{Hom}_{\mathrm{Ab}}(Y_{n+1}, M) \to \mathrm{Hom}_{\mathrm{Ab}}(X_n, M), \quad \phi \mapsto \phi \circ h_n,$$

as we have for any group homomorphism $\phi : Y_{n+1} \to M$

$$(d^n \circ k^n + k^{n+1} \circ d^n)(\phi) = \phi \circ (h_n \circ d_{n+1} + d_{n+1} \circ h_{n+1}) = \phi \circ (g_n - f_n).$$

As the singular chain complex functor $S_\bullet : \mathrm{Top}(2) \to \mathrm{Ch}_{\mathrm{Ab}}$ sends homotopic maps to chain homotopic chain maps, this proves the homotopy axiom.

3. The excision axiom follows by an argument analogous to the proof of Theorem 3.5.1. By constructing retractions $r'_n : S^{\mathcal{U}}_n(X) \to S_n(A)$ and $r_n : S_n(X) \to S_n(A)$ as in 1. one can show that both rows of the commuting diagram (23) are split exact. Applying the functor $\mathrm{Hom}(-, M)$ then yields again a commuting diagram with exact rows and an associated long exact cohomology sequence, in which all arrows except the middle one are isomorphisms. This proves a counterpart of Proposition 3.4.13. The excision isomorphism is obtained as in the proof of Theorem 3.5.1.

4. The additivity axiom follows from the fact that the functor $S_\bullet : \mathrm{Top}(2) \to \mathrm{Ch}_{\mathrm{Ab}}$ preserves coproducts and the functor $\mathrm{Hom}(-, M) : \mathrm{Ab}^{op} \to \mathrm{Ab}$ sends direct sums in Ab to direct products. Recall from Exercise 22 that for any family $(X_i, A_i)_{i\in I}$ of pairs in Top(2) and $n \in \mathbb{Z}$

$$S_n(\amalg_{i\in I} X_i, \amalg_{i\in I} A_i) = S_n(\amalg_{i\in I} X_i)/S_n(\amalg_{i\in I} A_i) = \oplus_{i\in I} S_n(X_i, A_i).$$

By Exercise 1 the functor $\mathrm{Hom}(-, M) : \mathrm{Ab}^{op} \to \mathrm{Ab}$ sends direct sums of abelian groups to products of abelian groups. Hence, the cochain complex $S^\bullet(\amalg_{i\in I} X_i, \amalg_{i\in I} A_i; M)$ is given by

$$\ldots \to \Pi_{i\in I} \mathrm{Hom}(S_n(X_i, A_i), M) \xrightarrow{d^{n-1} : (\phi_i)_{i\in I} \mapsto (\phi_i \circ d^i_{n+1})_{i\in I}} \Pi_{i\in I} \mathrm{Hom}(S_{n+1}(X_i, A_i), M) \to \ldots,$$

where $d^i_n : S_n(X_i, A_i) \to S_{n-1}(X_i, A_i)$ are the boundary operators of $S_\bullet(X_i, A_i)$. Equivalently, we have $d^n : (\phi_i)_{i\in I} \mapsto (d^n_i(\phi_i))_{i\in I}$, where $d^n_i : S^n(X_i, A_i; M) \to S^{n+1}(X_i, A_i; M)$, $\phi \mapsto \phi \circ d^i_{n+1}$ are the coboundary operators of $S^\bullet(X_i, A_i; M)$. It follows that the cohomologies are given by

$$H^n(\amalg_{i\in I} X_i, \amalg_{i\in I} A_i; M) = \frac{\ker d^n}{\mathrm{im}\, d^{n-1}} = \prod_{i\in I} \frac{\ker d^n_i}{\mathrm{im}\, d^{n-1}_i} = \prod_{i\in I} H^n(X_i, A_i; M).$$

5. To prove the dimension axiom, recall that the singular chain complexes for the one-point space are given by $S_k(\bullet) = 0$ for $k \neq 0$ and $S_0(\bullet) = \mathbb{Z}$. This yields $S^0(\bullet; M) = \mathrm{Hom}_{\mathrm{Ab}}(\mathbb{Z}, M) \cong M$ and $S^k(\bullet; M) = \mathrm{Hom}_{\mathrm{Ab}}(0, M) = 0$ for $k \neq 0$, and the boundary operators are given by

$$d^0 = 0 : M \to 0, \quad d^{-1} = 0 : 0 \to M, \quad d^k = 0 : 0 \to 0 \quad k \neq 0, -1.$$

This gives $H^0(\bullet; M) = \ker d^0 / \mathrm{im}\, d^{-1} = M$ and $H^k(\bullet; M) = \ker d^k / \mathrm{im}\, d^{k-1} = 0$ for $k \neq 0$. □

6.2 The ring structure of cohomology

In this section we show that the cohomology groups of a topological space X with coefficients in a commutative unital ring R combine into a graded ring, the *cohomology ring* $H^\bullet(X; R)$. This ring structure contains additional information about the topological spaces: even if all cohomology groups of two topological spaces are isomorphic, their cohomology rings may be non-isomorphic (Exercise 72). This shows in particular that the spaces are not homotopy equivalent.

We will then investigate another structure, the cap product, that combines cohomology groups and homology groups of a topological space to form homology groups. This can be seen as an action of the cohomology ring on the homology groups and gives the direct sum of the latter the structure of a graded module over the cohomology ring.

The multiplication of the cohomology ring is obtained from the multiplication of R, which allows one to form the pointwise product of any p-cochain $\phi : S_p(X) \to R$ and a q-cochain $\phi : S_q(Y) \to R$, and by pre-composition with an Eilenberg Zilber map $f : S_\bullet \times \Rightarrow \otimes(S_\bullet \times S_\bullet)$ as in Theorem 5.5.2. The latter combines a pair of a p-cochain $\phi : S_p(X) \to R$ and a q-cochain $\phi : S_q(Y) \to R$ into a $(p+q)$-cochain on $X \times Y$. Setting $X = Y$ and precomposing with the chain map induced by the diagonal map $D : X \to X \times X$, $x \mapsto (x, x)$ yields a multiplication map that combines singular p- and q-cochains of X into a singular $(p+q)$-cochain of X.

Definition 6.2.1: Let R be a commutative unital ring with multiplication map $\mu : R \otimes R \to R$, $r \otimes s \mapsto r \cdot s$ and let $f : S_\bullet \times \Rightarrow \otimes(S_\bullet \times S_\bullet)$ an Eilenberg-Zilber map.

For all topological spaces X, Y and $p, q \in \mathbb{N}_0$

- the **cohomology cross product** is the group homomorphism

  $$\times : S^p(X; R) \otimes S^q(Y; R) \to S^{p+q}(X \times Y; R), \quad \phi \otimes \psi \mapsto \mu \circ (\phi \otimes \psi) \circ \pi_{p,q} \circ f^{X,Y}_{p+q},$$

  where $\pi_{p,q} : (S_\bullet(X) \otimes S_\bullet(Y))_{p+q} \to S_p(X) \otimes S_q(Y)$ is the canonical projection.

- the **cup product** is the group homomorphism

  $$\cup : S^p(X; R) \otimes S^q(X; R) \to S^{p+q}(X; R), \quad \phi \otimes \chi \mapsto \mu \circ (\phi \otimes \chi) \circ \pi_{p,q} \circ f^{X,X}_{p+q} \circ S_{p+q}(D)$$

  where $D : X \to X \times X$, $x \mapsto (x, x)$ denotes the diagonal map.

Remark 6.2.2: If we identify singular simplexes $\tau : \Delta^n \to X \times Y$ with pairs $\tau = (\rho, \sigma)$ of singular simplexes $\rho = \pi_1 \circ \chi : \Delta^n \to X$ and $\sigma = \pi_2 \circ \chi : \Delta^n \to Y$ then the cup product and the cross product are related by

$$(\phi \cup \chi)(\rho) = (\phi \times \chi)(\rho, \rho) \qquad (\phi \times \psi)(\rho, \sigma) = (\phi \circ \pi_1 \cup \psi \circ \pi_2)(\rho, \sigma)$$

for $\phi \in S^p(X)$, $\chi \in S^q(X)$, $\psi \in S^q(Y)$ and singular simplexes $\rho : \Delta^p \to X$ and $\sigma : \Delta^q \to Y$.

We can also describe the cross and the cup product with the following commuting diagram

$$\begin{tikzcd}
S_n(X) \arrow[rr, "\phi\cup\chi"] \arrow[d, "S_n(D)"'] & & R \\
S_n(X\times X) \arrow[urr, "\phi\times\chi"'] \arrow[d, "f_n^{X,X}"'] & & R\otimes R \arrow[u, "\mu: r\otimes s\mapsto r\cdot s"'] \\
\oplus_{j+k=n} S_j(X)\otimes S_k(X) \arrow[rr, "\pi_{p,q}"'] & & S_p(X)\otimes S_q(X) \arrow[u, "\phi\otimes\chi"']
\end{tikzcd}$$

Clearly, the cross and cup products of cochains from Definition 6.2.1 depend on the choice of the Eilenberg-Zilber map. However, we are mainly interested in the induced maps on the cohomologies. As all Eilenberg-Zilber maps are related by natural chain homotopies, this dependence on the choice of the Eilenberg-Zilber map vanishes, once one passes to the cohomologies.

Proposition 6.2.3: For all topological spaces X, Y, commutative unital rings R and $p, q \in \mathbb{N}_0$ the cross product and the cup product induce group homomorphisms

$$\begin{aligned}
\times &: H^p(X;R)\otimes H^q(Y;R) \to H^{p+q}(X\times Y;R), && [\phi]\otimes[\chi]\mapsto[\phi\times\chi]\\
\cup &: H^p(X;R)\otimes H^q(X;R) \to H^{p+q}(X;R), && [\phi]\otimes[\psi]\mapsto[\phi\cup\psi]
\end{aligned}$$

that do not depend on the choice of the Eilenberg-Zilber map.

Proof:
1. To show that the cross and cup product are well-defined on the homologies, it is sufficient to consider the cross product by Remark 6.2.2. We have to show that the cross product of a coboundary with a cochain is again a coboundary. For this let $\phi \in Z^p(X;R)$, $\psi \in Z^q(Y;R)$, $\chi \in S^{p-1}(X;R)$ and $\xi \in S^{q-1}(X;R)$. With the identities

$$\pi_{p,q}\circ d_{p+q+1} = (d_{p+1}\otimes \mathrm{id})\circ \pi_{p+1,q} + (-1)^p(\mathrm{id}\otimes d_{q+1})\circ\pi_{p,q+1} \tag{57}$$

and with $d^p(\phi)=0$ and $d^q(\psi)=0$ we obtain

$$\begin{aligned}
d^{p-1}(\chi)\times\psi &= \mu\circ((\chi\circ d_p)\otimes\psi)\circ\pi_{p,q}\circ f^{X,Y}_{p+q} \overset{(57)}{=} \mu\circ(\chi\otimes\psi)\circ\pi_{p-1,q}\circ d_{p+q}\circ f^{X,Y}_{p+q}\\
&= \mu\circ(\chi\otimes\psi)\circ\pi_{p-1,q}\circ f^{X,Y}_{p+q-1}\circ d_{p+q} = d^{p+q-1}(\chi\times\psi)\\
\phi\times d^{q-1}(\xi) &= \mu\circ(\phi\otimes(\xi\circ d_q))\circ\pi_{p,q}\circ f^{X,Y}_{p+q} \overset{(57)}{=} (-1)^p\mu\circ(\phi\otimes\xi)\circ\pi_{p,q-1}\circ d_{p+q}\circ f^{X,Y}_{p+q}\\
&= (-1)^p\mu\circ(\phi\otimes\xi)\circ\pi_{p,q-1}\circ f^{X,Y}_{p+q-1}\circ d_{p+q} = (-1)^p d^{p+q-1}(\phi\times\xi)
\end{aligned}$$

2. That the cross product and cup product on the cohomologies is independent of the choice of the Eilenberg-Zilber map follows, because any two Eilenberg-Zilber maps are related by natural chain homotopies. If $f_\bullet, f'_\bullet : S_\bullet(X\times Y)\to S_\bullet(X)\otimes S_\bullet(Y)$ are Eilenberg-Zilber maps and $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ a natural family of chain homotopies, then we have

$$\begin{aligned}
\phi\times_{f'}\psi - \phi\times_f\psi &= \mu\circ(\phi\otimes\psi)\circ\pi_{p,q}\circ(f'^{X,Y}_{p,q} - f^{X,Y}_{p,q})\\
&= \mu\circ(\phi\otimes\psi)\circ\pi_{p,q}\circ(d_{p+q+1}\circ h^{X,Y}_{p+q} + h^{X,Y}_{p+q-1}\circ d_{p+q})\\
&\overset{(57)}{=} \mu\circ(d^p(\phi)\otimes\psi)\circ\pi_{p+1,q}\circ h^{X,Y}_{p+q+1} + (-1)^p\mu\circ(\phi\otimes d^q(\psi))\circ\pi_{p,q+1}\circ h^{X,Y}_{p+q+1}\\
&\quad + d^{p+q}(\mu\circ(\phi\otimes\psi)\circ\pi_{p,q}\circ h^{X,Y}_{p+q-1})
\end{aligned}$$

The last summand of the last term is trivial in the cohomology class $H^{p+q}(X\times Y;R)$, the other two summands vanish on cohomology classes $[\phi]\in H^p(X;R)$ and $[\psi]\in H^q(Y;R)$. $\square$

Although the cross product and the cup product are defined abstractly, in terms of Eilenberg-Zilber maps, it is useful for computations and proofs to have a more concrete description. This is achieved by considering a specific Eilenberg-Zilber map with a geometric interpretation, the *Alexander-Whitney map*. It is given in terms of front and rear faces of singular simplexes.

Definition 6.2.4: Let X be a topological space and $\sigma:\Delta^n\to X$ a singular n-simplex in X.

1. The $(n-q)$-dimensional **front face** of σ is the singular $(n-q)$-simplex

$$F^{n-q}(\sigma)=\sigma\circ i_{n-q}:\Delta^{n-q}\to X \qquad i_{n-q}:=f^n_n\circ\ldots\circ f^{n-q+1}_{n-q+1}:\Delta^{n-q}\to\Delta^n, e_i\mapsto e_i.$$

2. The q-dimensional **back face** or **rear face** of σ is the singular q-simplex

$$R^q(\sigma)=\sigma\circ r_q:\Delta^q\to X \qquad r_q:=f^n_0\circ\ldots\circ f^{q+1}_0:\Delta^q\to\Delta^n, e_i\mapsto e_{n-q+i}.$$

Remark 6.2.5:

1. The front and rear face define group homomorphisms $F^{n-q}:S_n(X)\to S_{n-q}(X)$ and $R^q:S_n(X)\to S_q(X)$ that are natural in X: for all continuous maps $f:X\to Y$ one has

$$F^{n-q}\circ S_n(f)=S_{n-q}(f)\circ F^{n-q} \qquad R^q\circ S_n(f)=S_q(f)\circ R^q.$$

2. A direct computation using the relations $f^n_i\circ f^{n-1}_j=f^n_j\circ f^{n-1}_{i-1}$ for $0\leq j<i\leq n$ from (9) shows that the front and rear face satisfy

$$F^{n-q-1}(\sigma\circ f^n_i)=\begin{cases}F^{n-q}(\sigma)\circ f^{n-q}_i & i\leq n-q\\ F^{n-q-1}(\sigma) & i\geq n-q\end{cases} \tag{58}$$
$$R^q(\sigma\circ f^n_i)=\begin{cases}R^q(\sigma) & i\leq n-q\\ R^{q+1}(\sigma)\circ f^{q+1}_{i-n+q} & i\geq n-q.\end{cases}$$

To define the Alexander-Whitney map, we again identify singular n-simplexes $\rho:\Delta^n\to X\times Y$ with pairs (σ,τ) of singular n-simplexes $\sigma=\pi_1\circ\rho:\Delta^n\to X$ and $\tau=\pi_2\circ\rho:\Delta^n\to Y$. The Alexander-Whitney map on a pair (σ,τ) of singular n-simplexes is then obtained by taking the tensor product of the p-dimensional front face of σ and the q-dimensional rear face of τ and summing over all p,q with $p+q=n$.

Proposition 6.2.6: The group homomorphisms

$$\mathrm{AW}^{X,Y}_n:S_n(X\times Y)\to\bigoplus_{p+q=n}S_p(X)\otimes S_q(Y), \quad (\sigma,\tau)\mapsto\sum_{p+q=n}F^p(\sigma)\otimes R^q(\tau)$$

define a natural transformation $\mathrm{AW}:S_\bullet\times\Rightarrow\otimes(S_\bullet\times S_\bullet)$, the **Alexander-Whitney map**.

Proof:
In degree 0, the Alexander-Whitney map is the canonical isomorphism

$$\mathrm{AW}^{X,Y}_0:S_0(X\times Y)=\langle X\times Y\rangle_{\mathbb{Z}}\xrightarrow{\cong}\langle X\rangle_{\mathbb{Z}}\otimes\langle Y\rangle_{\mathbb{Z}}=S_0(X)\otimes S_0(Y).$$

The naturality of the Alexander-Whitney map follows directly from Remark 6.2.5, 1. That its components are chain maps follows by a direct computation with Remark 6.2.5, 2:

$$\begin{aligned}
&d_n \circ \mathrm{AW}_n^{X,Y}(\sigma,\tau)\\
&= \sum_{q=0}^{n-1}\sum_{j=0}^{n-q}(-1)^j (F^{n-q}(\sigma)\circ f_j^{n-q})\otimes R^q(\tau) + \sum_{q=1}^{n}\sum_{j=0}^{q}(-1)^{j+n-q}F^{n-q}(\sigma)\otimes R^q(\tau)\circ f_j^q\\
&\overset{(58)}{=} \sum_{q=0}^{n-1}\sum_{j=0}^{n-q}(-1)^j F^{n-q-1}(\sigma\circ f_j^n)\otimes R^q(\tau) + \sum_{q=1}^{n}\sum_{j=0}^{q}(-1)^{j+n-q}F^{n-q}(\sigma)\otimes R^{q-1}(\tau\circ f_{j+n-q}^n)\\
&= \sum_{q=0}^{n-1}\sum_{j=0}^{n-q}(-1)^j F^{n-q-1}(\sigma\circ f_j^n)\otimes R^q(\tau) + \sum_{q=0}^{n-1}\sum_{j=n-q+1}^{n}(-1)^j F^{n-q-1}(\sigma)\otimes R^q(\tau\circ f_j^n)\\
&\overset{(58)}{=} \sum_{q=0}^{n-1}\sum_{j=0}^{n}(-1)^j F^{n-q-1}(\sigma\circ f_j^n)\otimes R^q(\tau\circ f_j^n) = \mathrm{AW}_{n-1}^{X,Y}\circ d_n(\sigma,\tau).
\end{aligned}$$

□

Corollary 6.2.7: The cross product and the cup product are given by

$$([\phi]\times[\psi])(\sigma,\tau) = \phi(F^p(\sigma))\cdot\psi(R^q(\tau)) \qquad ([\phi]\cup[\chi])(\sigma) = \phi(F^p(\sigma))\cdot\chi(R^q(\sigma))$$

for $\phi\in Z^p(X;R)$, $\psi\in Z^q(Y;R)$, $\chi\in Z^q(X;R)$ and $\sigma:\Delta^{p+q}\to X$, $\tau:\Delta^{p+q}\to Y$.

Using the simple description of the cup product in terms of the Alexander-Whitney map, we can investigate its properties. Its construction in terms of an Eilenberg-Zilber map implies that the cup product is natural in the underlying topological space X. Its concrete description in terms of the Alexander-Whitney map shows that it is also associative and commutative up to a sign that depends on the dimension of the cohomology groups. Analogous properties hold for the cross product, but we will not require them in the following, so we leave the formulation of the corresponding statements for cross products and their proof as an exercise.

Proposition 6.2.8: Let X, Y, Z be topological spaces, R a commutative unital ring and $\alpha\in Z^p(X;R)$, $\beta\in Z^q(X;R)$, $\gamma\in Z^r(X;R)$. The cup product has the following properties

1. **Naturality:** $H^{p+q}(f)([\alpha]\cup[\beta]) = H^p(f)[\alpha]\cup H^q(f)[\beta]$ for all continuous maps $f: X\to Y$,
2. **Associativity:** $[\alpha]\cup([\beta]\cup[\gamma]) = ([\alpha]\cup[\beta])\cup[\gamma]$,
3. **Graded commutativity:** $[\alpha]\cup[\beta] = (-1)^{pq}[\beta]\cup[\alpha]$.

Proof:
1. Naturality follows, due to the naturality of the Eilenberg-Zilber map, the naturality of the map $\pi_{p+q}: (S_\bullet(X)\otimes S_\bullet(Y))_{p+q}\to S_p(X)\otimes S_q(Y)$ and of the diagonal map $D: X\to X\times X$.

2. Associativity holds already for the cup product of cochains and follows from the formula for the Alexander-Whitney map and the formulas for the front and rear face in Definition 6.2.4. These formulas imply for all singular n-simplexes $\sigma:\Delta^n\to X$ with $n=p+q+r$

$$F^p(\sigma)\otimes F^q\circ R^{q+r}(\sigma)\otimes R^r\circ R^{q+r}(\sigma) = F^p\circ F^{p+q}(\sigma)\otimes R^q\circ F^{p+q}(\sigma)\otimes R^r(\sigma)$$

This gives for all $\phi\in S^p(X;R)$, $\psi\in S^q(X;R)$ and $\chi\in S^r(X;R)$ and singular n-simplexes $\sigma:\Delta^n\to X$ with $n=p+q+r$

$$\begin{aligned}
&(\phi\cup\psi)\cup\chi(\sigma) = (\phi\cup\psi)(F^{p+q}(\sigma))\cdot\chi(R^r(\sigma)) = \phi(F^p\circ F^{p+q}(\sigma))\cdot\psi(R^q\circ F^{p+q}(\sigma))\cdot\chi(R^r(\sigma))\\
&= \phi(F^p(\sigma))\cdot\psi(F^q\circ R^{q+r}(\sigma))\cdot\chi(R^r\circ R^{q+r}(\sigma)) = \phi(F^p(\sigma))\cdot(\psi\cup\chi)(R^{q+r}(\sigma)) = \phi\cup(\psi\cup\chi)(\sigma).
\end{aligned}$$

3. Graded commutativity holds only for the cup product of the cohomologies, not of the cochains. To prove it for the cohomologies, we consider the group homomorphisms

$$\phi_n^X : S_n(X) \to S_n(X), \quad \sigma \mapsto (-1)^{n(n+1)/2}\sigma \circ I_n$$

where $I_n : \Delta^n \to \Delta^n$, $[v_0, \ldots, v_n] \mapsto [v_n, \ldots, v_0]$ is the affine-linear map that reverses the ordering of the vertices of the standard n-simplex. They satisfy $\phi_n^X \circ \phi_n^X = \mathrm{id}_{S_n(X)}$ for all $n \in \mathbb{N}_0$ and topological spaces X as well as $\phi_0^X = \mathrm{id}_{S_0(X)}$.

3.(a) We show that these group homomorphisms relate the cup product and its opposite:

For abelian groups A, B, we denote by $\tau : A\otimes B \to B\otimes A$, $a\otimes b \mapsto b\otimes a$ the map that flips the factors in their tensor products. From the formulas in Definition 6.2.4 and the identity $\frac{1}{2}(p+q)(p+q+1) = pq + \frac{1}{2}p(p+1) + \frac{1}{2}q(q+1)$ we then have

$$(F^q\otimes R^p) \circ S_n(D) \circ \phi_n^X = (-1)^{pq}\tau \circ (\phi_p^X \circ F^p\otimes\phi_q^X \circ R^q) \circ S_n(D).$$

This implies for all $\psi \in S^p(X;R)$ and $\chi \in S^q(X;R)$

$$\begin{aligned}(\chi \cup \psi)(\sigma) &= \mu \circ (\chi\otimes\psi) \circ (F^q\otimes R^p)(\sigma\otimes\sigma) \\ &= (-1)^{pq}\mu \circ (\psi\otimes\chi) \circ ((\phi_p^X \circ F^p)\otimes(\phi_q^X \circ R^q))(\phi_n^X(\sigma)\otimes\phi_n^X(\sigma)) \\ &= (-1)^{pq}((\psi \circ \phi_p^X) \cup (\chi \circ \phi_q^X))(\phi_n^X(\sigma)\otimes\phi_n^X(\sigma)).\end{aligned} \tag{59}$$

3.(b) We show that these group homomorphisms define chain maps $\phi_\bullet^X : S_\bullet(X) \to S_\bullet(X)$ that are natural in X and chain homotopic to the identity.

Naturality in X follows directly: for all continuous maps $f : X \to Y$ and $n \in \mathbb{N}_0$ one has

$$S_n(f) \circ \phi_n^X(\sigma) = (-1)^{n(n+1)/2} f \circ \sigma \circ I_n = \phi_n^Y \circ S_n(f).$$

With the identity $I_n \circ f_j^n = f_{n-j}^n \circ I_{n-1}$ we compute for all singular n-simplexes $\sigma : \Delta^n \to X$

$$\begin{aligned}d_n \circ \phi_n^X(\sigma) &= (-1)^{n(n+1)/2}\sum_{j=0}^n(-1)^j\sigma \circ I_n \circ f_j^n = (-1)^{n(n+1)/2}\sum_{j=0}^n(-1)^j\sigma \circ f_{n-j}^n \circ I_{n-1} \\ &= (-1)^{n(n-1)/2}\sum_{j=0}^n(-1)^{n-j}\sigma \circ f_{n-j}^n \circ I_{n-1} = (-1)^{n(n-1)/2}\sum_{k=0}^n(-1)^k\sigma \circ f_k^n \circ I_{n-1} \\ &= \phi_{n-1}^X \circ d_n(\sigma).\end{aligned}$$

This shows that we have a natural family of chain maps $\phi_\bullet^X : S_\bullet(X) \to S_\bullet(X)$ with $\phi_0^X = \mathrm{id}_{S_0(X)}$. We inductively construct a construct a family of chain homotopies $h_\bullet^X : \mathrm{id}_{S_\bullet(X)} \Rightarrow \phi_\bullet^X$, natural in X, by setting $h_0^X = 0 : S_0(X) \to S_1(X)$, as in the proof of the Eilenberg-Zilber Theorem.

Suppose we constructed for all $k \leq n-1$ group homomorphisms $h_k^X : S_k(X) \to S_{k+1}(X)$ with

$$d_{k+1} \circ h_k^X + h_{k-1}^X \circ d_k = \phi_k^X \qquad S_{k+1}(f) \circ h_k^X = h_k^Y \circ S_k(f) \tag{60}$$

for all $k \leq n-1$ and continuous maps $f : X \to Y$. As $H_k(\Delta^n) = 0$ for $k \in \mathbb{N}$ and

$$d_n \circ (f_n^{\Delta^n} - h_{n-1}^{\Delta^n} \circ d_n) = d_n \circ f_n^{\Delta^n} - d_n \circ h_{n-1}^{\Delta^n} \circ d_n = f_{n-1}^{\Delta^n} \circ d_n - f_{n-1}^{\Delta^n} \circ d_n + h_{n-2}^{\Delta^n} \circ d_{n-1} \circ d_n = 0,$$

there is a $z \in S_{n+1}(\Delta^n)$ with $d_{n+1}(z) = (f_n^{\Delta^n} - h_{n-1}^{\Delta^n} \circ d_n)(\mathrm{id}_{\Delta^n})$, and we set

$$h_n^X : S_n(X) \to S_{n+1}(X), \quad \sigma \mapsto S_{n+1}(\sigma)(z) \qquad d_{n+1}(z) = (f_n^{\Delta^n} - h_{n-1}^{\Delta^n} \circ d_n)(\mathrm{id}_{\Delta^n}).$$

This gives for all singular n-simplexes $\sigma : \Delta^n \to X$ and continuous maps $f : X \to Y$

$$\begin{aligned} d_{n+1} \circ h_n^X(\sigma) &= d_{n+1} \circ S_{n+1}(\sigma)(z) = S_n(\sigma) \circ d_{n+1}(z) \\ &= S_n(\sigma) \circ f_n^{\Delta^n}(\mathrm{id}_{\Delta^n}) - S_n(\sigma) \circ h_{n-1}^{\Delta^n} \circ d_n(\mathrm{id}_{\Delta^n}) \\ &= f_n^X(\sigma) - h_{n-1}^X \circ S_{n-1}(\sigma) \circ d_n(\mathrm{id}_{\Delta^n}) = f_n^X(\sigma) - h_{n-1}^X \circ d_n(\sigma) \end{aligned}$$

$$S_{n+1}(f) \circ h_n^X(\sigma) = S_{n+1}(f \circ \sigma)(z) = h_n^Y(f \circ \sigma) = h_n^Y \circ S_n(f)(\sigma).$$

3.(c) As the chain maps $\phi_\bullet^X : S_\bullet(X) \to S_\bullet(X)$ are chain homotopic to the identity maps, they induce the identity morphisms on the cohomologies. Thus, formula (59) for cohomology classes $[\psi] \in H^p(X;R)$ and $[\chi] \in H^q(X;R)$ becomes $[\chi] \cup [\psi] = (-1)^{pq}[\psi] \cup [\chi]$. □

If we combine different cohomology groups of a topological space X by taking their direct sum, then we can view the cup products as a multiplication map on this direct sum. Proposition 6.2.8 then ensures the associativity of this multiplication. The fact that all cup products are group homomorphisms implies distributivity. The compatibility of the cup product with the dimensions of the cohomology groups is encoded in the notion of a graded ring.

Definition 6.2.9: A **graded ring** R is a unital ring R together with a direct sum decomposition $R = \oplus_{j=0}^\infty R_j$ of its additive group, such that $R_j \cdot R_k \subset R_{j+k}$ for all $j, k \in \mathbb{N}_0$.

It is called **graded commutative**, if $r \cdot s = (-1)^{jk} s \cdot r$ for all $r \in R_j$ and $s \in R_k$.

Corollary 6.2.10: Let X be a topological space X and R a commutative ring.

1. The cohomologies of X with coefficients in R form a graded commutative ring

$$H^\bullet(X;R) = \oplus_{k=0}^\infty H^k(X;R)$$

   with the addition from their abelian group structure and the multiplication by the cup product. The unit element is the cohomology class of $\phi : S_0(X) \to R$, $x \mapsto 1$.

2. Continuous maps $f : X \to Y$ induce ring homomorphisms $H^\bullet(f) : H^\bullet(Y) \to H^\bullet(X)$.

Proof:
The distributive law follows from the fact that the cup product induces group homomorphisms $\cup : H^p(X;R) \otimes H^q(X;R) \to H^{p+q}(X;R)$, the associativity of the multiplication and the graded commutativity from Proposition 6.2.8. That $[\phi] \in H^0(X;R)$ is the neutral element for the multiplication follows directly from Definition 6.2.1. That every continuous map $f : X \to Y$ induces a unital ring homomorphism, follows from the naturality of the cup product in Proposition 6.2.8 and the definition of the unit element. □

The cohomology rings often contain nilpotent elements. in particular, as the multiplication adds the degrees, any topological space X whose cohomologies $H^k(X;R)$ vanish for all $k \geq n$ and with $H^k(X;R) \neq 0$ for some $k > 0$ must contain nilpotent elements. This holds in particular for finite CW complexes. Likewise, the graded commutativity implies that elements of odd degrees must be nilpotent if R has no 2-torsion elements.

Example 6.2.11:

1. The cohomology ring $H^\bullet(S^n)$ of the n-sphere has a single generator $[z] \in H^n(S_n) \cong \mathbb{Z}$ of degree n and the unit element $1 \in H^0(S^n)$ with

$$[z] \cup [z] = [0] \qquad [z] \cup 1 = 1 \cup [z] = [z] \qquad 1 \cup 1 = 1.$$

2. If R is torsion free or a field of char $R \neq 2$, any $[z] \in H^p(X;R)$ of odd degree p is nilpotent

$$[z] \cup [z] = (-1)^{p^2}[z] \cdot [z] = -[z] \cup [z] \quad \Rightarrow \quad [z] \cup [z] = 0.$$

3. Let $X_1, \ldots, X_n$ be topological spaces with basepoints $x_i \in X_i$ and suppose that every basepoint $x_i \in X_i$ has a neighbourhood U_i that strongly deformation retracts to x_i.

   Then $H_k(X_1 \vee ... \vee X_n) \cong H_k(X_1) \oplus ... \oplus H_k(X_n)$ for $k \in \mathbb{N}$ by Proposition 3.5.9 and

$$H^k(X_1 \vee ... \vee X_n; R) = H^k(X_1;R) \times ... \times H^k(X_n;R) \qquad k \in \mathbb{N}$$
$$H^0(X_1 \vee ... \vee X_n; R) = \{(\phi_1, ..., \phi_n) \in H^0(X_1;R) \times ... \times H^0(X_n;R) \mid \phi_i[x_i] = \phi_j[x_j]\}.$$

   Thus, the cohomology ring of the wedge product is a subring

$$H^\bullet(X_1 \vee ... \vee X_n; R) \subset H^\bullet(X_1;R) \times ... \times H^\bullet(X_n;R).$$

We will now derive another algebraic structure, the cap product, that relates the homologies of a topological space X to its cohomologies. The starting point is the evaluation of n-cochains of X on its n-chains. This can be considered in more generality for chain complexes $X_\bullet$ and the associated chain complexes $\mathrm{Hom}(X_\bullet, M)$ for an abelian group M and is known under the name *Kronecker pairing*. It induces a pairing between the nth cohomology group and the nth homology group of the chain complex that takes values in M.

Definition 6.2.12: Let M be an abelian group. The **Kronecker pairing** between a chain complex $X_\bullet$ and the associated cochain complex $M^\bullet = \mathrm{Hom}_{\mathrm{Ab}}(X_\bullet, M)$ is given by the maps

$$\langle \, , \, \rangle : M^n \otimes X_n \to M, \quad \phi \otimes x \mapsto \langle \phi, x \rangle = \phi(x).$$

Lemma 6.2.13: Let $X_\bullet$ be a chain complex and $M^\bullet = \mathrm{Hom}_{\mathrm{Ab}}(X_\bullet, M)$ for an abelian group M. The Kronecker pairing induces a group homomorphism

$$\langle \, , \, \rangle : H^n(M^\bullet) \otimes H_n(X_\bullet) \to M, \quad [\phi] \otimes [x] \mapsto \langle \phi, x \rangle.$$

Proof:
We show that the Kronecker pairing of $\phi \in Z^n(M^\bullet)$ and $z \in Z_n(X)$ depends only on the cohomology class of ϕ and the homology class of z. Let $\phi : X_n \to M$ be a group homomorphism with $d^n(\phi) = \phi \circ d_{n+1} = 0$ and $\phi' = \phi + \psi \circ d_{n-1}$ with a group homomorphism $\psi : X_{n-1} \to M$. Let $z' = z + d_{n+1}(x)$ with $z \in Z_n(X)$ and $x \in X_{n+1}$. Then we have

$$\begin{aligned}\langle \phi', z' \rangle &= \langle \phi + \psi \circ d_n, z + d_{n+1}(x) \rangle = \langle \phi, z \rangle + \langle \psi \circ d_n, z \rangle + \langle \phi, d_{n+1}(x) \rangle + \langle \psi \circ d_n, d_{n+1}(x) \rangle \\ &= \phi(z) + \psi \circ d_n(z) + \phi \circ d_{n+1}(x) + \psi \circ d_n \circ d_{n+1}(x) = \phi(z) = \langle \phi, z \rangle,\end{aligned}$$

where the second term in the last line vanishes, because $z \in Z_n(X)$, the third due to $\phi \circ d_{n+1} = 0$ and the third due to $d_n \circ d_{n+1} = 0$. □

If we apply the Kronecker pairing to the chain complex $S_\bullet(X)$ and the cochain complex $S^\bullet(X;M)$ of a topological space X, then it defines a pairing between its nth cohomology and its nth homology group

$$\langle\ ,\ \rangle : H^n(X;M)\otimes H_n(X) \to M.$$

However, this pairing is too restrictive. Firstly, one would like to admit non-trivial coefficients in the homology groups as well, for symmetry reasons. This can be remedied by restricting attention to coefficients in a commutative ring R and using its multiplication.

Secondly, as with the cup product, we would like to pair cochains and chains of different degrees. More specifically, evaluating a q-cochain on an n-chain with $n \geq q$ should give an $(n-q)$-chain. As for the cup product, this is achieved by applying an Eilenberg-Zilber map to the n-chain. To obtain simple expressions, we work with the Alexander-Whitney map from Definition 6.2.6 that is given in terms of the front and rear face of a simplex from Definition 6.2.4.

Finally, we will need the pairing between homologies and cohomologies also for relative cohomology and homology groups. This is important for applications to oriented manifolds in the next two sections, where we will treat Poincaré duality.

Definition 6.2.14: Let (X, A) be a pair of topological spaces, R a unital ring and $0 \leq q \leq n$. The **cap product** is the group homomorphism

$$\cap : S^q(X, A; R)\otimes S_n(X, A; R) \to S_{n-q}(X; R), \qquad \phi\otimes(z\otimes r) \mapsto F^{n-q}(z)\otimes r\langle \phi, R^q(z)\rangle$$

Example 6.2.15:

1. For $q = 0$ we have $F^n(\sigma) = \sigma$ and $R^0(\sigma) = \sigma(e_n)$ for singular n-simplexes $\sigma : \Delta^n \to X$. As $S^0(X, A; R) = \mathrm{Hom}_{\mathrm{Ab}}(S_0(X, A), R)$, any 0-cochain $\phi \in S^0(X, A; R)$ corresponds to a map $\phi : X \to R$ with $\phi(a) = 0$ for $a \in A$. Thus, the cap product is given by

$$\cap : S^0(X, A; R)\otimes S_n(X, A; R) \to S_n(X; R), \quad \phi\otimes(\sigma\otimes r) \mapsto \sigma\otimes r\phi(\sigma(e_n)).$$

2. For $q = n$ we have $F^0(\sigma) = \sigma(e_0)$ and $R^n(\sigma) = \sigma$ for singular n-simplexes $\sigma : \Delta^n \to X$. Thus, the cap product is given by

$$\cap : S^n(X, A; R)\otimes S_n(X, A; R) \to S_0(X; R), \quad \phi\otimes(\sigma\otimes r) \mapsto \sigma(e_0)\otimes r\phi(\sigma).$$

As we work with pairs of topological spaces and relative (co)homologies, we need to show that the cap product is indeed well-defined on relative q-cochains and relative n-chains. The definition in terms of the front and rear face also suggests that it should be natural in X and compatible with coboundary and boundary operators. The latter is largely a consequence of the identities (58) that describe the interaction of the front and rear face with the face maps.

Lemma 6.2.16: (**Properties of the cap product**)
The cap product is well-defined and satisfies the following identities

1. **Leibniz formula:** for all $\phi \in S^q(X, A; R)$, $z \in S_n(X, A)$ and $r \in R$

$$d_{n-q}(\phi \cap (z\otimes r)) = (-1)^{n-q} d^q(\phi) \cap (z\otimes r) + \phi \cap (d_n(z)\otimes r).$$

2. **Naturality:**
for all morphisms $f : (X, A) \to (Y, B)$, $\phi \in S^q(X, A; R)$, $z \in S_n(X, A)$ and $r \in R$ one has

$$S_{n-q}(f)\big(S^q(f)\phi \cap (z \otimes r)\big) = \phi \cap (S_n(f)z \otimes r).$$

Proof:
To see that the cap product is well-defined, consider n-chains $x \in S_n(X)$, $a \in S_n(A)$. Then $R^q(a) \in S_q(A)$, which implies $\langle \phi, R^q(a)\rangle = 0$ for $\phi \in S^q(X, A; R) = \mathrm{Hom}(S_n(X)/S_n(A), R)$. Likewise, we have $F^{n-q}(a) \in S_{n-q}(A)$, which implies $F^{n-q}(a) = 0$ in $S_n(X, A)$. This gives

$$\begin{aligned}\phi \cap ((x + a)\otimes r) &= F^{n-q}(x + a)\otimes r\langle \phi, R^q(x + a)\rangle = (F^{n-q}(x) + F^{n-q}(a))\otimes r\langle \phi, R^q(x) + R^q(a)\rangle \\ &= F^{n-q}(x)\otimes r\langle \phi, R^q(x)\rangle = \phi \cap (x\otimes r).\end{aligned}$$

1. To prove the Leibniz formula, we compute for a singular n-simplex $\sigma : \Delta^n \to X$

$$\begin{aligned}&d_{n-q}(\phi \cap (\sigma\otimes r)) = d_{n-q}(F^{n-q}(\sigma)\otimes r\langle \phi, R^q(\sigma)\rangle = (d_{n-q}F^{n-q}(\sigma))\otimes r\langle \phi, R^q(\sigma)\rangle \\ &= \sum_{i=0}^{n-q}(-1)^i F^{n-q}(\sigma) \circ f_i^{n-q}\otimes r\langle \phi, R^q(\sigma)\rangle \overset{(58)}{=} \sum_{i=0}^{n-q}(-1)^i F^{n-q-1}(\sigma \circ f_i^n)\otimes r\langle \phi, R^q(\sigma)\rangle.\end{aligned}$$

To determine the right-hand-side of the Leibnitz identity we compute the first term

$$\begin{aligned}d^q(\phi) \cap (\sigma\otimes r) &= F^{n-q-1}(\sigma)\otimes r\langle d^q(\phi), R^{q+1}(\sigma)\rangle = F^{n-q-1}(\sigma)\otimes r\langle \phi, d_{q+1} \circ R^{q+1}(\sigma)\rangle \\ &= \sum_{i=0}^{q+1}(-1)^i F^{n-q-1}(\sigma)\otimes r\langle \phi, R^{q+1}(\sigma) \circ f_i^{q+1}\rangle.\end{aligned}$$

Applying again (58), we obtain for the second term on the right

$$\begin{aligned}&\phi \cap (d_n(\sigma)\otimes r) = F^{n-q-1} \circ d_n(\sigma)\otimes r\langle \phi, R^q \circ d_n(\sigma)\rangle \\ &= \sum_{i=0}^{n}(-1)^i F^{n-q-1}(\sigma \circ f_i^n)\otimes r\langle \phi, R^q(\sigma \circ f_i^n)\rangle \\ &= \sum_{i=0}^{n-q-1}(-1)^i F^{n-q-1}(\sigma \circ f_i^n)\otimes r\langle \phi, R^q(\sigma \circ f_i^n)\rangle + \sum_{i=n-q}^{n}(-1)^i F^{n-q-1}(\sigma \circ f_i^n)\otimes r\langle \phi, R^q(\sigma \circ f_i^n)\rangle \\ &\overset{(58)}{=} \sum_{i=0}^{n-q-1}(-1)^i F^{n-q-1}(\sigma \circ f_i^n)\otimes r\langle \phi, R^q(\sigma)\rangle + \sum_{i=n-q}^{n}(-1)^i F^{n-q-1}(\sigma)\otimes r\langle \phi, R^{q+1}(\sigma) \circ f_{i-n+q+1}^{q+1}\rangle \\ &= \sum_{i=0}^{n-q-1}(-1)^i F^{n-q-1}(\sigma \circ f_i^n)\otimes r\langle \phi, R^q(\sigma)\rangle + \sum_{i=1}^{q+1}(-1)^{i+n-q-1} F^{n-q-1}(\sigma)\otimes r\langle \phi, R^{q+1}(\sigma) \circ f_i^{q+1}\rangle\end{aligned}$$

Combining these terms with the appropriate signs yields

$$\begin{aligned}&(-1)^{n-q}d^q(\phi) \cap (\sigma\otimes r) + \phi \cap (d_n(\sigma)\otimes r) \\ &= (-1)^{n-q}F^{n-q-1}(\sigma)\otimes r\langle \phi, R^{q+1}(\sigma) \circ f_0^{q+1}\rangle + \sum_{i=0}^{n-q-1}(-1)^i F^{n-q-1}(\sigma \circ f_i^n)\otimes r\langle \phi, R^q(\sigma)\rangle \\ &\overset{(58)}{=} \sum_{i=0}^{n-q}(-1)^i F^{n-q-1}(\sigma \circ f_i^n)\otimes r\langle \phi, R^q(\sigma)\rangle = d_{n-q}(\phi \cap (\sigma\otimes r)).\end{aligned}$$

2. Naturality of the cap product follows by a direct computation. We have for any singular n-simplex $\sigma : \Delta^n \to X$ and morphism of pairs $f : (X, A) \to (Y, B)$

$$\begin{aligned}S_{n-q}(f)(S^q(f)\phi \cap (\sigma\otimes r)) &= (S_{n-q}(f) \circ F^{n-q}(\sigma))\otimes r\langle \phi \circ S^q(f), R^q(\sigma)) \\ &= (S_{n-q}(f) \circ F^{n-q}(\sigma))\otimes r\langle \phi, S_q(f) \circ R^q(\sigma)) \\ &= F^{n-q}(f \circ \sigma)\otimes r\langle \phi, R^q(f \circ \sigma)) = \phi \cap (S_n(f)\sigma\otimes r).\end{aligned}$$

$\square$

With the results from Lemma 6.2.16 it is now straightforward to show that the cap product induces a pairing between the relative cohomology and homology groups of a pair of topological spaces with coefficients in a commutative ring R. It has naturality properties analogous to the ones in Lemma 6.2.16 with respect to morphisms of pairs of topological spaces.

Proposition 6.2.17: The cap product induces group homomorphisms

$$\cap : H^q(X,A;R)\otimes H_n(X,A;R) \to H_{n-q}(X;R), \quad [\phi]\otimes([z]\otimes r) \mapsto [\phi \cap (z\otimes r)]$$

that satisfy for all morphisms $f : (X,A) \to (Y,B)$ in Top(2):

$$H_{n-q}(f)(H^q(f)[\phi] \cap ([z]\otimes r)) = [\phi] \cap (H_n(f)[z]\otimes r).$$

Proof:
The Leibniz formula implies $\phi \cap (z\otimes r) \in Z_{n-q}(X;R)$ for all $\phi \in Z^q(X,A;R)$ and $z \in Z_n(X,A)$ and $r \in R$. If we also have $\psi = d^{q-1}(\chi) \in B^q(X,A;R)$ and $b = d_{n+1}(x) \in B_n(X,A;R)$, then

$$\begin{aligned}
&d_{n-q+1}((-1)^{n-q+1}\chi \cap (z\otimes r)) = d^{q+1}(\chi) \cap (z\otimes r) = \psi \cap (z\otimes r) &&\Rightarrow\quad \psi \cap (z\otimes r) \in B_{n-q}(X;R)\\
&d_{n-q+1}(\phi \cap (x\otimes r)) = \phi \cap (d_{n+1}(x)\otimes r) = \phi \cap (b\otimes r) &&\Rightarrow\quad \phi \cap (b\otimes r) \in B_{n-q}(X;R).
\end{aligned}$$

This shows that the cap product induces a group homomorphism

$$\cap : H^q(X,A;R)\otimes H_n(X,A;R) \to H_{n-q}(X;R), \quad [\phi]\otimes([z]\otimes r) \mapsto [\phi \cap (z\otimes r)].$$

The naturality with respect to the morphisms in Top(2) follows from Lemma 6.2.16, 2. $\square$

We will now investigate the algebraic properties of the cap product. By comparing Definition 6.2.14 of the cap product for the case $(X,A) = (X,\emptyset)$ with the formula for the cup product in Definition 6.2.7 the similarities are apparent. Both are formulated in terms of the front and rear face of a simplex, and the only difference between the two is the presence of the Kronecker pairing in the latter. As the cap product of a singular cocycle and a singular cycle gives another singular cycle, it is natural to interpret it as an action of the cohomology group on the homology group. The fact that it respects degree is encoded in the notion of a graded module.

Definition 6.2.18: Let $R = \oplus_{j=0}^{\infty} R_j$ be a graded ring.

A **graded module** over R is an abelian group M with a direct sum decomposition $M = \oplus_{j=0}^{\infty} M_j$ and an R-module structure $\rhd : R\otimes M \to M$ such that $R_j \rhd M_k \subset M_{k+j}$ for all $j,k \in \mathbb{N}_0$.

Proposition 6.2.19: Let R be a commutative ring and X a topological space.

The cap product defines a graded module structure on $H_\bullet(X;R) = \oplus_{k=0}^{\infty} H_{-k}(X;R)$ over the cohomology ring $H^\bullet(X;R) = \oplus_{j=0}^{\infty} H^k(X;R)$.

Proof:
The cap product defines a map

$$\rhd = \cap : H^\bullet(X;R)\otimes H_\bullet(X;R) \to H_\bullet(X;R) \quad \text{with} \quad H^j(X;R) \cap H_k(X;R) \subset H_{k-j}(X;R).$$

By the formula in Example 6.2.15, 1. the unit element $1 \in H^0(X;R)$ acts by $1 \cap [z \otimes r] = [z \otimes r]$. It remains to show that for all $\phi \in Z^p(X;R)$, $\psi \in Z^q(X;R)$ and $z \otimes r \in Z_n(X;R)$

$$([\phi] \cup [\psi]) \cap [z \otimes r] = [\phi] \cap ([\psi] \cap [z \otimes r])$$

For this, we note that by Definition 6.2.4 of the front and rear face of a simplex we have

$$F^{n-p-q}(\sigma) \otimes F^p \circ R^{p+q}(\sigma) \otimes R^q \circ R^{p+q}(\sigma) = F^{n-p-q} \circ F^{n-q}(\sigma) \otimes R^p \circ F^{n-q}(\sigma) \otimes R^q(\sigma)$$

for all singular n-simplexes $\sigma : \Delta^n \to X$. Inserting this into the formulas for the cup product from Corollary 6.2.7 and the formula for the cap product from Definition 6.2.14 gives

$$\begin{aligned}
(\phi \cup \psi) \cap (\sigma \otimes r) &= F^{n-(p+q)}(\sigma) \otimes r \langle \phi \cup \psi, R^{p+q}(\sigma) \rangle \\
&= F^{n-(p+q)}(\sigma) \otimes r \cdot \phi(F^p \circ R^{p+q}(\sigma)) \cdot \psi(R^q \circ R^{p+q}(\sigma)) \\
&= F^{(n-q)-p}(F^{n-q}(\sigma)) \otimes r \cdot \phi(R^p \circ F^{n-q}(\sigma))) \cdot \psi(R^q(\sigma)) \\
&= \phi \cap (F^{n-q}(\sigma) \otimes r \cdot \psi(R^q(\sigma))) = \phi \cap (\psi \cap (\sigma \otimes r)).
\end{aligned}$$

□

6.3 Orientation of compact manifolds

In this and the following section, we will show that for compact oriented topological n-manifolds X, there is a relation between the homology groups $H_p(X;R)$ and cohomology groups $H^{n-p}(X;R)$, namely Poincaré duality, which states that for all $p \in \{0, \ldots, n\}$

$$H^{n-p}(X;R) \cong H_p(X;R).$$

This isomorphism is given by the cap product and a distinguished element $[X] \in H_n(X)$, defined by the orientation. For non-compact topological n-manifolds there is an analogous isomorphism for the cohomology group with compact support.

In this section we define *orientations* of *compact* topological manifolds in terms of their homology groups. The notion of an orientation is familiar from linear algebra and can easily generalised to *smooth* manifolds. Recall that an orientation on a finite-dimensional real vector space V is an equivalence class of ordered bases of V. Two bases $B = (v_1, \ldots, v_n)$ and $B' = (v'_1, \ldots, v'_n)$ are equivalent if the unique linear map $\phi : V \to V$ with $\phi(v_i) = v'_i$ for $i = 1, \ldots, n$ has determinant $\det(\phi) > 0$. Hence, a finite-dimensional real vector space V has exactly two orientations.

A smooth n-manifold is a Hausdorff space X with a choice of homeomorphisms $\phi : U \to V$, the charts, from open subsets $U \subset X$ to open subsets $V \subset \mathbb{R}^n$ such that the subsets U cover X and all maps $\phi_{UU'} : (\phi'|_{U \cap U'}) \circ (\phi|_{U \cap U'})^{-1} : \phi(U \cap U') \to \phi'(U \cap U')$ are smooth. The fact that the coordinate change maps are smooth allows one to define an orientation of X as a collection of charts whose domains cover X and such that all derivatives of coordinate change maps have positive determinant: $\det d((\phi'|_{U \cap U'}) \circ (\phi|_{U \cap U'})^{-1}) > 0$ for all $x \in \phi(U \cap U')$.

For topological manifolds, smoothness of the coordinate change maps is not required. They are just homeomorphisms, so one cannot define orientations by applying linear algebra to their derivatives. Instead, orientations are defined by their homologies.

Definition 6.3.1:

1. An n-dimensional **topological manifold** is a second countable Hausdorff space X such that every $x \in X$ has an open neighbourhood U homeomorphic to an open subset $V \subset \mathbb{R}^n$.
2. A homeomorphism $\phi : U \subset X \to V \subset \mathbb{R}^n$ for an open subset $U \subset X$ is called a **chart**. A family $\{\phi_i : U_i \to V_i \mid i \in I\}$ of charts with $X = \cup_{i\in I} U_i$ is called an **atlas**.

Remark 6.3.2:

1. Without loss of generality one can impose $V = \mathbb{R}^n$ or $V = \mathring{D}^n$ in Definition 6.3.1.

   If $\phi : U \to V$ is a chart around $x \in U$, then $U \subset \mathbb{R}^n$ contains an open n-ball B around $\phi(x)$, which is homeomorphic to $\mathring{D}^n$. Restricting the chart ϕ to the open neighbourhood $U' = \phi^{-1}(B) \subset U$ then yields a homeomorphism $\phi' : U' \to B \cong \mathring{D}^n \cong \mathbb{R}^n$.
2. A topological manifold of dimension n inherits the *local* topological properties from $\mathbb{R}^n$. In particular, it is locally compact, locally connected and locally path-connected.

   This follows, because every neighbourhood W of X contains an open neighbourhood $U \subset W$ of x that is the domain of a chart $\phi : U \to V$. If $W \subset V$ is a neighbourhood of $\phi(x) \in V$ with the required property, then $\phi^{-1}(W) \subset U$ also has this property.
3. As locally compact Hausdorff spaces, topological manifold are regular topological spaces.

In the following we write n-manifold for topological manifold of dimension n. The requirement of a countable base of the topology (second countability) is included in the definition of a topological n-manifold for convenience, as it lets us avoid the use of Zorn's Lemma in some of the proofs. In many references, this condition is not imposed.

Many of the topological spaces encountered so far are topological manifolds. Obvious counterexamples are any non-Hausdorff topological spaces, disjoint unions of topological manifolds of different dimensions and infinite-dimensional CW complexes such as $\mathbb{R}\mathrm{P}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$.

Example 6.3.3:

1. The empty topological space $\emptyset$ is an n-dimensional topological manifold for every $n \in \mathbb{N}_0$.
2. Every open subset of $\mathbb{R}^n$ is an n-dimensional topological manifold by construction.
3. Any open subset $W \subset X$ of an n-dimensional topological manifold X is a topological manifold. The charts of W are obtained by restricting charts $\phi : U \to V$ of X to the open subsets $U \cap W \subset W$.
4. The disjoint union of n-dimensional topological manifolds is a topological n-manifold.
5. The product $X \times Y$ of an n-dimensional topological manifold X and an m-dimensional topological manifold Y is a topological manifold of dimension $n + m$.
6. The sphere S^n and real projective space $\mathbb{R}\mathrm{P}^n$ are n-dimensional topological manifolds. Complex projective space $\mathbb{C}\mathrm{P}^n$ is an $2n$-dimensional topological manifold.
7. Any oriented surface of genus $g \geq 0$ is a 2-dimensional topological manifold.
8. The Klein bottle $K = [0,1] \times [0,1]/\sim$ with $(x,1) \sim (x,0)$ and $(0,y) \sim (1,1-y)$ for $x, y \in [0,1]$ and the open Möbius strip $M = [0,1] \times (0,1)/\sim$ with $(0,y) \sim (1,1-y)$ for $y \in (-1,1)$ are 2-dimensional topological manifolds.

To define orientations for topological manifolds, we consider the relative homologies $H_k(X, X \setminus \{x\})$ for points $x \in X$. We can compute these relative homology groups by restricting X to the domain of a chart $\phi : U \to V$ around x and show that they are given by $\mathbb{Z}$ for $k = n$ and trivial otherwise. The abelian group $\mathbb{Z}$ has exactly two generators, namely 1 and -1, and these generators will replace the sign of the determinant in the definition of orientations of vector spaces and orientations of smooth manifolds.

Proposition 6.3.4: For a topological n-manifold X and any point $x \in X$ one has

$$H_k(X, X \setminus \{x\}) = \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n. \end{cases}$$

Proof:
Let $\phi : U \to V$ a chart around $x \in X$. Then we have

$$\overline{X \setminus U} = X \setminus U \subset X \setminus \{x\} = (X \setminus \{x\})^\circ \qquad \overline{\mathbb{R}^n \setminus V} = \mathbb{R}^n \setminus V \subset \mathbb{R}^n \setminus \{\phi(x)\} = (\mathbb{R}^n \setminus \{\phi(x)\})^\circ.$$

By Theorem 3.5.1 excising $X \setminus U$ and $\mathbb{R}^n \setminus V$ yields

$$H_k(X, X \setminus \{x\}) \cong H_k(U, U \setminus \{x\}) \cong H_k(V, V \setminus \{\phi(x)\}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{\phi(x)\}).$$

Applying Exercise 21 and Example 3.5.6 then yields the result. □

Remark 6.3.5: By Proposition 6.3.4 the dimension of a non-empty topological manifold is unique. If $\emptyset \neq X$ is a topological n-manifold, then X is not a topological m-manifold for $m \neq n$.

The idea is now to define an orientation as a coherent choice of generators of $H_n(X, X \setminus \{x\}) \cong \mathbb{Z}$ for each point $x \in X$. At each point there are two choices, namely one corresponding to $1 \in \mathbb{Z}$ and one corresponding to $-1 \in \mathbb{Z}$, and we must relate these choices at least for points in a neighbourhood of a given point $x \in X$. To induce maps between the relative homologies, this relation must be given by morphisms in Top(2). Given a topological space X and subspaces $B \subset A \subset X$, we have a morphism of pairs induced by the identity map on X

$$i_{AB} : (X, X \setminus A) \to (X, X \setminus B), \quad x \mapsto x. \tag{61}$$

If $B = \{x\} \subset A$ contains a single point, we write $i_{Ax} := i_{A\{x\}} : X \setminus A \to X \setminus \{x\}$. Note that for all subsets $C \subset B \subset A \subset X$ these morphisms of pairs satisfy the compatibility conditions

$$i_{AA} = 1_{(X, X\setminus A)} : (X, X \setminus A) \to (X, X \setminus A) \qquad i_{AC} = i_{BC} \circ i_{AB} : (X, X \setminus A) \to (X, X \setminus C).$$

Given a subset $U \subset X$ we can use the maps $i_{Ux} : X \setminus U \to X \setminus \{x\}$ to compare the different choice of generators at points $x \in U$.

Definition 6.3.6: Let X be a topological n-manifold and $W \subset X$ a subset.

1. A choice of generators $[z_w] \in H_n(X, X \setminus \{w\})$ for each $w \in W$ is called **coherent**, if for every $w \in W$ there is an open neighbourhood U in X and an element $[z_U] \in H_n(X, X \setminus U)$ such that $H_n(i_{Uy})[z_U] = [z_y]$ for all $y \in W \cap U$.
2. X is called **orientable**, if there is a coherent choice of generators $[z_x] \in H_n(X, X \setminus \{x\})$ for $W = X$. A coherent choice of generators for X is called an **orientation** of X.

Remark 6.3.7:

1. If $\{[z_x]\}_{x\in X}$ is an orientation of X, then $\{-[z_x]\}_{x\in X}$ is also an orientation of X, the **opposite orientation**.
2. If $U \subset X$ is such that $H_n(i_{Ux}) : H_n(X, X \setminus U) \to H_n(X, X \setminus \{x\})$ is an isomorphism for all $x \in X$, then the generators at points $x, y \in U$ are related by
$$[z_y] = H_n(i_{Uy}) \circ H_n(i_{Ux})^{-1}[z_x].$$
This can be seen as the topological counterpart of the formula for orientations of smooth manifolds on the intersection of the domains of two charts.

Without further assumptions on the neighbourhood U, we do not have explicit results on the homology group $H_n(X, X \setminus U)$ or the map $H_n(i_{Ux}) : H_n(X, X \setminus U) \to H_n(X, X \setminus \{x\})$. In particular, the latter is in general not an isomorphism. However, there is one situation, where this is well-controlled, namely *compact* and *convex* subsets $K \subset \mathbb{R}^n$. In this case Exercise 26 implies that the maps $H_n(i_{Kx})$ are isomorphisms.

Example 6.3.8: If $K \subset \mathbb{R}^n$ is compact and convex, then the map
$$H_m(i_{Kx}) : H_m(\mathbb{R}^n, \mathbb{R}^n \setminus K) \to H_m(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$$
is an isomorphism for all $x \in K$ and $m \in \mathbb{N}_0$. In particular, we have
$$H_n(\mathbb{R}^n, \mathbb{R}^n \setminus K) \cong \mathbb{Z} \qquad\qquad H_m(\mathbb{R}^n, \mathbb{R}^n \setminus K) = 0 \text{ for } m \neq n.$$

Proof:
The inclusion defines a homotopy equivalence $i_{Kx} : \mathbb{R}^n \setminus K \to \mathbb{R}^n \setminus \{x\}$. This follows, because the inclusion $i : \mathring{D}^n \setminus K \to \mathring{D}^n \setminus \{0\}$ is a homotopy equivalence for any compact subset $K \subset \mathring{D}^n$ with $0 \in K$ by Exercise 26. For any $x \in K$ there is a homeomorphism $\phi : \mathbb{R}^n \to \mathring{D}^n$ with $\phi(x) = 0$ that sends K to a compact convex subset $K' \subset \mathring{D}^n$. This can be constructed by first applying a translation that sends x to 0, then a rescaling that sends K to a convex subset of an n-ball $B_{1/2}(0)$ and then a homeomorphism $\psi : \mathbb{R}^n \to \mathring{D}^n$ that is the identity on $B_{1/2}(0) \subset \mathring{D}^n$. This implies that $H_m(i_{Kx}) : H_m(\mathbb{R}^n \setminus K) \to H_m(\mathbb{R}^n \setminus \{x\})$ is an isomorphism. Applying the 5-Lemma to the long exact homology sequence

$$\begin{tikzcd}
\ldots \arrow[r] & H_m(\mathbb{R}^n \setminus K) \arrow[r, "H_m(\iota)"] \arrow[d, "H_m(i_{Kx})", "\cong"'] & H_m(\mathbb{R}^n) \arrow[r, "H_m(\pi)"] \arrow[d, "\mathrm{id}"] & H_m(\mathbb{R}^n, \mathbb{R}^n \setminus K) \arrow[r, "\partial_m"] \arrow[d, "H_m(i_{Kx})"] & H_{m-1}(\mathbb{R}^n \setminus K) \arrow[r, "H_{m-1}(\iota)"] \arrow[d, "H_{m-1}(i_{Kx})", "\cong"'] & H_{m-1}(\mathbb{R}^n) \arrow[r] \arrow[d, "\mathrm{id}"] & \ldots \\
\ldots \arrow[r] & H_m(\mathbb{R}^n \setminus \{x\}) \arrow[r, "H_m(\iota)"] & H_m(\mathbb{R}^n) \arrow[r, "H_m(\pi)"] & H_m(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \arrow[r, "\partial_m"] & H_{m-1}(\mathbb{R}^n \setminus \{x\}) \arrow[r, "H_{m-1}(\iota)"] & H_{m-1}(\mathbb{R}^n) \arrow[r] & \ldots
\end{tikzcd}$$

shows that the middle arrow is an isomorphism as well. □

Given a coherent choice of generators $\{[z_x]\}_{x\in W}$ for some subset $W \subset X$, it is natural to ask if they can be combined into a single generator of $H_n(X, X \setminus W)$. In other words, is there an element $[z_W] \in H_n(X, X \setminus W)$ such that $[z_x] = H_n(i_{Wx})[z_W]$ for all $x \in W$? In particular, does orientability of a topological n-manifold X imply that there is a $[z_X] \in H_n(X)$ such that $[z_x] = H_n(i_{Xx})[z_X]$ for all $x \in X$?

As orientability is defined *locally*, via neighbourhoods of points, it is clear that the proof of such a statement will involve a cover of X by open neighbourhoods of points and will be particularly simple if this cover is guaranteed to have a finite subcover. Hence, it makes sense to assume that the subsets are compact.

Definition 6.3.9: Let X be a topological n-manifold and $K \subset X$ compact.

1. An element $[z_K] \in H_n(X, X \setminus K)$ is called an **orientation of** X **along** K if the elements $H_n(i_{Kx})[z_K] \in H_n(X, X \setminus \{x\})$ for $x \in K$ form a coherent set of generators for K.
2. If $K = X$, then an orientation along K is called a **orientation class** or **fundamental class** of X and denoted $[X]$.

The aim is now to prove that every compact orientable manifold has a fundamental class. The first step is to investigate orientations along compact subsets $K \subset X$. To establish uniqueness of orientations along K, we need to investigate the kernels of all of the group homomorphisms $H_n(i_{Kx}) : H_n(X, X \setminus K) \to H_n(X, X \setminus \{x\})$ for $x \in K$ and to show that $H_n(i_{Kx})[z] = 0$ for some $[z] \in H_n(X, X \setminus K)$ and all $x \in K$ implies $[z] = 0$. We will prove this first for small compact subsets that are contained in the domains of charts and then extend it to their union with the relative Mayer-Vietoris sequence. For this, we also need control over the homology groups $H_m(X, X \setminus K)$ for $m > n$.

Lemma 6.3.10: Let X be a connected orientable n-manifold and $K \subset X$ compact. Then:

1. $H_m(X, X \setminus K) = 0$ for all $m > n$.
2. If $[z] \in H_n(X, X \setminus K)$ satisfies $H_n(i_{Kx})[z] = 0$ for all $x \in K$, then $[z] = 0$.

In particular, if X is compact, then $H_m(X) = 0$ for $m > n$.

Proof:
1. We first reduce the proof of 1. and 2. to the case of compact subsets $K \subset X$ with $K \subset U$ for some chart $\phi : U \to V$.

1.(a) As X is a locally compact topological space, every point $x \in K$ has a compact neighbourhood $K_x \subset U_x$ contained in the domain of a chart $\phi_x : U_x \to V_x \subset \mathbb{R}^n$. Any choice of open neighbourhoods $W_x \subset K_x \subset U_x$ for each $x \in K$ yields an open cover $K = \cup_{x \in K} W_x$, which has a finite subcover by compactness of K. Hence, there are compact subsets $K_{x_1}, \ldots, K_{x_r} \subset X$, each contained in the domain of a chart, such that $K \subset K_{x_1} \cup \ldots \cup K_{x_r}$. This implies $K = K_1 \cup \ldots \cup K_r$ with $K_i = K \cap K_{x_i}$ compact and contained in the domain of a chart.

1.(b) We show that if 1. and 2. hold for compact subsets $A, B \subset X$ and their intersection $A \cap B$, then they also hold for $A \cup B$. This follows with the relative Mayer-Vietoris sequence from Proposition 3.5.5, applied to $X \setminus A$ and $X \setminus B$, which are open in $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$. If 1. holds for A, B and $A \cap B$, one has

$$\ldots \to \underbrace{H_{m+1}(X, X \setminus (A \cap B))}_{=0\ m \geq n} \xrightarrow{\partial_{m+1}} H_m(X, X \setminus (A \cup B)) \xrightarrow{\phi_m} \underbrace{H_m(X, X \setminus A)}_{=0\ m>n} \oplus \underbrace{H_m(X, X \setminus B)}_{=0\ m>n} \to \ldots$$

with $\phi_n = (H_n(i_A), -H_n(i_B))$ for the morphisms $i_A : (X, X \setminus (A \cup B)) \to (X, X \setminus A)$ and $i_B : (X, X \setminus (A \cup B)) \to (X, X \setminus B)$ by Proposition 3.5.5.

This implies $H_m(X, X \setminus (A \cup B)) = 0$ for $m > n$ and injectivity of ϕ_n for $m = n$. As we have $H_n(i_{(A \cup B)x}) = H_n(i_{Ax}) \circ H_n(i_A)$ for all $x \in A$ and $H_n(i_{Bx}) \circ H_n(i_B)$ for all $x \in B$, the identity $H_n(i_{(A \cup B)x})[z] = 0$ for all $x \in A \cup B$ implies

$$H_n(i_{Ax}) \circ H_n(i_A)[z] = 0\, \forall x \in A, \qquad H_n(i_{Bx}) \circ H_n(i_B)[z] = 0\, \forall x \in B.$$

As 2. holds for A and B, this gives $H_n(i_A)[z] = H_n(i_B)[z] = 0$ and $[z] = 0$ by injectivity of ϕ_n.

2. If $K \subset U$ for some chart $\phi : U \to V$, we have isomorphisms of pairs

$$\phi : (U, U \setminus K) \to (V, V \setminus \phi(K)) \qquad \phi : (U, U \setminus \{x\}) \to (V, V \setminus \{\phi(x)\}) \text{ for } x \in K.$$

By excising $X \setminus U$ and $\mathbb{R}^n \setminus V$ we then obtain for all $m \in \mathbb{N}_0$ and $x \in K$

$$H_m(X, X \setminus K) \cong H_m(U, U \setminus K) \xrightarrow[\cong]{H_m(\phi)} H_m(V, V \setminus \phi(K)) \cong H_m(\mathbb{R}^n, \mathbb{R}^n \setminus \phi(K))$$
$$H_m(X, X \setminus \{x\}) \cong H_m(U, U \setminus \{x\}) \xrightarrow[\cong]{H_m(\phi)} H_m(V, V \setminus \{\phi(x)\}) \cong H_m(\mathbb{R}^n, \mathbb{R}^n \setminus \{\phi(x)\})$$

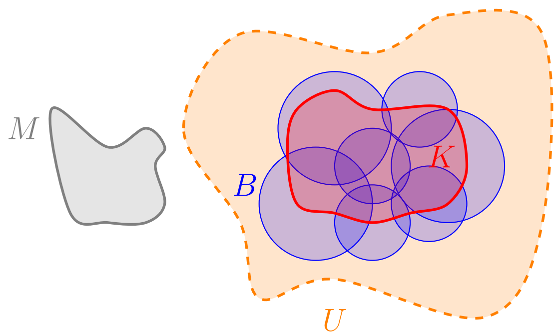
and a commuting diagram

$$\begin{tikzcd} H_m(X, X \setminus K) \arrow[r, "\cong"] \arrow[d, "H_m(i_{Kx})"'] & H_m(\mathbb{R}^n, \mathbb{R}^n \setminus \phi(K)) \arrow[d, "H_m(i_{\phi(K)\phi(x)})"] \\ H_m(X, X \setminus \{x\}) \arrow[r, "\cong"'] & H_m(\mathbb{R}^n, \mathbb{R}^n \setminus \{\phi(x)\}). \end{tikzcd}$$

This shows that it is sufficient to prove claims 1. and 2. for compact subsets $K \subset \mathbb{R}^n$.

3. We prove 1. and 2. for compact subsets $K \subset \mathbb{R}^n$.

Let $z \in Z_m(\mathbb{R}^n, \mathbb{R}^n \setminus K)$ be a relative m-cycle. Then we have $d_m(z) = \Sigma_{j=1}^r \lambda_j \tau_j$ for singular $(m-1)$-simplexes $\tau_1, \ldots, \tau_r : \Delta^{m-1} \to \mathbb{R}^n \setminus K$. By compactness of Δ^{m-1} and continuity of the simplexes, the union of their images $M := \cup_{j=1}^r \tau_j(\Delta^{m-1}) \subset \mathbb{R}^n \setminus K$ is compact. Consequently, there is an open neighbourhood U of K such that $M \subset \mathbb{R}^n \setminus U$. As K is compact, there are closed balls $B_1, \ldots, B_s$ in $\mathbb{R}^n$ with $K \subset B := \cup_{j=1}^s B_j \subset U$.



As $M \subset \mathbb{R}^n \setminus U$, the relative m-cycle z also defines relative m-cycles $z' \in Z_m(\mathbb{R}^n, \mathbb{R}^n \setminus U)$ and $z'' \in Z_m(\mathbb{R}^n, \mathbb{R}^n \setminus B)$ with $H_m(i_{UB})[z'] = [z'']$ and $H_m(i_{BK})[z''] = [z]$. As the balls $B_i \subset \mathbb{R}^n$ are compact and convex, claims 1. and 2. hold for B_j by Example 6.3.8 and follow for B with 1.(b).

Claim 1. for B implies $[z''] = H_n(i_{UB})[z'] \in H_m(\mathbb{R}^n, \mathbb{R}^n \setminus B) = \{0\}$ for $m > n$ and hence $[z] = H_n(i_{BK})[z''] = 0$. This shows claim 1. for K.

For claim 2. let $m = n$ and $H_n(i_{Kx})[z] = 0$ for all $x \in K$. Then we have for all $x \in K \cap B_j$

$$0 = H_n(i_{Kx})[z] = H_n(i_{Kx}) \circ H_n(i_{BK})[z''] = H_n(i_{Bx})[z''] = H_n(i_{B_j x}) \circ H_n(i_{BB_j})[z''].$$

As $H_n(i_{B_j x})$ is an isomorphism by Example 6.3.8, this implies $H_n(i_{BB_j})[z''] = 0$. It follows that $H_n(i_{By})[z''] = H_n(i_{B_j y}) \circ H_n(i_{BB_j})[z''] = 0$ for all $y \in B_j$. As $K \subset \cup_{j=1}^s B_j$, we can apply this argument for all $j = 1, \ldots, s$ and obtain $H_n(i_{By})[z''] = 0$ for all $y \in B = \cup_{j=1}^s B_j$. As claim 2. holds for B, it this yields $[z''] = 0$ and $[z] = H_n(i_{BK})[z''] = 0$, which proves claim 2. for K. □

We can now apply Lemma 6.3.10 to show that for any orientation $\{[z_x]\}_{x\in X}$ on X and any compact subset $K \subset X$, there is at most one orientation $[z_K]$ along K with $H_n(i_{Kx})[z_K] = [z_x]$ for all $x \in K$. To prove that there is at least one such orientation along K is more difficult. For this one works with open neighbourhoods U of points $x \in X$ as in the definition of an orientation and chooses compact subsets contained in such neighbourhoods. One then extends the result to a compact subset K by covering K with neighbourhoods contained in these small compacta and applying the relative Mayer-Vietoris sequence.

Proposition 6.3.11: Let X be an n-manifold and $\mathcal{O} = \{[z_x]\}_{x\in X}$ an orientation on X. Then for any compact subset $K \subset X$ there is a unique orientation $[z_K] \in H_n(X, X \setminus K)$ along K compatible with $\mathcal{O}$: $H_n(i_{Kx})[z_K] = [z_x]$ for all $x \in K$.

Proof:
1. To show uniqueness, let $[z_K], [z'_K] \in H_n(X, X \setminus K)$ two orientations of X along K that are compatible with $\mathcal{O}$. Then we have for all $x \in K$

$$H_n(i_{Kx})([z_K] - [z'_K]) = H_n(i_{Kx})[z_K] - H_n(i_{Kx})[z'_K]) = [z_x] - [z_x] = 0,$$

and this implies $[z_K] = [z'_K] = 0$ by Lemma 6.3.10. This shows uniqueness.

2. Let $K_1, K_2 \subset X$ compact subsets and $[z_{K_1}] \in H_n(X, X \setminus K_1)$, $[z_{K_2}] \in H_n(X, X \setminus K_2)$ orientations along K_1 and K_2 compatible with $\mathcal{O}$. We show that there is an orientation compatible with $\mathcal{O}$ on $K_1 \cup K_2$.

The identity $i_{K_jy} = i_{K_jy} \circ i_{K_j(K_1\cap K_2)}$ for $j = 1, 2$ implies for all $y \in K_1 \cap K_2$

$$[z_y] = H_n(i_{K_jy})[z_{K_j}] = H_n(i_{(K_1\cap K_2)y}) \circ H_n(i_{K_j(K_1\cap K_2)})[z_{K_j}].$$

With 1. this implies $H_n(i_{K_1(K_1\cap K_2)})[z_{K_1}] = H_n(i_{K_2(K_1\cap K_2)})[z_{K_2}] = [z_{K_1\cap K_2}]$ is an orientation compatible with $\mathcal{O}$ along $K_1 \cap K_2$. The relative Mayer-Vietoris sequence from Proposition 3.5.5 yields an exact sequence, where the first term on the left vanishes by Lemma 6.3.10

$$0 \xrightarrow{\partial_{n+1}} H_n(X, X \setminus (K_1 \cup K_2)) \xrightarrow{\phi_n} H_n(X, X \setminus K_1) \oplus H_n(X, X \setminus K_2) \xrightarrow{\psi_n} H_n(X, X \setminus (K_1 \cap K_2)) \to \ldots$$
$$\phi_n = (H_n(i_{(K_1\cup K_2)K_1}), -H_n(i_{(K_1\cup K_2)K_2})) \qquad \psi_n = H_n(i_{K_1(K_1\cap K_2)}) + H_n(i_{K_2(K_1\cap K_2)}).$$

Then we have $([z_{K_1}], -[z_{K_2}]) \in \ker \psi_n = \mathrm{im}\, \phi_n$, as

$$\psi_n([z_{K_1}], -[z_{K_2}]) = H_n(i_{K_1(K_1\cap K_2)})[z_{K_1}] - H_n(i_{(K_2(K_1\cap K_2))})[z_{K_2}] = [z_{K_1\cap K_2}] - [z_{K_1\cap K_2}] = 0.$$

Thus, there is an element $[z_{K_1\cup K_2}] \in H_n(X, X \setminus (K_1 \cup K_2))$ with

$$H_n(i_{(K_1\cup K_2)K_1})[z_{K_1\cup K_2}] = [z_{K_1}] \qquad H_n(i_{(K_1\cup K_2)K_1})[z_{K_1\cup K_2}] = [z_{K_2}].$$

This implies for all $y \in K_j$

$$H_n(i_{(K_1\cup K_2)y})[z_{K_1\cup K_2}] = H_n(i_{K_jy}) \circ H_n(i_{(K_1\cup K_2)y})[z_{K_1\cup K_2}] = H_n(i_{K_jy})[z_{K_j}] = [z_y]$$

and hence $H_n(i_{(K_1\cup K_2)y})[z_{K_1\cup K_2}] = [z_y]$ for all $y \in K_1 \cup K_2$, which shows that $[z_{K_1\cup K_2}]$ is an orientation compatible with $\mathcal{O}$ along $K_1 \cup K_2$.

3. To show existence, choose for each $x \in K$ a neighbourhood U_x and a $[z_{U_x}] \in H_n(X, X \setminus U_x)$ with $H_n(i_{U_x y})[z_{U_x}] = [z_y]$ for all $y \in U_x$, which exist by definition of an orientation. As X is locally compact, there are compact neighbourhoods K_x and open neighbourhoods W_x of x such that $W_x \subset K_x \subset U_x$. The open cover $K \subset \cup_{x\in K} W_x$ has a finite subcover $K \subset \cup_{i=1}^r W_{x_i}$, and this yields $K = \cup_{i=1}^r K_i$, where $K_i = K_{x_i} \cap K \subset U_{x_i}$ is compact. By defining

$$[z_{K_i}] = H_n(i_{U_{x_i} K_i})[z_{U_{x_i}}],$$

we obtain an orientation compatible with $\mathcal{O}$ along K_i, as we have for all $y \in K_i \subset U_{x_i}$

$$H_n(i_{K_i y})[z_{K_i}] = H_n(i_{K_i y} \circ i_{U_{x_i} K_i})[z_{K_i}] = H_n(i_{U_{x_i} y})[z_{U_{x_i}}] = [z_y].$$

Applying step 2. we obtain an orientation $[z_K]$ along K that is compatible with $\mathcal{O}$. □

By combining these results, we can relate the existence of an orientation on a *compact connected* topological n-manifold X to the existence of an orientation class and to the nth homology group of the manifold. The latter gives a simple criterion for the orientability of the manifold.

Theorem 6.3.12: Let X be a connected compact n-manifold.
Then the following are equivalent:

(i) The manifold X is orientable.
(ii) There is an orientation class of X.
(iii) $H_n(X) \cong \mathbb{Z}$.

Proof:
(i)⇒(ii): follows from Proposition 6.3.11.

(iii)⇒(i): If $H_n(X) \cong \mathbb{Z}$, then there are two possible choices of generators of $H_n(X)$. For any generator $[z] \in H_n(X)$, the set $\{H_n(i_{Xx})[z] \mid x \in X\}$ is a coherent set of generators for X and thus an orientation of X.

(ii)⇒(iii): Suppose $[z_X] \in H_n(X)$ is an orientation class. Then the elements $[z_x] = H_n(i_{Xx})[z_X]$ for $x \in X$ form a coherent set of generators for X. As $[z_x]$ generates $H_n(X, X \setminus \{x\}) \cong \mathbb{Z}$, the element $[z_X]$ cannot be of finite order. We show that it generates $H_n(X)$.

Let $[z] \in H_n(X)$. As $[z_x]$ generates $H_n(X, X \setminus \{x\}) \cong \mathbb{Z}$, for any there is a unique $k_x \in \mathbb{Z}$ with $H_n(i_{Xx})[z] = k_x[z_x]$. This is equivalent to $[z] - k_x[z_X] \in \ker H_n(i_{Xx})$ for all $x \in X$.

For every $x \in X$ we can choose a chart $\phi : U \to \mathring{D}^n \subset \mathbb{R}^n$ and a compact neighbourhood K of x such that $\phi(K)$ is a compact convex neighbourhood of $\phi(x)$. Example 6.3.8 yields isomorphisms

$$H_n(i_{Kx}) : H_n(X, X \setminus K) \xrightarrow{\cong} H_n(X, X \setminus \{x\})$$

for all $x \in K$. This implies $\ker H_n(i_{Xx}) = \ker H_n(i_{Ky}) \circ H_n(i_{XK}) = \ker H_n(i_{XK})$ for $x \in K$ and shows that $k_x = k_y$ for all $x, y \in K$. This shows that the map $f : X \to \mathbb{Z}$, $x \mapsto k_x$ is locally constant and hence constant, as X is connected. Thus, for every $[z] \in H_n(X)$ there is a unique $k \in \mathbb{Z}$ with $H_n(i_{Xx})([z] - k[z_X]) = 0$ for all $x \in X$. This implies $[z] = k[z_X]$ by compactness of X with Lemma 6.3.10, 2. and shows that $[z_X]$ generates $H_n(X)$. As $[z_X]$ is of infinite order, it follows that $H_n(X) \cong \mathbb{Z}$. □

Example 6.3.13:

1. The n-sphere is orientable, as $H_n(S^n) \cong \mathbb{Z}$. The same holds for any orientable surface of genus $g > 0$.
2. The Klein bottle K is a not orientable, because $H_2(K) = 0$ (Exercise 20).
3. Real projective space $\mathbb{RP}^n$ has the homologies $H_n(\mathbb{RP}^n) = \mathbb{Z}$ for n odd and $H_n(\mathbb{RP}^n) = 0$ for n even by Example 4.2.9. Hence, $\mathbb{RP}^n$ is orientable if and only if n is odd.
4. By Example 4.2.6 complex projective space $\mathbb{CP}^n$ has the homology group $H_{2n}(\mathbb{CP}^n) = \mathbb{Z}$. Hence, $\mathbb{CP}^n$ is an orientable $2n$-dimensional topological manifold for all $n \in \mathbb{N}$.

Note that the fundamental class in Theorem 6.3.12 must be a generator of $H_n(X) \cong \mathbb{Z}$, which is unique up to a minus sign. Each connected compact orientable topological manifold has exactly two orientation classes. Specifying an orientation amounts to selecting an orientation class, and the opposite orientation corresponds to its inverse. Note also that all non-orientable compact topological manifolds X in Example 6.3.13 have $H_n(X) = 0$. This is not a coincidence. One can show that any non-orientable compact n-manifold has a trivial nth homology group.

A nice application of Theorem 6.3.12 is that it allows us to extend the concept of the mapping degree from continuous maps $f : S^n \to S^n$ to continuous maps $f : X \to Y$ between compact connected oriented n-manifolds X and Y. As $H_n(X) \cong H_n(Y) = \mathbb{Z}$ by Theorem 6.3.12 and the orientation classes $[X]$ and $[Y]$ generate the homology groups, there is a number $\deg(f) \in \mathbb{Z}$ with $H_n(f)[X] = \deg(f)[Y]$.

Definition 6.3.14: Let X, Y be compact connected oriented manifolds of dimension n and $[X]$ and $[Y]$ the fundamental classes compatible with the chosen orientations.

The **mapping degree** of a continuous map $f : X \to Y$ is the unique integer $\deg(f) \in \mathbb{Z}$ with

$$H_n(f)[X] = \deg(f)[Y].$$

Note that the mapping degree of a continuous map $f : X \to Y$ depends on the choice of an orientation class or, equivalently, of an orientation of X and Y. For $X = Y$, it is independent of the orientation choice, because $H_n(f)[X] = \deg(f)[X]$ implies $H_n(f)(-[X]) = -\deg(f)[X]$. It is also clear from the definition that for $X = Y = S^n$ the mapping degree reduces to the one from Definition 3.6.5. The properties of the mapping degree of maps between spheres in Proposition 6.3.15 also generalise to the mapping degree of maps between compact connected oriented manifolds.

Proposition 6.3.15: Let X, Y, Z be oriented connected topological manifolds of dimension n and denote by $\overline{X}$ the manifold X with the opposite orientation. Then

1. The mapping degree is multiplicative:
   $\deg(g \circ f) = \deg(g) \cdot \deg(f)$ for all continuous maps $f : X \to Y$ and $g : Y \to Z$,
2. $\deg(\mathrm{id}_X : X \to X) = 1$ and $\deg(\mathrm{id}_X : X \to \overline{X}) = -1$,
3. If $\deg(f) \neq 0$ then f is surjective.

Proof:
1. and 2. follow from the definition. If $[X]$, $[Y]$, $[Z]$ are orientation classes of X, Y, Z, then

$$\deg(g\circ f)[Z] = H_n(g\circ f)[X] = H_n(g)\circ H_n(f)[X] = \deg(f)H_n(g)[Y] = \deg(f)\cdot\deg(g)[Z],$$

which implies $\deg(g\circ f) = \deg(g)\cdot\deg(f)$. As we have $H_n(\mathrm{id}_X)[X] = [X] = -(-[X])$, it follows that $\deg(\mathrm{id}_X : X \to X) = 1$ and $\deg(\mathrm{id}_X : X \to \overline{X}) = -1$.

3. Suppose that $f : X \to Y$ is not surjective and let $y \in Y \setminus \mathrm{im}\,(f)$. Then we have

$$0 = H_n(i_{Yy})\circ H_n(f) : H_n(X) \to H_n(Y, Y\setminus\{y\}), \quad [\sigma] \mapsto [f\circ\sigma]$$

because $f\circ\sigma(\Delta^n) \subset Y\setminus\{y\}$ for any singular n-simplex $\sigma : \Delta^n \to Y$. As Y is compact connected and oriented, the group homomorphism $H_n(i_{Yy}) : H_n(Y) \to H_n(Y, Y\setminus\{y\})$ is an isomorphism for any $y \in Y$ by the proof of Theorem 6.3.12. This implies $H_n(f) = 0$ and $\deg(f) = 0$. $\square$

Remark 6.3.16: Analogously, one defines orientations for a commutative unital ring R.

1. A topological n-manifold is called R**-orientable** is there is a coherent choice of generators of the homology group $H_n(X, X\setminus\{x\}; R) \cong R$ for all $x \in X$.

   That $H_n(X, X\setminus\{x\}; R) \cong R$ for all $x \in X$ follows with the universal coefficient theorem in Corollary 5.4.7 from Proposition 6.3.4.

2. Lemma 6.3.10 also holds for homologies with coefficients in a commutative unital ring R.

   The reduction to the claims for $(\mathbb{R}^n, \mathbb{R}^n\setminus B)$ for compact balls $B \subset \mathbb{R}^n$ works analogously. At the end of the proof, it is sufficient to note that Example 6.3.8 generalises to coefficients in R with the universal coefficient theorem. In particular, $H_m(\mathbb{R}^n, \mathbb{R}^n\setminus B; R) = 0$ for $m > n$.

3. Orientations along a compact subset $K \subset X$ with coefficients in R are defined analogously to Definition 6.3.9 as elements $[z_K] \in H_n(X, X\setminus K; R)$.

4. Proposition 6.3.11 and Theorem 6.3.12 hold analogously for R-orientations $[z_K] \in H_n(X, X\setminus K; R)$ and for $H_n(X) \cong R$.

Remark 6.3.17: More generally, one can show [H, Th. 3.26] that for a non R-orientable topological n-manifold X the maps $H_n(i_{Xx}) : H_n(X; R) \to H_n(X, X\setminus\{x\}; R)$ are always injective with image $\mathrm{Tor}_2(R) = \{r \in R \mid 2r = 0\}$. This implies $H_n(i_{xX}) = 0$ for $R = \mathbb{Z}$.

6.4 Poincaré duality

In this section, we prove that any R-orientation of a compact orientable topological n-manifold X with associated fundamental class $[X] \in H_n(X; R) \cong R$ for induces group isomorphisms

$$\mathrm{PD}_k : -\cap[X] : H^k(X; R) \xrightarrow{\cong} H_{n-k}(X; R),$$

the Poincaré duality maps. This is fairly straightforward to prove for compact manifolds X. However, we also want to formulate an analogous result for non-compact orientable topological

n-manifolds. In this case, we need to replace the cohomology groups with the cohomology groups of compact support. They are constructed by considering only cocycles and coboundaries that are trivial on all simplexes whose image does not intersect a certain compactum.

Definition 6.4.1: Let X be a topological space and R a commutative unital ring.

1. The subcomplex $S^\bullet_c(X;R) \subset S^\bullet(X;R)$ of **singular cochains with compact support** is given by the subgroups

   $$\begin{aligned} S^n_c(X;R) =&\{\phi \in \mathrm{Hom}_{\mathrm{Ab}}(S_n(X), R) \mid \\ &\exists K_\phi \subset X \text{ compact with } \phi(\sigma) = 0 \text{ for all } \sigma : \Delta^n \to X \text{ with } \sigma(\Delta^n) \cap K_\phi = \emptyset\}. \end{aligned}$$

2. The nth **cohomology group with compact support** is

   $$H^n_x(X;R) = H^n S^\bullet_c(X;R).$$

That $S^\bullet_c(X;R) \subset S^\bullet(X;R)$ is a subcomplex can be seen as follows. Let $\phi : S_n(X) \to R$ be a group homomorphism with $\phi(\sigma) = 0$ for all singular simplexes $\sigma : \Delta^n \to X$ with $\sigma(\Delta^n) \cap K_\phi = \emptyset$. For all singular $(n+1)$-simplexes $\tau : \Delta^{n+1} \to X$ with $\tau \cap K_\phi = \emptyset$, every n-simplex in $d_{n+1}(\tau)$ is of the form $\tau \circ f^n_j : \Delta^n \to X$ and satisfies $\tau \circ f^n_j(\Delta^n) \cap K_\phi \subset \tau(\Delta^{n+1}) \cap K_\phi = \emptyset$. This implies $d^n(\phi)(\tau) = \phi(d_{n+1}(\tau)) = 0$ for all singular $(n+1)$-simplexes $\tau : \Delta^{n+1} \to X$ with $\tau(\Delta^{n+1}) \cap K_\phi = \emptyset$. Hence, we can set $K_{d^n(\phi)} = K_\phi$, and $d^n(\phi)$ has compact support as well.

We will now relate the singular cochain complex $S^\bullet_c(X;R)$ with compact support to the relative cochain complexes $S^\bullet(X, X \setminus K; R)$ for compact subsets $K \subset X$. By construction, a relative cochain $\phi \in S^n(X, X\setminus K; R)$ is a group homomorphism $\phi : S_n(X, X \setminus K) \to R$. By the universal property of the factor group $S_n(X, X \setminus K) = S_n(X)/S_n(X \setminus K)$, such group homomorphisms are in bijection with group homomorphisms $\phi : S_n(X) \to R$ such that $S_n(X \setminus K) \subset \ker \phi$ or, equivalently, with $\phi(\sigma) = 0$ for all singular simplexes $\sigma : \Delta^n \to X$ with $\sigma(\Delta^n) \cap K = \emptyset$. Thus, every relative cochain $\phi \in S^n(X, X \setminus K; R)$ is a cochain with compact support.

More abstractly, we can express this in terms of the inclusion morphisms in Top(2) associated to subsets of X from (61). For all subsets $B \subset A \subset X$ of a topological space X, the identity map induces a morphism $i_{AB} : (X, X \setminus A) \to (X, X \setminus B)$, $x \mapsto x$ in Top(2) with

$$i_{AA} = 1_{(X, X\setminus A)} \qquad i_{AC} = i_{BC} \circ i_{AB} \quad \text{for} \quad C \subset B \subset A \subset X. \tag{62}$$

These morphisms of pairs induce cochain maps between the associated relative cochain complexes and group homomorphisms between their cohomologies. This allows us to rephrase our observation from the last paragraph as the following lemma.

Lemma 6.4.2: Let X be a topological space and R a commutative unital ring.

1. For any compact subset $K \subset X$ the morphism of pairs $i_{XK} : (X, \emptyset) \to (X, X \setminus K)$ induces a cochain map $S^n(i_{XK}) : S^n(X, X \setminus K; R) \to S^n_c(X;R)$ and group homomorphisms

   $$H^n(i_{XK}) : H^n(X, X \setminus K; R) \to H^n_c(X;R)$$

2. For compact subsets $K \subset L \subset X$ there is a commuting diagram of cochain complexes

$$\begin{tikzcd} S^\bullet(X, X\setminus K; R) \arrow[rr, "S^\bullet(i_{LK})"] \arrow[dr, "S^\bullet(i_{KX})"'] & & S^\bullet(X, X \setminus L; R) \arrow[dl, "S^\bullet(i_{LX})"] \\ & S^\bullet_c(X;R) & \end{tikzcd}$$

and for each $n \in \mathbb{N}_0$ a commuting diagram of cohomology groups

$$\begin{tikzcd} H^n(X, X\setminus K; R) \arrow[rr, "H^n(i_{LK})"] \arrow[dr, "H^n(i_{KX})"'] & & H^n(X, X\setminus L; R) \arrow[dl, "H^n(i_{LX})"] \\ & H^n_c(X; R) & \end{tikzcd}$$

Proof:
The first statement follows, because the morphisms $i_{XK} : (X, \emptyset) \to (X, X \setminus K)$ induce cochain maps $S^\bullet(i_{XK}) : S^\bullet(X, X \setminus K; R) \to S^\bullet(X; R)$ and every relative cochain $\phi \in S^n(X, X \setminus K; R)$ is a cochain with compact support in $S^n(X; R)$. The second statement follows from (62). □

We will now build up the singular cochain complex $S^\bullet_c(X; R)$ with compact support from the relative cochain complexes $S^\bullet(X, X \setminus K; R)$ for compact subsets $K \subset X$ and relate their cohomologies. Naively, one could attempt to combine the cochain complexes $S^\bullet(X, X \setminus K; R)$ for different compact subsets $K \subset X$ by taking their direct sum. However, this does not yield correct result, as we need to identify relative cochains in $S^n(X, X \setminus L; R)$ with relative cochains in $S^n(X, X \setminus K; R)$ for compact subsets $L \subset K \subset X$ and all $n \in \mathbb{Z}$. Hence, we need to take a quotient module of the direct sum of all R-modules $S^n(X, X \setminus K; R)$, where cochains are identified by the inclusions $S^n(i_{LK}) : S^n(X, X \setminus L; R) \to S^n(X, X \setminus K; R)$. This is a special case of a *categorical colimit*, associated to a *poset category*.

Definition 6.4.3:
A **partially ordered set** or **poset** is a set I together with a relation $\preceq$ that is

(i) **reflexive:** $i \preceq i$ for all $i \in I$,
(ii) **antisymmetric:** $i \preceq j$ and $j \preceq i$ implies $i = j$,
(iii) **transitive:** $i \preceq j$ and $j \preceq k$ implies $i \preceq k$.

A poset $(M, \preceq)$ is called **direct**, if for all $i, j \in I$ there is a $k \in I$ with $i, j \preceq k$.

Lemma 6.4.4: Every poset $(I, \preceq)$ defines a category $\mathcal{I}$, the **poset category** for I, whose

- objects are elements $i \in I$,
- morphism sets are given by $|\mathrm{Hom}_{\mathcal{I}}(i, j)| = 1$ if $i \preceq j$ and $\mathrm{Hom}_{\mathcal{I}}(i, j) = \emptyset$ else.

Proof:
The existence of identity morphisms is guaranteed by the reflexivity of $\preceq$ and the associativity of the composition of morphisms by the transitivity of $\preceq$. □

Example 6.4.5: Let X be a topological space and $\mathcal{K} \subset \mathcal{P}(X)$ the set of its compact subsets $K \subset X$. Then $\mathcal{K}$ with the relation $\preceq = \subset$ is a poset. It is direct, because for all compact subsets $K, K' \subset X$ the union $K \cup K'$ is a compact subset of K with $K, K' \subset K \cup K'$.

When considering the relative cochain complexes with respect to complements of compact subsets, we assign to each compact subset $K \subset X$ a relative cochain complex $S^\bullet(X, X \setminus K; R)$ and to every inclusion $L \subset K \subset X$ a chain map $S^\bullet(i_{LK}) : S^\bullet(X, X \setminus L; R) \to S^\bullet(X, X \setminus K; R)$. It is plausible that these assignments should define a functor from the poset category for $\mathcal{K}$ of X into the category of cochain complexes. Such functors from directed poset categories are known as *direct systems*. The notion that we need to combine the relative cochain complexes of compacta $K \subset X$ into the cochain complex with compact support is a categorical colimit, in this case known as a *direct limit*.

Definition 6.4.6: Let $(I, \preceq)$ a direct poset and $\mathcal{C}$ a category.

1. A **direct system** for $(I, \preceq)$ in $\mathcal{C}$ is a functor $F : \mathcal{I} \to \mathcal{C}$:
   - a family $(F_i)_{i \in I}$ of objects in $\mathcal{C}$,
   - a family $(f_{ij})_{i \preceq j}$ of morphisms $f_{ij} : F_i \to F_j$ for all $i, j \in I$ with $i \preceq j$,

   such that for all $i \preceq j \preceq k$
   $$f_{ii} = 1_{F_i} \qquad f_{ik} = f_{jk} \circ f_{ik} : F_i \to F_k.$$
2. A **map of direct systems** from $F : \mathcal{I} \to \mathcal{C}$ to $F' : \mathcal{I} \to \mathcal{C}$ is a natural transformation $\gamma : F \Rightarrow F'$, a family $(\gamma_i)_{i \in I}$ of morphisms $\gamma_i : F_i \to F'_i$ with $f'_{ij} \circ \gamma_i = \gamma_j \circ f_{ij}$ for all $i \preceq j$.
3. A **direct limit** of a direct system $F : \mathcal{I} \to \mathcal{C}$ is a colimit of F:
   - an object $\lim_\to F$ of $\mathcal{C}$,
   - a family $(f_i)_{i \in I}$ of morphisms $f_i : F_i \to \lim_\to F_i$ with $f_j \circ f_{ij} = f_i$ for all $i \preceq j$,

   that have the following **universal property**:
   for every object C and family $(f'_i)_{i \in I}$ of morphisms $f'_i : F_i \to C$ with $f'_j \circ f_{ij} = f'_i$ for all $i \preceq j$ there is a unique morphism $f : \lim_\to F \to C$ with $f'_i = f \circ f_i$ for all $i \in I$
   $$\begin{tikzcd}
   F_i \arrow[rr, "f_{ij}"] \arrow[dr, "f_i"] \arrow[ddr, bend right, "f'_i"'] & & F_j \arrow[dl, "f_j"'] \arrow[ddl, bend left, "f'_j"] \\
   & \lim_\to F \arrow[d, dashed, "\exists! f"] & \\
   & C. &
   \end{tikzcd}$$

Remark 6.4.7:

1. The name *direct limit* for a categorical colimit predates the notion of a colimit.
2. As it is defined by a universal property, the direct limit of a direct system is unique up to unique isomorphism:

   If $(f_i : F_i \to C)_{i \in I}$ and $(f'_i : F_i \to C')_{i \in I}$ are two families of morphisms that define direct limits of $F : \mathcal{I} \to \mathcal{C}$, then there are unique morphisms $f : C \to C'$ with $f'_i = f \circ f_i$ for all $i \in I$ and $f' : C' \to C$ with $f_i = f' \circ f'_i$ for all $i \in I$, and $f' = f^{-1}$.
3. If $F, F' : \mathcal{I} \to \mathcal{C}$ have direct limits, any morphism $\gamma : F \Rightarrow F'$ of direct systems induces a unique morphism $\lim_\to \gamma : \lim_\to F_i \to \lim_\to F'_i$ with $\gamma \circ f_i = f'_i \circ \gamma_i$ for $i \in I$. This follows with the universal property of the direct limit
   $$\begin{tikzcd}
   F'_i \arrow[rrrr, "f'_{ij}"] \arrow[dddrr, bend right, "f'_i"'] & & & & F'_j \arrow[dddll, bend left, "f'_j"] \\
   & F_i \arrow[ul, "\gamma_i"'] \arrow[rr, "f_{ij}"] \arrow[dr, "f_i"'] & & F_j \arrow[ur, "\gamma_j"] \arrow[dl, "f_j"] & \\
   & & \lim_\to F \arrow[d, dashed, "\exists! \lim_\to \gamma"] & & \\
   & & \lim_\to F'. & &
   \end{tikzcd}$$

   If all direct limits of functors $F : \mathcal{I} \to \mathcal{C}$ exist, this defines a functor $\lim_\to : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ from the functor category $\mathcal{C}^{\mathcal{I}} = \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ into the category $\mathcal{C}$ (Exercise 74).

4. Concretely, the direct limit of a direct system $F : \mathcal{I} \to \mathrm{Ab}$ is the factor group

$$\lim_{\to} F = \left(\oplus_{i\in I} F_i\right)/U \qquad U = \langle \cup_{i\preceq j\in I}\{\iota_j \circ f_{ji}(m) - \iota_i(m) \mid m \in F_i\}\rangle$$

where $\iota_i : F_i \to \oplus_{i\in I} F_i$ are the inclusions. The morphisms $f_i = \pi \circ \iota_i : F_i \to \lim_{\to} F$ are the composites of the inclusions ι_i with the canonical surjection $\pi : \oplus_{i\in I} F_i \to \lim_{\to} F$ (Exercise 75).

5. The direct limit of a direct system $F : \mathcal{I} \to \mathrm{Ch}_{\mathrm{Ab}}$ is given degreewise by

$$(\lim_{\to} F)_n = \lim_{\to} F_n \qquad d_n = \lim_{\to} d_n : (\lim_{\to} F)_n \to (\lim_{\to} F)_{n-1}.$$

6. Every direct system in the categories Set, Top and in the categories Ab or $\mathrm{Ch}_{\mathrm{Ab}}$ has a direct limit. This follows, because these categories are **cocomplete**: every functor into these categories has a colimit.

Example 6.4.8:
Let X be a topological space and $\mathcal{K}$ the associated direct poset category of compact subsets.

1. There is a direct system $S^\bullet(X, X \setminus -; R) : \mathcal{K} \to \mathrm{Ch}^{\mathrm{Ab}}$ of cochain complexes that
   - assigns to a compact subset $K \subset X$ the cochain complex $S^\bullet(X, X \setminus K; R)$
   - assigns to compact subsets $L \subset K \subset X$ the cochain map

   $$S^\bullet(i_{LK}) : S^\bullet(X, X \setminus L; R) \to S^\bullet(X, X \setminus K; R).$$

2. The cohomologies define a direct system $H^n(X, X \setminus -; R) : \mathcal{K} \to \mathrm{Ab}$ that
   - assigns to a compact subset $K \subset X$ the abelian group $F_K = H^n(X, X \setminus K; R)$
   - assigns to compact subsets $L \subset K \subset X$ the group homomorphism

   $$H^n(i_{LK}) : H^n(X, X \setminus L; R) \to H^n(X, X \setminus K; R).$$

We will now show that the direct limit of the direct system $S^\bullet(X, X \setminus -; R)$ is precisely the singular cochain complex with compact support, and that the direct limit of the direct system $H^n(X, X \setminus -; R)$ are isomorphic to its cohomologies. This requires some auxiliary results on the interaction of direct limits with short exact sequences.

Lemma 6.4.9: Let $(I, \preceq)$ a directed poset.

1. If $A, B, C : \mathcal{I} \to \mathrm{Ab}$ are direct systems of abelian groups and $\iota : A \Rightarrow B$ and $\pi : B \Rightarrow C$ maps of direct systems that form for each $i \in I$ a short exact sequence

   $$0 \to A_i \xrightarrow{\iota_i} B_i \xrightarrow{\pi_i} C_i \to 0,$$

   then they induce a induced short exact sequence

   $$0 \to \lim_{\to} A \xrightarrow{\lim_{\to}\iota} \lim_{\to} B \xrightarrow{\lim_{\to}\pi} \lim_{\to} C \to 0.$$

2. If $A_\bullet : \mathcal{I} \to \mathrm{Ch}_{\mathrm{Ab}}$ is a direct system of chain complexes, then for all $n \in \mathbb{Z}$

   $$\lim_{\to} H_n(A_\bullet) \cong H_n(\lim_{\to} A_\bullet).$$

Proof:
1. We denote by $a_{ij}: A_i \to A_j$ the group homomorphisms maps that characterise the direct systems, and by $a_i: A_i \to \lim_\to A$ the maps that characterise the direct limits. We use the description in Remark 6.4.7, 4. and write $[x]$ for the equivalence class of an element $x \in A_i$ in $\lim_\to A = \oplus_{i\in I} A_i/U_A$. We then have $a_i: A_i \to \lim_\to A$, $x \mapsto [x]$ with $[a_{ij}(x)] = [x]$ for all $x \in A_i$, $i \preceq j$ and analogous identities for B and C.

1. (a) We show that $\lim_\to \iota$ is injective:

Let $a \in \lim_\to A$ with $\lim_\to \iota(a) = 0$. By Remark 6.4.7, 4. it is of the form $a = [\Sigma_{k=1}^n x_k]$ with $x_k \in A_{i_k}$ and $i_k \in I$. As I is direct, there is a $j \in I$ with $i_1, \ldots, i_n \preceq j$, which implies $a = [x]$, where $x = \sum_{i=1}^n a_{i_k j}(x_k) \in A_j$. Then we have $0 = \lim_\to \iota(a) = [\iota_j(x)]$. By Remark 6.4.7, 4. there is an $l \in I$ with $j \preceq l$ and $0 = b_{jl} \circ \iota_j(x)$. As ι is a map of direct systems, this implies $0 = b_{jl} \circ \iota_j(x) = \iota_l \circ a_{jl}(x)$ and by injectivity of ι_l that $a_{jl}(x) = 0$. Hence, $a = [x] = [a_{jl}(x)] = 0$.

1. (b) We show that $\lim_\to \pi$ is surjective:

Let $c \in \lim_\to C$. As in case (a), there is an $x \in C_j$ for some $j \in I$ with $c = [x]$. By surjectivity of $\pi_j: B_j \to C_j$ there is a $y \in B_j$ with $x = \pi_j(y)$. This implies $c = [x] = [\pi_j(y)] = \lim_\to \pi([x])$.

1. (c) We show that $\ker \lim_\to \pi = \mathrm{im} \lim_\to \iota$:

If $b = \lim_\to \iota(c)$ for some $c \in \lim_\to C$, there is an $x \in C_j$ for some $j \in I$ with $c = [x]$. This implies $b = \lim_\to \iota[x] = [\iota_j(x)]$ and $\lim_\to \pi(b) = \lim_\to \pi[\iota_j(x)] = [\pi_j \circ \iota_j(x)] = 0$ and $b \in \ker \lim_\to \pi$.

Let $b \in \ker \lim_\to \pi$. As in (a), there is an $x \in B_j$ for some $j \in I$ with $b = [x]$ and $\lim_\to \pi(b) = [\pi_j(x)] = 0$. Hence, there is a $k \in I$ with $j \preceq k$ such that $0 = c_{jk} \circ \pi_j(x) = \pi_k \circ b_{jk}(x)$. By exactness of the sequence for k, there is an $y \in A_k$ with $\iota_k(y) = b_{jk}(x)$. This implies $\lim_\to \iota[y] = [\iota_k(y)] = [b_{jk}(x)] = [x] = b$ and $b \in \mathrm{im} \lim_\to \iota$.

2. A direct system $A_\bullet: \mathcal{I} \to \mathrm{Ch}_{\mathrm{Ab}}$ of chain complexes consists of chain complexes $A^i_\bullet$ for $i \in I$ and chain maps $a^{ij}_\bullet: S^i_\bullet \to A^j_\bullet$ for all $i \preceq j$ such that $a^{ii}_\bullet = \mathrm{id}_{A^i_\bullet}$ for all $i \in \mathcal{I}$ and $a^{jk}_\bullet \circ a^{ij}_\bullet = a^{ik}_\bullet$ for all $i \preceq j \preceq k$. Its direct limit is a chain complex $\lim_\to A_\bullet$ with chain maps $a^i_\bullet: A^i_\bullet \to \lim_\to A_\bullet$ such that $a^j_\bullet \circ a^{ij}_\bullet = a^i_\bullet$ for all $i \preceq j$.

We consider the functors $H_n, Z_n, B_n: \mathrm{Ch}_{\mathrm{Ab}} \to \mathrm{Ab}$ that assign to a chain complex $X_\bullet$ the abelian groups $H_n(X_\bullet)$, $Z_n(X_\bullet)$ and $B_n(X_\bullet)$, respectively, and to a chain map $f_\bullet: X_\bullet \to Y_\bullet$ the induced maps $H_n(f_\bullet): H_n(X_\bullet) \to H_n(Y_\bullet)$, $Z_n(f_\bullet): Z_n(X_\bullet) \to Z_n(Y_\bullet)$ and $B_n(f_\bullet): B_n(X_\bullet) \to B_n(Y_\bullet)$. The inclusions $\iota_n: B_n(X_\bullet) \to Z_n(X_\bullet)$ and projections $\pi_n: Z_n(X_\bullet) \to H_n(X_\bullet)$ define natural transformations $\iota_n: B_n \Rightarrow Z_n$ and $\pi_n: Z_n \Rightarrow H_n$.

Composing these functors and natural transformations with the direct system yields direct systems $H_nA_\bullet, Z_nA_\bullet, B_nA_\bullet: \mathcal{I} \to \mathrm{Ab}$ and maps of direct systems $\iota_nA_\bullet: B_nA_\bullet \Rightarrow Z_nA_\bullet$ and $\pi_nA_\bullet: Z_nA_\bullet \Rightarrow H_nA_\bullet$ for all $n \in \mathbb{Z}$. For each $i \in I$ they form a short exact sequence

$$0 \to B_n(A^i_\bullet) \xrightarrow{\iota^i_n} Z_n(A^i_\bullet) \xrightarrow{\pi^i_n} H_n(A^i_\bullet) \to 0.$$

By 1. this yields for all $n \in \mathbb{Z}$ a short exact sequence

$$0 \to \lim_{\to} B_n(A_\bullet) \xrightarrow{\lim_\to \iota_n} \lim_{\to} Z_n(A_\bullet) \xrightarrow{\lim_\to \pi_n} \lim_{\to} H_n(A_\bullet) \to 0,$$

which implies

$$\lim_{\to} H_n(A_\bullet) = \frac{\lim_\to Z_n(A_\bullet)}{\lim_\to B_n(A_\bullet)}.$$

The maps $Z_n(a^i_\bullet) : Z_n(A^i_\bullet) \to Z_n(\lim_\to A_\bullet)$ and $B_n(a^i_\bullet) : B_n(A^i_\bullet) \to B_n(\lim_\to A_\bullet)$ satisfy

$$Z_n(a^j_\bullet) \circ Z_n(a^{ij}_\bullet) = Z_n(a^i_\bullet) \qquad B_n(a^j_\bullet) \circ B_n(a^{ij}_\bullet) = B_n(a^i_\bullet)$$

for all $i \preceq j$. By the universal property of the direct limit they induce R-linear maps

$$z_n : \lim_\to Z_n(A_\bullet) \to Z_n(\lim_\to A_\bullet),\ [x]' \mapsto [x] \qquad b_n : \lim_\to B_n(A_\bullet) \to B_n(\lim_\to A_\bullet),\ [y]' \mapsto [y],$$

where $[x]'$, $[y]'$ denote the equivalence classes of $x \in Z_n(A^i_\bullet)$ and $y \in B_n(A^i_\bullet)$ in $\lim_\to Z_n(A_\bullet)$ and $\lim_\to B_n(A_\bullet)$ and $[x]$, $[y]$ their equivalence classes in $Z_n(\lim_\to A_\bullet)$ and $B_n(\lim_\to A_\bullet)$.

It remains to show that z_n and b_n are isomorphisms, which yields

$$\lim_\to H_n(A_\bullet) = \frac{\lim_\to Z_n(A_\bullet)}{\lim_\to B_n(A_\bullet)} \cong \frac{Z_n(\lim_\to A_\bullet)}{B_n(\lim_\to A_\bullet)} = H_n(\lim_\to A_\bullet).$$

To show that z_n is surjective, let $a \in Z_n(\lim_\to A_\bullet)$. As in 1. (a), we see that there is an $x \in A^j_n$ with $a = [x]$ and $d_n(a) = [d^j_n(x)] = 0$. This implies that there is a $k \in I$ with $j \preceq k$ such that $0 = a^{jk}_{n-1} \circ d^j_n(x) = d^k_n \circ a^{jk}_n(x)$. Hence, we have $a = [x] = [a^{jk}_n(x)]$ with $a^{jk}_n(x) \in Z_n(a^k_\bullet)$ and hence $z_n[a^{jk}_n(x)]' = [a^{jk}_n(x)] = [x] = a$. This shows z_n is surjective.

To show z_n is injective, let $a \in \lim_\to Z_n(A_\bullet)$ with $z_n(a) = 0$. Then there is an $x \in Z_n(A^i_\bullet)$ with $a = [x]'$, and we have $z_n(c) = [x] = 0$. Hence, there is a $k \in I$ with $i \preceq k$ such that $a^{ik}_n(x) = 0$, and we have $a^{ik}_n(x) \in Z_n(A^k_\bullet)$, as $d^k_n \circ a^{ik}_n(x) = a^{ik}_{n-1} \circ d^i_n(x) = 0$. This implies $a = [x]' = [a^{ik}_n(x)] = 0$, and hence z_n is injective.

To show that b_n is surjective, let $a \in B_n(\lim_\to A_\bullet)$. Then there is a $b \in \lim_\to A_{n+1}$ with $a = d_{n+1}(b)$ and a $y \in A^j_{n+1}$ for some $j \in I$ with $b = [y]$. This implies $a = d_{n+1}(b) = [d_{n+1^j(y)}] = b_n[d^j_{n+1}(y)]'$ with $d^j_{n+1}(y) \in B_n(A^j_\bullet)$, and thus b_n is surjective.

To show that b_n is injective, let $c \in \lim_\to B_n(A_\bullet)$ with $b_n(c) = 0$. Then there is an $x \in A^i_{n+1}$ for some $i \in I$ with $c = [d^i_{n+1}(x)]'$, and we have $0 = b_n(c) = b_n[d^i_{n+1}(x)]' = [d^i_{n+1}(x)]$. Hence, there is a $k \in I$ with $i \preceq k$ such that $a^{ik}_n \circ d^i_{n+1}(x) = d^k_{n+1} \circ a^{ik}_{n+1}(x) = 0$. This implies $c = [d^i_{n+1}(x)]' = [d^k_{n+1} \circ a^{ik}_{n+1}(x)]' = [a^{ik}_n \circ d^i_{n+1}(x)]' = 0$, and b_n is injective. □

Proposition 6.4.10: Let R be a commutative unital ring, X a topological space and $\mathcal{K}$ the directed poset of compact subsets from Example 6.4.8. Then we have isomorphisms

$$\lim_\to S^\bullet(X, X \setminus -; R) \cong S^\bullet_c(X; R) \qquad \lim_\to H^n(X, X \setminus -; R) \cong H^n_c(X; R).$$

Proof:
The cochain maps $S^\bullet(i_{KX}) : S^\bullet(X, X \setminus K; R) \to S^\bullet_c(X; R)$ for compact subsets K satisfy

$$S^\bullet(i_{KX}) \circ S^\bullet(i_{LK}) = S^\bullet(i_{LX}) : S^\bullet(X, X \setminus L; R) \to S^\bullet_c(X; R)$$

for all compact subsets $L \subset K \subset X$. By the universal property of the direct limit, they induce a cochain map

$$f^\bullet : \lim_\to S^\bullet(X, X \setminus -; R) \to S^\bullet_c(X; R).$$

Its component $f^n_K : \lim_{\to} S^n(X, X \setminus K; R) \to S^n_c(X; R)$ for a compactum $K \subset X$ sends each equivalence class $[\phi]$ of a cochain $\phi \in S^n(X, X \setminus K; R)$ in the direct limit to ϕ. This is well-defined, as ϕ is a group homomorphism $\phi : S_n(X) \to R$ with $\phi(\sigma) = 0$ for all $\sigma : \Delta^n \to X$ with $\sigma(\Delta^n) \cap K = \emptyset$. Clearly, all maps f^n_K is injective for all $n \in \mathbb{Z}$ and compacta $K \subset X$. It is surjective, by definition of the cochains with compact support: If $\phi \in S^n_c(X; R)$, there is a compactum $K_\phi \subset X$ with $\phi(\sigma) = 0$ for all singular simplexes $\sigma : \Delta^n \to X$ with $\sigma(\Delta^n) \cap K_\phi = 0$. This shows that $\phi \in S^n(X, X \setminus K_\phi; R)$ and $f^n_{K_\phi}[\phi] = \phi$. Hence $f^\bullet$ is an isomorphism of cochain complexes. With Lemma 6.4.9, 2. we obtain

$$H^n_c(X; R) = H^n S^\bullet_c(X; R) \cong H^n(\lim_{\to} S^\bullet(X, X \setminus -; R)) \cong \lim_{\to} H^n(X, X \setminus -; R).$$

□

Example 6.4.11: The cohomologies with compact support of $\mathbb{R}^n$ with coefficients in a commutative unital ring R are given by

$$H^k_c(\mathbb{R}^n; R) = H^k_c(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) \cong \begin{cases} R & k = n \\ 0 & k \neq n \end{cases}$$

Proof:
We compute the cohomology groups with compact support with Proposition 6.4.10. This amounts to computing the direct limit of the functor $H^k(\mathbb{R}^n, \mathbb{R}^n \setminus -; R) : \mathcal{K} \to \mathrm{Ab}$, where $\mathcal{K}$ is the direct poset of compact subsets $K \subset \mathbb{R}^n$. By Remark 6.4.7, 4. it is given by

$$\lim_{\to} H^k(\mathbb{R}^n, \mathbb{R}^n \setminus -; R) = \bigoplus_{\substack{K \subset \mathbb{R}^n \\ \text{compact}}} H^n(\mathbb{R}^n, \mathbb{R}^n \setminus K; R)/\sim$$

$$[z] \sim H^k(i_{KL})[z] \quad \forall [z] \in H^k(\mathbb{R}^n, \mathbb{R}^n \setminus L; R) \text{ and } L \subset K \subset \mathbb{R}^n.$$

As every compact subset $K \subset \mathbb{R}^n$ is contained in a closed ball $B_r = \{x \in \mathbb{R}^n \mid ||x|| \leq r\}$ of integer radius $r \in \mathbb{N}_0$, every element $[z] \in H^k(\mathbb{R}^n, \mathbb{R}^n \setminus K; R)$ is equivalent to an element $H^k(i_{KB_r})[z] \in H^k(\mathbb{R}^n, \mathbb{R}^n \setminus B_r; R)$, and we can replace the direct poset $\mathcal{K}$ by the direct poset $\mathcal{B}$ whose objects are balls B_r of integer radius $r \in \mathbb{N}$ in $\mathbb{R}^n$.

As $B_r \subset \mathbb{R}^n$ is compact and convex, we can show analogously to Example 6.3.8 that we have a commuting diagram, in which all arrows are isomorphisms

$$\begin{tikzcd}
H^k(\mathbb{R}^n, \mathbb{R}^n \setminus B_r; R) \arrow[rr, "H^k(i_{B_r B_{r+1}})"] & & H^k(\mathbb{R}^n, \mathbb{R}^n \setminus B_{r+1}; R) \\
& H^k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) \arrow[ul, "\cong", "H^k(i_{\mathbb{R}^n \setminus \{0\} B_r})"'] \arrow[ur, "\cong"', "H^k(i_{\mathbb{R}^n \setminus \{0\} B_{r+1}})"] &
\end{tikzcd}$$

and obtain with Proposition 6.3.4

$$\lim_{\to} H^k(\mathbb{R}^n, \mathbb{R}^n \setminus -; R) \cong H^k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R) \cong \begin{cases} R & k = n \\ 0 & k \neq n. \end{cases}$$

□

Lemma 6.4.12: Let X be a connected topological n-manifold and $\{[z_x]\}_{x \in X}$ an R-orientation of X. For each compact subset $K \subset X$ denote by $[z_K] \in H_n(X, X \setminus K; R)$ the associated orientation along K from Proposition 6.3.11 with $H_n(i_{Kx})[z_K] = [z_x]$ for all $x \in K$.

Then the maps $-\cap[z_K]: H^{n-k}(X, X\setminus K; R)\to H_k(X;R)$ induce group homomorphisms

$$\mathrm{PD}_k: H_c^{n-k}(X;R)\to H_k(X;R) \quad \text{for} \quad k\in\{0,\ldots,n\},$$

with $\mathrm{PD}_k\circ H^{n-k}(i_{KX})=-\cap[z_K]$ for all compacta $K\subset X$, the **Poincaré duality maps**.

Proof:
By Proposition 6.4.10 we have $\lim_{\to} H^{n-k}(X,X\setminus -;R)\cong H_c^{n-k}(X;R)$. The direct limit is characterised by the maps $H^{n-k}(i_{KX}): H^{n-k}(X,X\setminus K;R)\to H_c^{n-k}(X;R)$. By Proposition 6.3.11 we have $H_n(i_{LK})[z_K]=[z_L]$ for all compact subsets $L\subset K\subset X$. With the naturality property of the cap product from Proposition 6.2.17 it follows that the maps

$$f_K=-\cap[z_K]: H^{n-k}(X,X\setminus K;R)\to H_k(X;R)$$

satisfy for all $[\phi]\in H^{n-k}(X,X\setminus K;R)$

$$\begin{aligned} f_K\circ H^{n-k}(i_{LK})[\phi] &= (H^{n-k}(i_{LK})[\phi])\cap[z_K]=H_k(i_{LK})(H^{n-k}(i_{LK})[\phi]\cap[z_K])\\ &\overset{6.2.17}{=}[\phi]\cap(H_n(i_{LK})[z_K])\overset{6.3.11}{=}[\phi]\cap[z_L]=f_L[\phi]. \end{aligned}$$

By the universal property of the direct limit of $H^{n-k}(X,X\setminus -;R):\mathcal{K}\to \mathrm{Ab}$, there is a unique group homomorphism $\mathrm{PD}_k: H_c^{n-k}(X;R)\cong \lim_{\to} H^{n-k}(X,X\setminus -;R)\to H_k(X;R)$ with $PD_k\circ H^{n-k}(i_{KX})=-\cap[z_K]$ for all compact subsets $K\subset X$. □

Theorem 6.4.13: Let X be a connected topological n-manifold. Then for all $k\in\mathbb{Z}$ the Poincaré duality map is a group isomorphism.

$$\mathrm{PD}_k: H_c^{n-k}(X;R)\xrightarrow{\cong} H_k(X;R).$$

Proof:
1. We show: If $X=U\cup V$ with $U,V\subset X$ open, and the claim holds for U, V and $U\cap V$, then it holds for X.

Suppose $U,V\subset X$ are open and equipped with the R-orientations induced by the R-orientation of X and that the Poincaré duality maps for U, V and $U\cap V$ are isomorphisms. For compact $K\subset U$ and $L\subset V$, we consider the relative Mayer-Vietoris sequence for cohomology for $A_1=X\setminus K$ and $A_2=X\setminus L$ with $A_1\cup A_2=X\setminus(K\cap L)$ and $A_1\cap A_2=X\setminus(K\cup L)$. Excising the closed subspaces $X\setminus U\subset X\setminus K$ and $X\setminus V\subset X\setminus L$ yields

$$H^k(X,X\setminus K;R)\cong H^k(U,U\setminus K;R) \qquad H^k(X,X\setminus L;R)\cong H^k(V,V\setminus L;R).$$

Excision of the closed subspace $X\setminus(U\cap V)\subset X\setminus(K\cap L)$ gives

$$H^k(X,X\setminus(K\cap L);R)\cong H^k(U\cap V,(U\cap V)\setminus(K\cap L);R).$$

so that we have a long exact sequence of cohomology groups

$$\begin{tikzcd}
\ldots \arrow[d, "\partial^{k-1}"] & \ldots \arrow[d, "\partial^{k-1}"] \\
H^k(X, X\setminus (K\cap L); R) \arrow[r, "\cong"] \arrow[d] & H^k(U\cap V, (U\cap V)\setminus (K\cap L); R) \arrow[d] \\
H^k(X, X\setminus K; R)\oplus H^k(X, X\setminus L; R) \arrow[r, "\cong"] \arrow[d] & H^k(U, U\setminus K; R)\oplus H^k(V, V\setminus L; R) \arrow[d] \\
H^k(X, X\setminus (K\cup L); R) \arrow[r, "\mathrm{id}"] \arrow[d, "\partial^k"] & H^k(X, X\setminus (K\cup L); R) \arrow[d, "\partial^k"] \\
H^{k+1}(X, X\setminus (K\cap L); R) \arrow[r, "\cong"] \arrow[d] & H^k(U\cap V, (U\cap V)\setminus (K\cap L); R) \arrow[d] \\
\ldots & \ldots
\end{tikzcd}$$

Taking the direct limit over the column on the right and applying the Poincaré duality maps for U, V, $U\cap V$ and X yields the following commuting diagram with exact columns, in which the right-hand-side is the relative Mayer-Vietoris sequence from Proposition 3.5.5 for homology

$$\begin{tikzcd}
\ldots \arrow[d] & \ldots \arrow[d] \\
H_c^{n-k}(U\cap V; R) \arrow[r, "\mathrm{PD}_k^{U\cap V}", "\cong"'] \arrow[d] & H_k(U\cap V; R) \arrow[d] \\
H_c^{n-k}(U; R)\oplus H_c^{n-k}(V; R) \arrow[r, "\mathrm{PD}_k^U+\mathrm{PD}_k^V", "\cong"'] \arrow[d] & H_k(U; R)\oplus H_k(V; R) \arrow[d] \\
H_c^{n-k}(X; R) \arrow[r, "\mathrm{PD}_k^X"] \arrow[d] & H_k(X; R) \arrow[d] \\
H_c^{n-k+1}(U\cap V; R) \arrow[r, "\mathrm{PD}_{k-1}^{U\cap V}", "\cong"'] \arrow[d] & H_{k-1}(U\cap V; R) \arrow[d] \\
H_c^{n-k+1}(U; R)\oplus H_c^{n-k+1}(V; R) \arrow[r, "\mathrm{PD}_{k-1}^U+\mathrm{PD}_{k-1}^V", "\cong"'] \arrow[d] & H_{k-1}(U; R)\oplus H_{k-1}(V; R) \arrow[d] \\
\ldots & \ldots
\end{tikzcd}$$

As all horizontal arrows except the middle one are isomorphisms by assumption, the middle one is an isomorphism as well by the 5-lemma.

2. We show: If $X=\cup_{i=1}^\infty U_i$ with open subsets $U_1\subset U_2\subset\ldots\subset X$ and the claim holds for U_i, then it holds for X.

For open subsets $U\subset V\subset X$ and compact subsets $L\subset K\subset U$ we denote by

$$i_{LK}^U:(U, U\setminus K)\to (U, U\setminus L),\ u\mapsto u \qquad j_K^{UV}:(U, U\setminus K)\to (V, V\setminus K),\ u\mapsto u$$

the associated morphism in Top(2). By the excision axiom, applied to $V\setminus U\subset V\setminus K$, the map

$$H^{n-k}(j_K^{UV}): H^{n-k}(V, V\setminus K; R)\to H^{n-k}(U, U\setminus K; R)$$

is an isomorphism. If we equip all open subsets of X with the induced orientations, we have the following commuting diagram, where we suppress R for better legibility

$$\begin{tikzcd}
 & & H^{n-k}(U,U\setminus L) \arrow[dll, bend right, "f_L^U=-\cap[z_L]"'] \arrow[dl, "H^{n-k}(i^U_{LU})"'] \arrow[dd, "H^{n-k}(i^U_{LK})"] & H^{n-k}(V,V\setminus L) \arrow[l, "H^{n-k}(j_L^{UV})"', "\cong"] \arrow[dd, "H^{n-k}(i^V_{LK})"'] \arrow[dr, "H^{n-k}(i^V_{LV})"] \arrow[drr, bend left, "f_L^V=-\cap[z_L]"] & & \\
H_k(U) & H_c^{n-k}(U) \arrow[l, "\mathrm{PD}_k^U"'] & & & H_c^{n-k}(V) \arrow[r, "\mathrm{PD}_k^V"] & H_k(V) \\
 & & H^{n-k}(U,U\setminus K) \arrow[ull, bend left, "f_K^U=-\cap[z_K]"] \arrow[ul, "H^{n-k}(i^U_{KU})"] & H^{n-k}(V,V\setminus K) \arrow[l, "\cong"', "H^{n-k}(j_K^{UV})"] \arrow[ur, "H^{n-k}(i^V_{KV})"'] \arrow[urr, bend right, "f_K^V=-\cap[z_K]"'] & &
\end{tikzcd}$$

The commutativity of the diagram implies

$$\psi_K^{UV}\circ H^{n-k}(i^U_{LK})=\psi_L^{UV}\quad\text{for}\quad \psi_L^{UV}:=H^{n-k}(i^V_{LV})\circ H^{n-k}(j_L^{UV})^{-1}$$

and hence induce a unique map

$$\psi^{n-k}_{UV}:H_c^{n-k}(U;R)=\lim_{\rightarrow}H^{n-k}(U,U\setminus -;R)\to H_c^{n-k}(V,R)$$

with $\psi^{n-k}_{UV}\circ H^{n-k}(i^U_{KU})=\psi^{UV}_K$ for all compact subsets $K\subset U$. By composing two such commuting diagrams for open subsets $U\subset V\subset W$ and by setting $U=V$ we obtain the identities

$$\psi^{n-k}_{UW}=\psi^{n-k}_{VW}\circ\psi^{n-k}_{UV}\qquad \psi^{n-k}_{UU}=\mathrm{id}_{H_c^{n-k}(U)}. \tag{63}$$

The diagram still commutes if we add arrows labelled by $H_k(\iota_{UV}):H_k(U;R)\to H_k(V;R)$, where $\iota_{UV}:U\to V$ is the inclusion. The universal property of the direct limit then yields commuting diagrams for all open subsets $U\subset V\subset X$ and $k\in\{0,\ldots,n\}$

$$\begin{tikzcd}
H_c^{n-k}(U;R) \arrow[r, "\psi^{n-k}_{UV}"] \arrow[d, "\mathrm{PD}_k^U"] & H^{n-k}(V;R) \arrow[d, "\mathrm{PD}_k^V"] \\
H_k(U;R) \arrow[r, "H_k(\iota_{UV})"'] & H_k(V;R).
\end{tikzcd}$$

Suppose now that $X=\cup_{i=1}^\infty U_i$ with $U_1\subset U_2\subset\ldots$ open and $\mathrm{PD}_k^{U_i}:H_c^{n-k}(U_i;R)\to H_k(U_i;R)$ an isomorphism for all $n\in\mathbb{N}$. Then from (63) and the properties of the inclusion we have

$$H_k(\iota_{U_jX})=H_k(\iota_{U_iX})\circ H_k(\iota_{U_jU_i})\qquad \psi^{n-k}_{U_jX}=\psi^{n-k}_{U_iX}\circ\psi^{n-k}_{U_jU_i}\qquad 1\leq j<i\in\mathbb{N}.$$

Thus, we have a direct systems $H_k(U_-;R):\mathbb{N}\to\mathrm{Ab}$ and $H_c^{n-k}(U_-;R):\mathbb{N}\to\mathrm{Ab}$ with commuting diagrams, in which the left arrow is an isomorphism by assumption

$$\begin{tikzcd}
H_c^{n-k}(U_i;R) \arrow[r, "\psi^{n-k}_{U_iX}"] \arrow[d, "\mathrm{PD}_k^{U_i}", "\cong"'] & H_c^{n-k}(X;R) \arrow[d, "\mathrm{PD}_k^X"] \\
H_k(U_i;R) \arrow[r, "H_k(\iota_{U_iX})"'] & H_k(X;R).
\end{tikzcd}$$

Taking the direct limit yields a commuting diagram

$$\begin{tikzcd}
\lim_{\to} H_c^{n-k}(U_-;R) \arrow[d, "\lim_{\to} \mathrm{PD}_k^U \cong"'] \arrow[rr, bend left, "\psi^{n-k}", "\cong"'] & H_c^{n-k}(U_i;R) \arrow[l, "f_i"] \arrow[r, "\psi^{n-k}_{U_i X}"'] \arrow[d, "\cong"', "\mathrm{PD}_k^{U_i}"] & H_c^{n-k}(X;R) \arrow[d, "\mathrm{PD}_k^X"] \\
\lim_{\to} H_k(U_-;R) \arrow[rr, bend right, "\phi_k"', "\cong"] & H_k(U_i;R) \arrow[l, "g_i"'] \arrow[r, "H_k(\iota_{U_i X})"] & H_k(X;R)
\end{tikzcd}$$

where the maps f_i and g_i characterise the direct limits and the curved arrows are induced by the universal property of the limit. The upper curved arrow is an isomorphism, because every compact subset $K \subset X$ is contained in some subset $U_i \subset X$ and then in all subsequent subsets $K \subset U_i \subset U_{i+1} \subset \ldots$.

The lower curved arrow is an isomorphism, because every cocycle $z = \Sigma_{j=1}^r z_j \sigma_j \in Z_k(X;R)$ is a linear combination of k-simplexes $\sigma_j : \Delta^k \to X$ and $\cup_{j=1}^r \sigma_j(\Delta^k)$ is compact. Thus, its open cover $X = \cup_{j=1}^\infty U_j$ has a finite subcover, and $\cup_{j=1}^r \sigma_j(\Delta^k) \subset U_i \subset U_{i+1} \subset \ldots$ for some $i \in \mathbb{N}$.

The left vertical arrow is an isomorphism, because PD_k is an isomorphism for all $i \in \mathbb{N}$, and this shows that $\mathrm{PD}_k^X : H_c^{n-k}(X) \to H_k(X)$ is an isomorphism as well.

3. We show that the claim holds for $X = \mathbb{R}^n$.

Because $\mathbb{R}^n$ is contractible and by Example 6.4.11 we have

$$H_k(\mathbb{R}^n;R) = \begin{cases} R & k = 0 \\ 0 & k \neq 0 \end{cases} \qquad\qquad H_c^k(\mathbb{R}^n;R) = H_c^k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\};R) \cong \begin{cases} R & k = n \\ 0 & k \neq n. \end{cases}$$

Thus, we only need to show that the map $\mathrm{PD}_0 : H_c^n(\mathbb{R}^n;R) \to H_0(\mathbb{R}^n;R)$ is an isomorphism. By Exercise 77 it is induced by the maps

$$f_r = - \cap [z_r] : H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_r;R) \to H_0(\mathbb{R}^n;R),$$

where $B_r = \{x \in \mathbb{R}^n \mid ||x|| \leq r\}$ is the closed ball of radius $r > 0$ and $[z_r] \in H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B_r)$ the induced orientation along B_r. By the universal coefficient theorem in Corollary 6.1.7 we have

$$H^n(\mathbb{R}^n, \mathbb{R}^n \setminus B_r;R) \cong \mathrm{Hom}_{\mathrm{Ab}}(H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B_r), R) \oplus \mathrm{Ext}(H_{n-1}(\mathbb{R}^n, \mathbb{R}^n \setminus B_r), R).$$

By Example 6.4.11 we have $H_{n-1}(\mathbb{R}^n, \mathbb{R}^n \setminus B_r) \cong H_{n-1}(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) = 0$, which shows that the second summand is trivial. Combining this with Example 6.2.15, 2. gives

$$f_r = - \cap [z_r] : \mathrm{Hom}_{\mathrm{Ab}}(H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B_r), R) \to H_0(\mathbb{R}^n;R), \quad \phi \mapsto \phi([z_r]).$$

As $H_0(\mathbb{R}^n, R) \cong R$ and $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B_r) = \langle [z_r] \rangle_{\mathbb{Z}} \cong \mathbb{Z}$, this map is an isomorphism for all $r > 0$. With Exercise 76 it follows that they induce an isomorphism $\mathrm{PD}_0 : H_c^n(\mathbb{R}^n;R) \to H_0(\mathbb{R}^n;R)$.

4. We show that the claim holds for open subsets $U \subset \mathbb{R}^n$.

Let $U \subset \mathbb{R}^n$ be an open subset. As $\mathbb{R}^n$ is second countable, U is a *countable* union

$$U = \bigcup_{i=1}^\infty \mathring{B}_i = \bigcup_{j=1}^\infty U_j \qquad U_j = \bigcup_{k=1}^j \mathring{B}_k.$$

of open n-balls $\mathring{B}_i \subset \mathbb{R}^n$. As each open n-ball $\mathring{B}_i$ is homeomorphic to $\mathbb{R}^n$, the claim holds for $\mathring{B}_i$ by 3. With 1. it follows that the claim holds for the open sets U_j for all $j \in \mathbb{N}$ and with 2. it follows that the claim holds for U.

5. We show that the claim holds in general:
Let X be a connected n-manifold with a fixed R-orientation.

5.(a) We show that X is covered by a *countable* set of coordinate charts:

As X is second countable there is a countable family $(B_i)_{i\in\mathbb{N}}$ of open subsets $B_i \subset X$ such that for every open subset $U \subset X$ and every point $x \in U$, there is an $i \in \mathbb{N}$ with $x \in B_i \subset U$.

As X is an n-manifold, every point $x \in X$ has an open neighbourhood U_x homeomorphic to an open subset $V_x \subset \mathbb{R}^n$. Any choice of such open neighbourhoods defines an open cover $X = \cup_{x\in X} U_x$. Consider the set $J = \{i \in \mathbb{N} \mid \exists x \in X : B_i \subset U_x\} \subset \mathbb{N}$ and choose for each $j \in J$ a point $x_j \in X$ with $B_j \subset U_{x_j}$. As U_x is open for all $x \in X$, there is an $i_x \in \mathbb{N}$ and a basis element B_{i_x} with $x \in B_{i_x} \subset U_x$ for all $x \in X$, and $i_x \in J \subset \mathbb{N}$ by assumption. Then we have $X = \cup_{x\in X} B_{i_x} \subset \cup_{j\in J} B_j$, and every B_j for $j \in J$ is a domain of a coordinate chart.

5.(b) Suppose $X = \cup_{i=1}^{\infty} U_i$, where $U_i \subset X$ is open and homeomorphic to an open subset $V_i \subset \mathbb{R}^n$. By adjusting the subset $V_i \subset \mathbb{R}^n$ we may assume that the homeomorphism preserves R-orientations. Then the claim holds for V_i by 4. and hence for U_i for all $i \in \mathbb{N}$. By 1. it also holds for the sets $W_j = \cup_{i=1}^{j} U_i$ for $j \in \mathbb{N}$ and by 2. for $X = \cup_{j=1}^{\infty} W_j$. □

Corollary 6.4.14: Let X be a compact connected R-oriented topological n-manifold with associated fundamental class $[X] \in H_n(X;R)$.

Then for all $k \in \mathbb{Z}$ the Poincaré duality map is given by the group isomorphism

$$PD_k = - \cap [X] : H^{n-k}(X;R) \xrightarrow{\cong} H_k(X;R).$$

In particular, one has $H_0(X;R) \cong H^n(X;R) \cong R$.

Example 6.4.15: By Example 5.4.8, 4. and and Example 6.1.8 the singular homologies and cohomologies of $\mathbb{RP}^n$ with coefficients in a commutative unital ring R are given by

$$H_k(\mathbb{RP}^n;R) = \begin{cases} R & k=0 \text{ or } k=n \text{ odd} \\ R/2R & 1 \le k < n \text{ odd} \\ \mathrm{Tor}_2(R) & 1 \le k \le n \text{ even} \\ 0 & k \ge n+1, \end{cases} \qquad H^k(\mathbb{RP}^n;R) = \begin{cases} R & k=0 \text{ or } k=n \text{ odd} \\ \mathrm{Tor}_2(R) & 1 \le k < n \text{ odd} \\ R/2R & 2 \le k \le n \text{ even} \\ 0 & k < n. \end{cases}$$

For n odd, these formulas imply together with Theorem 6.3.12 and Remark 6.3.16 that $\mathbb{RP}^n$ is R-orientable for any commutative unital ring R, as we have $H_n(\mathbb{RP}^n;R) = R$ for n odd. In this case, $k \in \{1,\ldots,n\}$ is odd if and only if $n-k \in \{1,\ldots,k\}$ is even. Comparing the formulas for homologies and cohomologies then shows the Poincaré duality isomorphisms.

For n even, these formulas imply together with Theorem 6.3.12 and Remark 6.3.16 that $\mathbb{RP}^n$ is R-orientable if and only if $\mathrm{Tor}_2(R) = R$, as $H_n(\mathbb{RP}^n;R) = \mathrm{Tor}_2(R)$. In this case the formulas for the homologies and cohomologies reduce to

$$H_k(\mathbb{RP}^n;R) = H^k(\mathbb{RP}^n,R) = \begin{cases} R & 0 \le k \le n \\ 0 & \text{else.} \end{cases}$$

7 Exercises

7.1 Exercises for Section 1

Exercise 1: Let A be an abelian group.

(a) Show that A defines a functor $\mathrm{Hom}(-, A) : \mathrm{Ab}^{op} \to \mathrm{Ab}$, if we equip the sets $\mathrm{Hom}_{\mathrm{Ab}}(B, A)$ with the pointwise addition of group homomorphisms.

(b) Show that $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, A)$ gives the n-torsion elements of A and $\mathrm{Hom}(\mathbb{Z}, A) \cong A$.

(c) Show that $\mathrm{Hom}(-, A) : \mathrm{Ab}^{op} \to \mathrm{Ab}$ sends direct sums of abelian groups to direct products

$$\mathrm{Hom}(\oplus_{i\in I} B_i, A) \cong \Pi_{i\in I}\mathrm{Hom}(B_i, A).$$

Exercise 2: Let A, B, C be sets.

- A **relation** between A and B is a subset $R \subset A \times B$.
- A relation $R \subset A \times B$ is called a **map** from A to B, if for every $a \in A$ there is a unique $b \in B$ with $(a, b) \in R$.
- The composite of two relations $R \subset A \times B$ and $S \subset B \times C$ is the relation

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B : \ (a, b) \in R, (b, c) \in S\} \subset A \times C.$$

(a) Show that sets and relations form a category **Rel** with $\mathrm{Hom}_{\mathrm{Rel}}(A, B) = \mathcal{P}(A \times B)$.

(b) Determine the isomorphisms in **Rel**.

(c) Show that the disjoint union of sets defines both, a product and a coproduct in Rel.

Exercise 3: Let G be a group. The **commutator subgroup** of G is the normal subgroup

$$[G, G] = \{[g, h] \mid g, h \in G\} \subset G \quad [g, h] := g \cdot h \cdot g^{-1} \cdot h^{-1}.$$

(a) Show that the commutator subgroup is indeed a normal subgroup of G and that the factor group $\mathrm{Ab}(G) = G/[G, G]$ is abelian.

(b) Denote by $\pi_G : G \to G/[G, G]$, $g \to g[G, G]$ the canonical surjection. Show that for every group homomorphism $f : G \to H$ there is a unique group homomorphism $\mathrm{Ab}(f) : G/[G, G] \to H/[H, H]$ with $\mathrm{Ab}(f) \circ \pi_G = \pi_H \circ f$.

(c) Show that this defines a functor $\mathrm{Ab} : \mathrm{Grp} \to \mathrm{Ab}$, the **abelisation functor**.

(d) Let $I : \mathrm{Ab} \to \mathrm{Grp}$ be the inclusion functor. Show that the canonical surjections $\pi_G : G \to \mathrm{Ab}(G)$ define a natural transformation between $\mathrm{id}_{\mathrm{Grp}} : \mathrm{Grp} \to \mathrm{Grp}$ and the functor $I\mathrm{Ab} : \mathrm{Grp} \to \mathrm{Grp}$.

Exercise 4: Give a presentation of the following abelian groups in terms of generators and relations:

(a) $\mathbb{Z} \times \mathbb{Z}$,
(b) $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ with $n, m \in \mathbb{N}$,
(c) the abelianisation $\mathrm{Ab}(S_n)$ for $n \in \mathbb{N}$.

Exercise 5: Let $(A_i)_{i\in I}$ a family of objects in Ab. Show with the universal property of the product and coproduct that there is a canonical morphism $I : \amalg_{i\in I} A_i \to \Pi_{i\in I} A_i$. Is there an analogous morphism in Top or Set?

Exercise 6: Let G be a group and $\mathcal{C} = BG$ be the groupoid with a single object $\bullet$ and $\mathrm{Hom}_{\mathcal{C}}(\bullet, \bullet) = G$, where the composition of morphisms is given by the group multiplication.

(a) Show that functors $F : \mathcal{C} \to \mathrm{Set}$ correspond to G-sets, sets with an action of G.
(b) Show that a functors $F : \mathcal{C} \to \mathrm{Vect}_{\mathbb{F}}$ correspond to representations of G
(c) Characterise functors $F : \mathcal{C} \to \mathrm{Grp}$ in a similar way.
(d) For (a), (b), (c), characterise the natural transformations between such functors.

Exercise 7: Let G be a group and $\rhd : G \times X \to X$ an action of G on a set X. Show that the category $\mathcal{C}$ with $\mathrm{Ob}\mathcal{C} = X$, with $\mathrm{Hom}_{\mathcal{C}}(x, y) = \{(x, g) \mid g \in G \text{ with } g \rhd x = y\}$ and the composition of morphisms given by the multiplication in G is a groupoid.

7.2 Exercises for Section 2

Exercise 8: One says a short exact sequence $0 \to A \xrightarrow{\iota} M \xrightarrow{p} B \to 0$ of abelian groups **splits** if there is a group isomorphism $\phi : M \to A \oplus B$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota"] \arrow[dr, "\iota'"'] & M \arrow[r, "p"] \arrow[d, "\phi"] & B \arrow[r] & 0\,, \\
& & A \oplus B \arrow[ur, "p'"'] & &
\end{tikzcd}$$

where $\iota' : A \to A \oplus B$, $a \mapsto (a, 0)$ and $p' : A \oplus B \to B$, $(a, b) \to b$. Show that the following statements are equivalent:

(i) The short exact sequence $0 \to A \xrightarrow{\iota} M \xrightarrow{p} B \to 0$ splits.
(ii) The projection $p : M \to B$ has a right inverse, a **section** of p.
(iii) the injection $\iota : A \to M$ has a left inverse, a **retraction** of r.

Exercise 9: Let $(A_n)_{n\in\mathbb{Z}}$ be a family of abelian groups. Is there a chain complex $X_\bullet$ such that is a X_n a free abelian group for all $n \in \mathbb{Z}$ and $H_n(X_\bullet) = A_n$ for all $n \in \mathbb{Z}$?

Exercise 10: Show that the following chain complexes are not chain homotopy equivalent

$$\ldots \xrightarrow{z\mapsto \bar{2}z} \mathbb{Z}/4\mathbb{Z} \xrightarrow{z\mapsto \bar{2}z} \mathbb{Z}/4\mathbb{Z} \xrightarrow{z\mapsto \bar{2}z} \ldots \qquad \ldots \to 0 \to 0 \to \ldots .$$

Exercise 11: Prove the **5-Lemma**: For any commutative diagram of abelian groups and group homomorphisms of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & D \arrow[r, "k"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "k'"] & E' \end{tikzcd}$$

with exact rows, the following implications hold:

(i) If β, δ are injective and α surjective, then γ is injective.

(ii) If β, δ are surjective and ϵ injective, then γ is surjective.

(iii) If α is surjective, ϵ injective and β, δ are isomorphisms, then γ is an isomorphism.

Exercise 12: Prove the **9-Lemma**: Consider the following commutative diagram of abelian groups with exact rows

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A_1 \arrow[r, "\iota_A"] \arrow[d, "\iota_1"] & A_2 \arrow[r, "\pi_A"] \arrow[d, "\iota_2"] & A_3 \arrow[r] \arrow[d, "\iota_3"] & 0 \\ 0 \arrow[r] & B_1 \arrow[r, "\iota_B"] \arrow[d, "\pi_1"] & B_2 \arrow[r, "\pi_B"] \arrow[d, "\pi_2"] & B_3 \arrow[r] \arrow[d, "\pi_3"] & 0 \\ 0 \arrow[r] & C_1 \arrow[r, "\iota_C"] \arrow[d] & C_2 \arrow[r, "\pi_C"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

Show:

(a) If the first two columns are short exact sequences, the third column is also a short exact sequence.
(b) If the last two columns are short exact sequences, the first column is also a short exact sequence.

Exercise 13: Let $k \in \mathbb{Z}$ and $X_\bullet$ a chain complex. Consider the subcomplex $A_\bullet \subset X_\bullet$ with $A_n = X_n$ for $n \leq k$ and $A_n = 0$ for $n > k$.

(a) Express the homologies of the chain complexes $A_\bullet$ and $X_\bullet/A_\bullet$ in terms of data for $X_\bullet$.
(b) Compute the long exact homology sequence and all connecting homomorphisms for the short exact sequence $0 \to A_\bullet \xrightarrow{\iota_\bullet} X_\bullet \xrightarrow{\pi_\bullet} X_\bullet/A_\bullet \to 0$.

Exercise 14: Let $X_\bullet$ the chain complex whose non-trivial abelian groups are

$$X_2 = \langle a, b \rangle_{\mathbb{Z}} \qquad X_1 = \langle c, d, e \rangle_{\mathbb{Z}} \qquad X_0 = \langle f, g, h \rangle_{\mathbb{Z}}$$

and with boundary operators given by

$$d_2(a) = d_2(b) = e, \quad d_1(c) = f - g, \quad d_1(d) = h - g, \quad d_1(e) = 0.$$

and $A_\bullet \subset X_\bullet$ the subcomplex given by

$$A_2 = \langle a \rangle \subset X_2 \qquad A_1 = \langle c, e \rangle \subset X_1 \qquad A_0 = \langle f, g \rangle \subset X_0.$$

Compute the homologies of $X_\bullet$ and $A_\bullet$ and use the long exact homology sequence to compute the homologies of $X_\bullet / A_\bullet$.

Exercise 15: Let $X_\bullet$ be a chain complex. Show that this defines exact sequences

$$0_\bullet \to Z_\bullet \xrightarrow{f_\bullet} X_\bullet \xrightarrow{g_\bullet} B_\bullet^{(-1)} \to 0_\bullet$$
$$0_\bullet \to H_\bullet \xrightarrow{h_\bullet} X_\bullet/B_\bullet \xrightarrow{k_\bullet} Z_\bullet^{(-1)} \xrightarrow{l_\bullet} H_\bullet^{(-1)} \to 0_\bullet,$$

where $Z_\bullet = (Z_n(X_\bullet))_{n\in\mathbb{Z}}$, $B_\bullet = (B_n(X_\bullet))_{n\in\mathbb{Z}}$ and $H_\bullet = (H_n(X_\bullet))_{n\in\mathbb{Z}}$ and $Y_\bullet^{(-1)} = (Y_{n-1})_{n\in\mathbb{Z}}$ denotes the shifted chain complex. Determine the boundary operators of $Z_\bullet$, $B_\bullet$ and $H_\bullet$ and the chain maps $f_\bullet, g_\bullet, h_\bullet, k_\bullet, l_\bullet$. Determine the long exact homology sequence for the first sequence.

Exercise 16: Let $f_\bullet : X_\bullet \to Y_\bullet$ be a chain map. The **mapping cone** of $f_\bullet$ is the chain complex $C_\bullet(f_\bullet) = (C_n)_{n\in\mathbb{Z}}$ with $C_n = X_{n-1} + Y_n$ and boundary operator

$$d_n : C_n \to C_{n-1}, \quad (x, y) \mapsto (-d_{n-1}(x), d_n(y) - f_{n-1}(x)).$$

(a) Show that $C_\bullet(f_\bullet)$ is a chain complex and the inclusions $\iota_n : Y_n \to C_n$ define a chain map $i_\bullet : Y_\bullet \to C_\bullet(f_\bullet)$.
(b) Show that $f_\bullet \sim 0_\bullet$, if and only if there is a chain map $g_\bullet : C_\bullet(\mathrm{id}_{X_\bullet}) \to Y_\bullet$ with $g_\bullet \circ i_\bullet = f_\bullet$.

7.3 Exercises for Section 3

Exercise 17: Let X, Y be topological spaces with associated singular chain complexes $S_\bullet(X)$ and $S_\bullet(Y)$. Is every chain map $g_\bullet : S_\bullet(X) \to S_\bullet(Y)$ of the form $g_\bullet = S_\bullet(f)$ for a continuous map $f : X \to Y$?

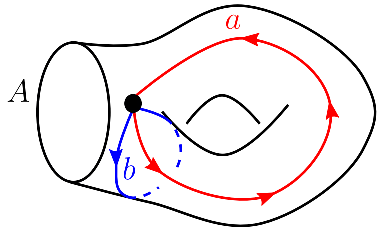
Exercise 18: Let $X = \amalg_{i\in I} X_i$ be a topological sum. Show that the inclusions $\iota^i : X_i \to X$ induce isomorphisms

$$I_n : \oplus_{i\in I} H_n(X_i) \xrightarrow{\sim} H_n(X) \qquad n \in \mathbb{N}_0.$$

Exercise 19: Let $X = T \setminus \mathring{D}^2$ be the torus with an open disc $\mathring{D}^2$ removed and $A \subset T \setminus \mathring{D}^2$ the boundary of the disc. The fundamental group of X is the free group with two generators

$$\pi_1(X) = F_2 = \langle a, b \rangle.$$

Show that there is no retraction $r : X \to A$.



Hint: A **retraction** $r : X \to A$ is a continuous map with $r \circ \iota_A = \mathrm{id}_A$, where $\iota_A : A \to X$ is the inclusion. Consider the group homomorphism $H_1(\iota_A) : H_1(A) \to H_1(X)$.

Exercise 20: A **Δ-complex** or **semisimplicial complex** is a topological space X, together with a family $\{\sigma_\alpha\}_{\alpha \in I}$ of continuous maps $\sigma_\alpha : \Delta^{n_\alpha} \to X$ such that:

(S1) The maps $\sigma_\alpha|_{\mathring{\Delta}^{n_\alpha}} : \mathring{\Delta}^{n_\alpha} \to X$ are injective for all $\alpha \in I$.

(S2) For every point $x \in X$ there is a unique $\alpha \in I$ with $x \in \sigma_\alpha(\mathring{\Delta}^{n_\alpha})$.

(S3) For every $\alpha \in I$ and $i \in \{0, ..., n_\alpha\}$ there is a $\beta \in I$ with $\sigma_\alpha \circ f_i^{n_\alpha} = \sigma_\beta : \Delta^{n_\alpha - 1} \to X$.

(S4) The topology on X is the final topology induced by the family $\{\sigma_\alpha\}_{\alpha \in I}$: A subset $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A) \subset \Delta^{n_\alpha}$ is open for all $\alpha \in I$.

The abelian group of **simplicial n-chains** is is trivial for $n < 0$ and the free abelian group

$$S_n(\Delta, k) = \langle \{\sigma_\alpha \mid \alpha \in I, n_\alpha = n\} \rangle_{\mathbb{Z}} \qquad n \in \mathbb{N}_0.$$

The **simplicial boundary operators** are

$$d_n : S_n(\Delta, k) \to S_{n-1}(\Delta, k), \quad \sigma_\alpha \mapsto \sum_{i=0}^{n} (-1)^i \, \sigma_\alpha \circ f_i^n$$

The nth **simplicial homology** of Δ is

$$H_n(\Delta, k) = \frac{\ker(d_n)}{\mathrm{im}\,(d_{n+1})}.$$

(a) Specify a semisimplicial complex structure on the topological space X formed by the edges and vertices of the standard 3-simplex. Compute its simplicial homologies.

(b) Specify a semisimplicial complex structure on the Klein bottle K and compute its simplicial homologies.

Hint: The Klein bottle is the quotient space $[0,1] \times [0,1] \sim$ with respect to the equivalence relation $(x, 0) \sim (x, 1)$ for all $x \in [0,1]$ and $(0, y) \sim (1, 1-y)$ for all $y \in [0,1]$.

Exercise 21: Let $n \in \mathbb{N}$. Show that $H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong H_k(D^n, S^{n-1})$ for all $k \in \mathbb{Z}$, but the pairs $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ and (D^n, S^{n-1}) are not homotopy equivalent in Top(2).

Exercise 22: Let I be an index set, $(X_i, A_i)_{i\in I}$ a family of objects in Top(2). Show that the relative homologies of the pair $(\amalg_{i\in I} X_i, \amalg_{i\in I A_i})$ are given by

$$H_n(\amalg_{i\in I} X_i, \amalg_{i\in I} A_i) \cong \oplus_{i\in I} H_n(X_i, A_i).$$

Exercise 23:
Let X be a topological space and $x \in X$. Use the long exact homology sequence to show

(a) $H_n(X) \cong H_n(X, \{x\})$ for all $n \in \mathbb{N}$.
(b) $H_0(X, \{x\}) \cong \ker \epsilon$, where $\epsilon : H_0(X) \to \mathbb{Z}$ with $\epsilon([x]) = 1$ and $\epsilon([x']) = 0$, if $x' \in X$ is not in the same path component as $x \in X$.

Exercise 24:
Show that $H_0(\mathbb{R}, \mathbb{Q}) = 0$ and that $H_1(\mathbb{R}, \mathbb{Q})$ is a free abelian group. Give a basis of $H_1(\mathbb{R}, \mathbb{Q})$.

Exercise 25: Let (X, A) be a pair of topological spaces.

(a) Show that $H_0(X, A) = 0$ if and only if every path-component of X contains a point in A.
(b) Show that $H_1(X, A) = 0$ if and only if the map $H_1(\iota) : H_1(A) \to H_1(X)$ is surjective and every path-component of X contains at most one path-component of A.

Exercise 26: Let $\mathring{D}^n = \{x \in \mathbb{R}^n \mid ||x|| < 1\}$ the open n-disc and $K \subset \mathring{D}^n$ a compact convex subset with $0 \in K$. Show that the inclusion $i : \mathring{D}^n \setminus K \to \mathring{D}^n \setminus \{0\}$ is a homotopy equivalence.

Exercise 27: The **cone** over a topological space X is the space

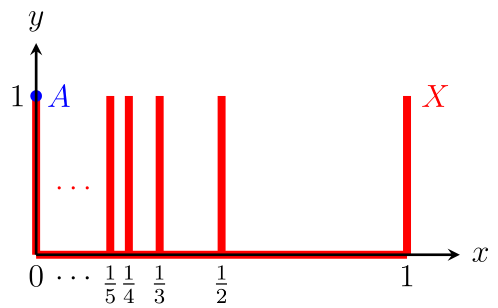
$$C(X) = (X \times [0,1])/\sim \qquad (x, 0) \sim (x', 0) \text{ for } x, x' \in X.$$

True or false?

(a) $S^{n-1} \times \{0\}$ is a strong deformation retract of $S^n \setminus \{e_{n+1}, -e_{n+1}\}$.
(b) $S^2 \setminus \{e_2, -e_2\}$ is a retract of $S^2 \setminus \{e_2\}$.
(c) For all topological spaces X the space $X \times \{1\}$ is a retract of $C(X)$.
(d) For all topological spaces X the space $C(X)$ is contractible.

Exercise 28: We consider the subspace

$$X = ([0,1] \times \{0\}) \cup (\{0\} \times [0,1]) \cup \bigcup_{n=1}^{\infty} \{\tfrac{1}{n}\} \times [0,1] \subset \mathbb{R}^2 \qquad A = \{e_2\} \subset X$$



(a) Show that A is a deformation retract of X.
(b) Show that A is not a strong deformation retract of X.

Exercise 29: Show that that for any affine-linear simplex $\sigma = [v_0, \ldots, v_n] : \Delta^n \to \Delta^p$ the barycentric subdivision is given by

$$B_n(\sigma) = \sum_{\pi \in S_{n+1}} (-1)^n \mathrm{sgn}(\pi)[v_0^\pi, \ldots, v_n^\pi] \qquad v_r^\pi = \frac{1}{r+1} \sum_{j=0}^{r} v_{\pi(j)},$$

where we identify $S_{n+1} = \mathrm{Aut}\{0, \ldots, n\}$.

Exercise 30: Compute the relative homologies $H_k(R, T)$ for $k \in \mathbb{N}_0$ and

$$R = \{x \in \mathbb{R}^n \mid 1 \le ||x|| \le 3\}$$
$$T = \{x \in \mathbb{R}^n \mid 1 \le ||x|| \le \tfrac{3}{2}\} \cup \{x \in \mathbb{R}^n \mid \tfrac{5}{2} \le ||x|| \le 3\} \quad n \in \mathbb{N}.$$

Exercise 31: Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2 - y^2$ and for $a \in \mathbb{R}$ the sets

$$M_a = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) \le a\}.$$

Compute the relative homologies $H_n(M_1, M_{-1})$ for $n \in \mathbb{N}_0$.

Exercise 32: Give an example of a pair (X, A) of topological spaces that is not a good pair.

Exercise 33:
Let $\emptyset \neq U \subset \mathbb{R}^n$ be an open subset. Compute for $x \in U$ the relative homologies $H_k(U, U \setminus \{x\})$.

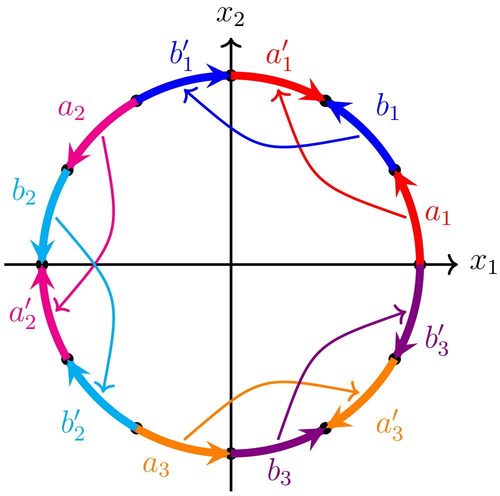
Exercise 34: Prove that the wedge sum has the following universal property:

The maps $i_j = \pi \circ \iota_j : X_j \xrightarrow{\iota_j} \amalg_{i \in I} X_i \xrightarrow{\pi} \vee_{i \in I} X_i$ are continuous for all $j \in I$.
For every family $(f_i)_{i \in I}$ of continuous maps $f_i : X_i \to Y$ with $f_i(x_i) = f_j(x_j)$ for all $i, j \in I$ there is a unique continuous map $f : \vee_{i \in I} X_i \to Y$ with $f \circ i_j = f_j$ for all $j \in I$.

Exercise 35: Compute the homology groups of an oriented surface Σ_g of genus $g \ge 1$ without using any results on fundamental groups.

You can realise Σ_g as the quotient $\Sigma_g = D^2/\sim$, where $D^2 = \{z \in \mathbb{C} \mid |z| \le 1\}$ and

$$e^{2\pi \mathrm{i}(4k+x)/4g} \sim e^{2\pi \mathrm{i}(4k+3-x)/4g}, \quad e^{2\pi \mathrm{i}(4k+1+x)/4g} \sim e^{2\pi \mathrm{i}(4k+4-x)/4g} \quad \text{für } x \in [0,1], k \in \{0, ..., g-1\}.$$



Exercise 36: Show that every continuous map $f : \mathbb{RP}^{2m} \to \mathbb{RP}^{2m}$, $m \in \mathbb{N}$, has a fixed point.

Exercise 37: Show that for any continuous map $f : S^n \to S^n$ the map $f' : D^{n+1} \to D^{n+1}$

$$f'(x) = \begin{cases} 0 & x = 0 \\ ||x|| \cdot f\left(\frac{x}{||x||}\right) & x \neq 0 \end{cases}$$

is continuous and induces a continuous map $\tilde{f} : S^{n+1} \cong D^{n+1}/\partial D^{n+1} \to S^{n+1} \cong D^{n+1}/\partial D^{n+1}$ with $\deg(\tilde{f}) = \deg(f)$.

Exercise 38: The **Riemann sphere** $S^2 = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the one-point compactification of $\mathbb{C}$. Any normalised complex polynomial $p = a_0 + \ldots + a_{n-1}z^{n-1} + z^n$ defines a continuous map

$$f_p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \quad z \mapsto a_0 + \ldots + a_{n-1}z^{n-1} + z^n$$

with $f_p(\infty) = \infty$ for $n > 0$ and $f_p(\infty) = 1$ for $n = 0$.

(a) Show that for any normalised complex polynomial $p = a_0 + \ldots + a_{n-1}z^{n-1} + z^n$ the map $f_p : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is homotopic to f_{z^n}.

(b) Show that the mapping degree of the map f_p for $p = a_0 + \ldots + a_{n-1}z^{n-1} + z^n$ is $\deg(p) = n$.

Hint: In (b) you can assume without proof that the map $f : S^1 \to S^1$, $z \mapsto z^n$ has mapping degree $\deg(f) = n$. This follows from Huréwicz's Theorem.

Exercise 39: An action of a group G on a topological space X is a group homomorphism $\rho : G \to \mathrm{Homeo}(X)$ into the group of homeomorphisms of X. It is called **free** if $\rho(g) : X \to X$ has no fixed points for all $g \in G \setminus \{e\}$.

(a) Show: an action of a group G on S^n, $n \geq 1$, induces a group homomorphism $\phi : G \to \mathbb{Z}/2\mathbb{Z}$.

(b) Show: for even $n \in \mathbb{N}$ the group $\mathbb{Z}/2\mathbb{Z}$ is the only non-trivial group that acts freely on S^n.

Exercise 40: A continuous map $f : S^n \to S^n$ is called **even** if $f(-x) = f(x)$ for all $x \in S^n$. Prove the following claims:

(a) For n even: if f is even then $\deg(f) = 0$.
(b) For n odd: if f is even then $\deg(f)$ even.
(c) For n odd: for any even $k \in \mathbb{Z}$ there is an even map $f : S^n \to S^n$ with $\deg(f) = k$.

Hint: Use that any even map $f : S^n \to S^n$ can be expressed as $f = h \circ \bar{f}$ with continuous maps $\bar{f} : S^n \to \mathbb{R}\mathrm{P}^n$ and $h : \mathbb{R}\mathrm{P}^n \to S^n$. Consider the group homomorphism $H_n(f) : H_n(S^n) \to H_n(S^n)$.

Exercise 41: In this exercise we prove that the Mayer-Vietoris sequence can be derived directly from the excision axiom.

(a) Consider a commuting diagram of abelian groups with exact rows, in which all group homomorphisms ϕ''_n are isomorphisms

$$\begin{tikzcd}
\ldots \arrow[r, "p_{n+1}"] & A''_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\phi''_{n+1}", "\cong"'] & A_n \arrow[r, "i_n"] \arrow[d, "\phi_n"] & A'_n \arrow[r, "p_n"] \arrow[d, "\phi'_n"] & A''_n \arrow[r, "\partial_n"] \arrow[d, "\phi''_n", "\cong"'] & A_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "\phi_{n-1}"] & \ldots \\
\ldots \arrow[r, "q_{n+1}"'] & B''_{n+1} \arrow[r, "\partial_{n+1}"'] & B_n \arrow[r, "j_n"'] & B'_n \arrow[r, "q_n"'] & B''_n \arrow[r, "\partial'_n"'] & B_{n-1} \arrow[r, "j_{n-1}"'] & \ldots
\end{tikzcd}$$

Show that this defines a long exact sequence

$$\ldots \to A_n \xrightarrow{(i_n,\phi_n)} A'_n \oplus B_n \xrightarrow{\phi'_n - j_n} B'_n \xrightarrow{\partial_n \circ (\phi''_n)^{-1} \circ q_n} A_{n-1} \to \ldots$$

(b) Let X be a topological space and $U_1, U_2 \subset X$ open with $X = U_1 \cup U_2$. Apply (a) and the excision axiom to derive the associated Mayer-Vietoris sequence.

Hint: Consider the long exact homology sequences of the pairs (X, U_1) and $(U_2, U_1 \cap U_2)$

7.4 Exercises for Section 4

Exercise 42: Show that pushouts of topological spaces are determined uniquely up to unique homeomorphisms by their universal property:

If $(X_1 \amalg_A X_2, i_1, i_2)$ and $(X_1 \amalg_A X_2, i'_1, i'_2)$ are both pushouts of continuous maps $f_1 : A \to X_1$ and $f_2 : A \to X_2$, then is a unique continuous map $\phi : X_1 \amalg_A X_2 \to X_1 \amalg'_A X_2$ with $\phi \circ i_j = i'_j$ and ϕ is a homeomorphism.

Exercise 43: Consider the following commuting diagram in Top

$$\begin{tikzcd}
Y & Y_1 \arrow[l, "i'_1"'] & X_1 \arrow[l, "i_1"'] \\
Y_2 \arrow[u, "i'_2"] & X_2 \arrow[l, "g_2"] \arrow[u, "i_2"'] & A \arrow[l, "f_2"] \arrow[u, "f_1"']
\end{tikzcd}$$

and suppose that the right square is a pushout. Show that the left square is a pushout if and only if the outer rectangle is a pushout.

Exercise 44: Let I be an index set and suppose we have for all $i \in I$ pushouts

$$\begin{tikzcd} Z_i & Y_i \arrow[l, "h_i"'] \\ X_i \arrow[u, "k_i"] & A_i. \arrow[l, "f_i"] \arrow[u, "g_i"'] \end{tikzcd}$$

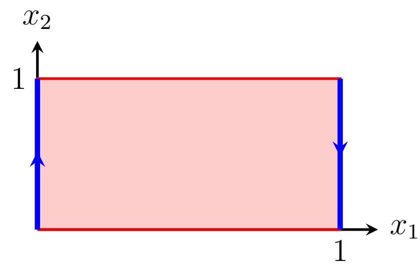
Show that this defines a pushout

$$\begin{tikzcd} \amalg_{i\in I} Z_i & \amalg_{i\in I} Y_i \arrow[l, "h"'] \\ \amalg_{i\in I} X_i \arrow[u, "k"] & \amalg_{i\in I} A_i. \arrow[l, "f"] \arrow[u, "g"'] \end{tikzcd}$$

Exercise 45: Let X be a CW complex. True or false?

(a) Each n-cell of X is open in X.
(b) The closure of an n-cell $\sigma = c_i(\mathring{D}^n)$ in X is $\overline{\sigma} = c_i(D^n)$.
(c) if σ is an n-cell of X and τ an m-cell of X with $n \neq m$, then σ and τ are not homeomorphic.

Exercise 46: The **Möbius strip** is the quotient space $M = [0,1]^{\times 2}/\sim$ with $(0,x) \sim (1, 1-x)$ for all $x \in [0,1]$. Show that it has a CW complex structure with a single 0-cell, two 1-cells and a single 2-cell.



Exercise 47: Let $X_0 \subset X_1 \subset X_2 \subset \ldots$ a sequence of subspaces. Suppose that X_i is a strong deformation retract of X_{i+1} for all $i \in \mathbb{N}_0$. Show that X_0 is a strong deformation retract of X_i for all $i \in \mathbb{N}_0$.

Exercise 48: Let $X_0 \subset X_1 \subset \ldots$ be a family of subspaces and $X = \cup_{n=0}^{\infty} X_n$ equipped with the final topology induced by the inclusions $\iota_n : X_n \to X$.

Show that X has the following universal property: for every family $(f_n)_{n\in\mathbb{N}}$ of continuous maps $f_n : X_n \to Y$ with $f_{n+1}|_{X_n} = f_n$ for all $n \in \mathbb{N}_0$, there is a unique continuous map $f : X \to Y$ with $f \circ \iota_n = f_n$ for all $n \in \mathbb{N}_0$.

Exercise 49: Let $X_0 \subset X_1 \subset \ldots$ be a sequence of subspaces and $X = \cup_{n=0}^{\infty} X_n$ their union with the final topology induced by the inclusions $\iota_n : X_n \to X$. Let $A \subset X$ be an *open* subspace and $A_n = A \cap X_n$ for all $n \in \mathbb{N}_0$. Show that the topology on A is the final topology induced by the inclusions $\iota'_n : A_n \to A$.

Exercise 50: Let X, Y be CW-complexes and $f : X \to Y$ a **cellular map**: a continuous map $f : X \to Y$ with $f(X^n) \subset Y^n$ for all $n \in \mathbb{N}_0$. Show that f induces a chain map $C_\bullet(f) : C_\bullet(X) \to C_\bullet(Y)$ between the cellular complexes $C_\bullet(X)$ and $C_\bullet(Y)$ such that the following diagram commutes for all $n \in \mathbb{N}_0$

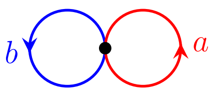
$$\begin{tikzcd}
H_n(C_\bullet(X)) \arrow[r, "H_n(C_\bullet(f))"] \arrow[d, "\cong"] & H_n(C_\bullet(Y)) \arrow[d, "\cong"] \\
H_n(X) \arrow[r, "H_n(f)"'] & H_n(Y).
\end{tikzcd}$$

Exercise 51:

(a) Let $f : S^n \to S^n$ be a continuous map of degree $\deg(f) = m$ and X the topological space obtained by attaching D^{n+1} to S^n with f. Compute the homologies of X.

(b) Let $(A_n)_{n\in\mathbb{N}}$ be a family of finitely generated abelian groups A_n. Show that there is a path-connected topological space X with $H_n(X) = A_n$ for all $n \in \mathbb{N}$.

Exercise 52: Let X be the CW complex obtained from the following bouquet $S^1 \vee S^1$



by attaching two 2-discs D^2 with the attaching maps given by the words a^5b^{-3} and $b^3(ab)^{-2}$.

(a) compute the homologies $H_n(X)$ for $n = 1, 2$.

(b) **bonus exercise:** show that X is not contractible.

Exercise 53: The **rank** of a finitely generated abelian group is

$$A \cong \mathbb{Z}^n \times \mathbb{Z}/p_1^{n_1}\mathbb{Z} \times \ldots \times \mathbb{Z}/p_k^{n_k}\mathbb{Z} \quad \Rightarrow \quad \mathrm{rk}(A) = n.$$

The **Euler characteristic** of a finite CW complex X is

$$\chi(X) = \sum_{n\in\mathbb{N}_0} (-1)^n \mathrm{rk}(H_n(X)).$$

(a) Let $I_n(X)$ be the set of n-cells of X. Show that

$$\chi(X) = \sum_{n\in\mathbb{N}_0} (-1)^n |I_n(X)|.$$

(b) Compute the Euler characteristic of an oriented surface of genus $g \geq 0$ and of $\mathbb{RP}^2$.

7.5 Exercises for Section 5

Exercise 54: Let $m, n \in \mathbb{N}$, $m, n \geq 2$, denote by $\gcd(m,n)$ the greatest common divisor of m and n and by $\mathrm{Tor}_n(A) = \{a \in A \mid n \cdot a = 0\}$ the n-torsion subgroup of an abelian group A.

(a) Show that the tensor product of the abelian groups $\mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ is given by

$$\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}.$$

(b) Show that $\mathrm{Tor}_n(\mathbb{Z}/m\mathbb{Z}) \cong \mathrm{Tor}_m(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(n,m)\mathbb{Z}$.

Exercise 55: True or False?

(a) If A, B are free abelian groups, then $A \otimes B$ is a free abelian group.
(b) For every non-trivial abelian group A there is an abelian group B with $A \otimes B \cong \mathbb{Z}$.
(c) If A, B are finite abelian groups then $|A \otimes B| \leq |A| \cdot |B|$.
(d) There is a non-trivial abelian group A with $A \otimes B = 0$ for all finite abelian groups B.

Exercise 56: Give an example of an abelian group A and a family $(B_i)_{i\in I}$ of abelian groups with $A \otimes (\Pi_{i\in I} B_i)$ not isomorphic to $\Pi_{i\in I}(A \otimes B_i)$.

Exercise 57: For a chain complex $X_\bullet$ and an abelian group A, we consider the chain complex

$$X_\bullet \otimes A = \ldots \xrightarrow{d_{n+2}\otimes \mathrm{id}_A} X_{n+1} \otimes A \xrightarrow{d_{n+1}\otimes \mathrm{id}_A} X_n \otimes A \xrightarrow{d_n \otimes \mathrm{id}_A} X_{n-1} \otimes A \xrightarrow{d_{n-1}\otimes \mathrm{id}_A} \ldots$$

Give an example of a chain complex $X_\bullet$ with $H_n(X_\bullet) = 0$ for all $n \in \mathbb{Z}$ and an abelian group A such that $H_n(X_\bullet \otimes A) \neq 0$ for all $n \in \mathbb{Z}$.

Exercise 58: Consider for a prime number $p \in \mathbb{N}$ and $n, m, k \in \mathbb{N}$ the short exact sequence

$$0 \to \mathbb{Z}/p^m\mathbb{Z} \xrightarrow{\iota: \bar{z} \mapsto p^n \bar{z}} \mathbb{Z}/p^{m+n}\mathbb{Z} \xrightarrow{\pi: \bar{z} \mapsto \bar{z}} \mathbb{Z}/p^n\mathbb{Z} \to 0.$$

(a) Compute chain maps between free resolutions that extend ι and π.
(b) Compute for $k \in \mathbb{N}$ the induced maps

$$\begin{aligned} \mathrm{Tor}(\iota, \mathbb{Z}/p^k\mathbb{Z}) &: \mathrm{Tor}(\mathbb{Z}/p^m\mathbb{Z}, \mathbb{Z}/p^k\mathbb{Z}) \to \mathrm{Tor}(\mathbb{Z}/p^{n+m}\mathbb{Z}, \mathbb{Z}/p^k\mathbb{Z}) \\ \mathrm{Tor}(\pi, \mathbb{Z}/p^k\mathbb{Z}) &: \mathrm{Tor}(\mathbb{Z}/p^{m+n}\mathbb{Z}, \mathbb{Z}/p^k\mathbb{Z}) \to \mathrm{Tor}(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{Z}/p^k\mathbb{Z}). \end{aligned}$$

Exercise 59: Let $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ a short exact sequence of abelian groups and $A_\bullet = 0 \to K_A \xrightarrow{i_A} F_A \xrightarrow{p_A} A \to 0$ and $C_\bullet = 0 \to K_C \xrightarrow{i_C} F_C \xrightarrow{p_C} C \to 0$ free resolutions of A and B. Show that there is a free resolution $B_\bullet$ of B such that the short exact sequence lifts to a short exact sequence of chain complexes $0 \to A_\bullet \xrightarrow{\iota_\bullet} B_\bullet \xrightarrow{\pi_\bullet} C_\bullet \to 0$.

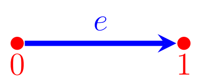
Exercise 60: Let $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ a short exact sequence of abelian groups. Show that for any abelian group D there is an exact sequence

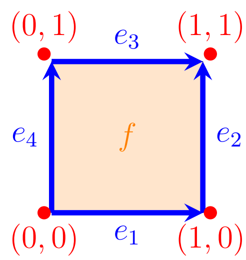
$$0 \to \mathrm{Tor}(A, D) \to \mathrm{Tor}(B, D) \to \mathrm{Tor}(C, D) \to A \otimes D \to B \otimes D \to C \otimes D \to 0.$$

Hint: Use Exercise 59.

Exercise 61: Let $X_\bullet$ be a chain complex in Ab. Show that there is a free chain complex $F_\bullet$ with $H_n(F_\bullet) = H_n(X_\bullet)$.

Exercise 62: Let $I_\bullet$ be the cellular chain complex for $[0, 1]$ with two 0-cells and one 1-cell and $S_\bullet$ the cellular chain complex for $[0, 1]^{\times 2}$ with four 0-cells, four 1-cells and one 2-cell.





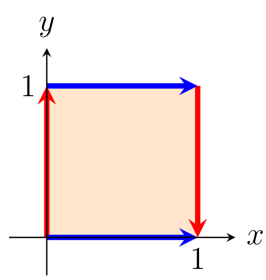
Show that there is a chain isomorphism $f_\bullet : I_\bullet \otimes I_\bullet \to S_\bullet$.

Exercise 63: Let $f_\bullet, f'_\bullet : X_\bullet \to X'_\bullet$ and $g_\bullet, g'_\bullet : Y_\bullet \to Y'_\bullet$ chain maps.

(a) Show that every pair $(h_\bullet, k_\bullet)$ of chain homotopies $h_\bullet : f_\bullet \Rightarrow f'_\bullet$ and $k_\bullet : g_\bullet \Rightarrow g'_\bullet$ induces a chain homotopy $h_\bullet \otimes k_\bullet : f_\bullet \otimes g_\bullet \Rightarrow f'_\bullet \otimes g'_\bullet$.

(b) Show that for chain homotopy equivalent chain complexes $X_\bullet \simeq X'_\bullet$ and $Y_\bullet \simeq Y'_\bullet$ the chain complexes $X_\bullet \otimes Y_\bullet$ and $X'_\bullet \otimes Y'_\bullet$ are chain homotopy equivalent.

Exercise 64: The Klein bottle is the quotient space $K = [0, 1]^{\times 2} / \sim$ with $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, 1 - y)$ for all $x, y \in [0, 1]$.



(a) Compute the homologies of K with coefficients in $\mathbb{Z}$ using cellular homology.

(b) Compute the homologies of K with coefficients in a general abelian group M.

Exercise 65: Let X be a CW complex of finite type and M a torsion free abelian group. Show that $H_n(X;M) = H_n(X) \otimes M$ for all $n \in \mathbb{N}_0$.

Exercise 66: Prove the third claim of the Eilenberg-Zilber Theorem.

Exercise 67: Let A be a free abelian group and X a finite CW complex with

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0 \\ A & k = n \\ 0 & \text{else} \end{cases}$$

for some $n \in \mathbb{N}$. Compute the homologies of the space $X^{\times m} = \underbrace{X \times \ldots X}_{m\times}$ for all $m \in \mathbb{N}$.

7.6 Exercises for Section 6

Exercise 68: Show that for all abelian groups M and families $(A_i)_{i\in I}$ of abelian groups

$$\mathrm{Ext}(\oplus_{i\in I} A_i, M) = \Pi_{i\in I}\mathrm{Ext}(A_i, M) \qquad \mathrm{Ext}(M, \Pi_{i\in I} A_i) \cong \Pi_{i\in I}\mathrm{Ext}(M, A_i).$$

Exercise 69: An abelian group M is called **injective**, if for every injective group homomorphism $\iota : X \to Y$ and group homomorphism $f : X \to M$ there is a group homomorphism $f' : Y \to M$ with $f' \circ \iota = f$

$$\begin{tikzcd} & M & \\ Y \arrow[ur, dashed, "\exists f'"] & X \arrow[l, hook', "\iota"'] \arrow[u, "f"'] & 0. \arrow[l] \end{tikzcd}$$

(a) Show that $\mathrm{Ext}(A, M) = 0$ for every abelian group A and injective abelian group M.
(b) Show that if M is injective, every short exact sequence $0 \to m \xrightarrow{\iota} N \xrightarrow{\pi} P \to 0$ splits.
(c) Show that every injective abelian group M is divisible.
(d) Show that the axiom of choice implies that every divisible abelian group M is injective.

Hint: In (c) consider the poset of all pairs (W, f'_W) of subgroups $W \subset Y$ and group homomorphisms $f_W : W \to M$ with $f_W \circ \iota = f$. Apply Zorn's Lemma.

Exercise 70: Compute the cohomology rings $H^\bullet(T, \mathbb{Z})$ and $H^\bullet(K, \mathbb{Z})$ of the torus T and of the Klein bottle K with coefficients in $\mathbb{Z}$.

Exercise 71: Let X be a finite path-connected CW complex with only even-dimensional cells and F a field with $\mathrm{char}\,\mathbb{F} = 0$. Show that the cohomology ring $H^\bullet(X, \mathbb{F})$ of X is a quotient of a polynomial algebra $\mathbb{F}[x_1, \ldots, x_k]$ for some $k \in \mathbb{N}$.

Hint: Use the universal property of the polynomial ring over a field $\mathbb{F}$:

For any commutative algebra A over $\mathbb{F}$ and any k-tuple $(a_1, \ldots, a_k)$ of elements $a_i \in A$, there is a unique algebra homomorphism $\phi : \mathbb{F}[x_1, \ldots, x_n] \to A$ with $f(x_i) = a_i$ for all $i = 1, \ldots, k$.

Exercise 72: Let $\mathbb{F}$ be a commutative unital ring and $n \in \mathbb{N}$.

(a) Show that the cohomology ring $H^\bullet(\mathbb{C}\mathrm{P}^n, \mathbb{F})$ is isomorphic to $\mathbb{F}[x]/(x^{n+1})$.

(b) Show that the topological spaces $\mathbb{C}\mathrm{P}^2$ and $S^2 \vee S^4$ have isomorphic cohomology groups $H^k(\mathbb{C}\mathrm{P}^2; \mathbb{F}) \cong H^k(S^2 \vee S^4; \mathbb{F})$ for all $k \in \mathbb{N}_0$, but that they are not homotopy equivalent.

Hint: In (a) assume without proof that the $2n$th cohomology group $H^{2n}(\mathbb{C}\mathrm{P}^n, \mathbb{F})$ is generated by a cup product of elements in lower-dimensional cohomology groups.

Exercise 73: Compute the cohomology ring of the Lie group

$$U(2) = \{M \in \mathrm{Mat}(2 \times 2, \mathbb{C}) \mid M^\dagger = M^{-1}\}.$$

Exercise 74: Let $(I, \preceq)$ be a direct poset and $\mathcal{C}$ a category such that the direct limit $\lim_{\rightarrow} F$ exists for all direct systems $F : \mathcal{I} \to \mathcal{C}$.

Show that the direct limit defines a functor $\lim_{\rightarrow} : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ that assigns

- to a direct system $F : \mathcal{I} \to \mathcal{C}$ its direct limit $\lim_{\rightarrow} F$,
- to a map $\gamma : F \Rightarrow F'$ of direct systems the map $\lim_{\rightarrow} \gamma : \lim_{\rightarrow} F \to \lim_{\rightarrow} F'$.

Exercise 75: Let $(I, \preceq)$ be a direct poset and $F : \mathcal{I} \to \mathrm{Ab}$ a direct system in Ab for some unital ring R.

Show that the direct limit of F is the quotient module

$$\lim_{\rightarrow} F = (\oplus_{i \in I} F_i)/U \qquad U = \langle \cup_{i \preceq j \in I} \{\iota_j \circ f_{ji}(m) - \iota_i(m) \mid m \in F_i\} \rangle$$

where $\iota_i : F_i \to \oplus_{i \in I} F_i$ are the inclusions for the direct sum

Exercise 76: Let R be a unital ring and $F : \mathcal{I} \to \mathrm{Ab}$ be a direct system with direct limit $\lim_{\rightarrow} F$. Suppose that $(g_i)_{i \in I}$ is a family of R-linear *isomorphisms* $g_i : F_i \to M$. Show that the induced R-linear map $g : \lim_{\rightarrow} F \to M$ is an isomorphism as well.

Exercise 77: Let $\mathcal{K}$ be the direct poset of compact subsets $K \subset \mathbb{R}^n$ and $\mathcal{B}$ the direct poset of closed r-balls $B_r = \{x \in \mathbb{R}^n \mid ||x|| \leq r\} \subset \mathbb{R}^n$ for $r > 0$, both with the inclusions.

For a direct system $F : \mathcal{K} \to \mathrm{Ab}$ we denote by $F' = FI : \mathcal{B} \to \mathrm{Ab}$ the direct system induced by the inclusion functor $I : \mathcal{B} \to \mathcal{K}$. Show that

$$\lim_{\rightarrow} F \cong \lim_{\rightarrow} F'$$

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