Quantum Cellular Automata and their Classification

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LQP49, Erlangen

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Italiadomani PIANO NAZIONALE DI RIPRESA E RESILIENZA

Outline

1) **Quantum Cellular Automata (QCAs):**
   origins, motivations, definition, and examples

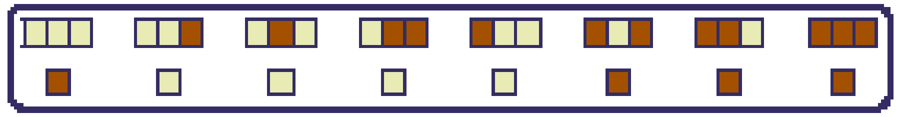
2) **Classification of QCAs:**
   index theory

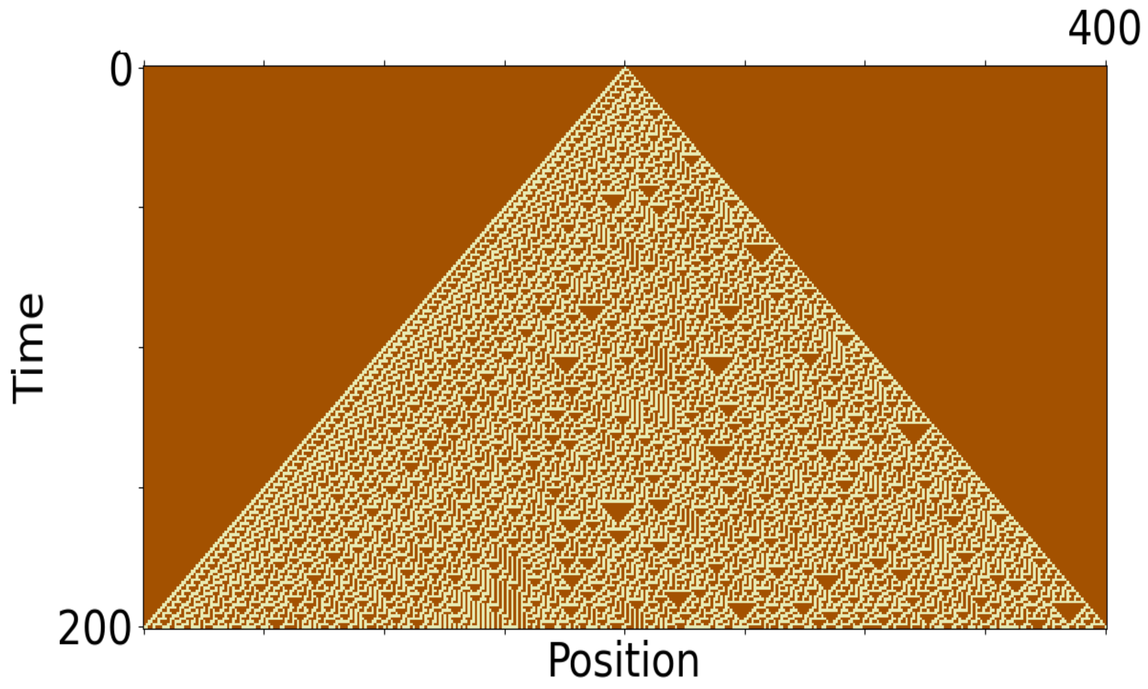
3) **Our results:**
   fermionic and higher dimensional index

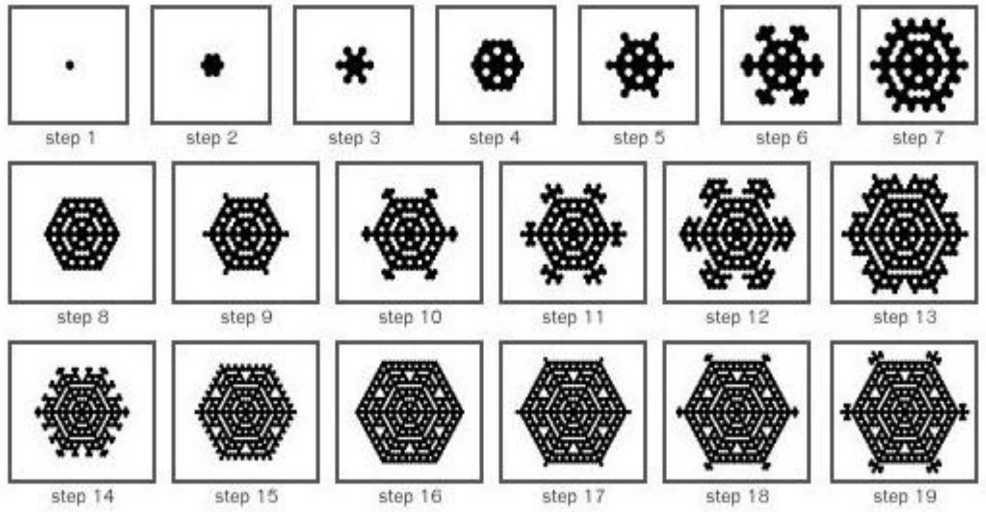
Cellular Automata: origins

J. von Neumann, and A. W. Burks,
Theory of self-reproducing automata,
Urbana, University of Illinois Press (1966)

A Cellular Automaton is a lattice of systems
with some local update rule



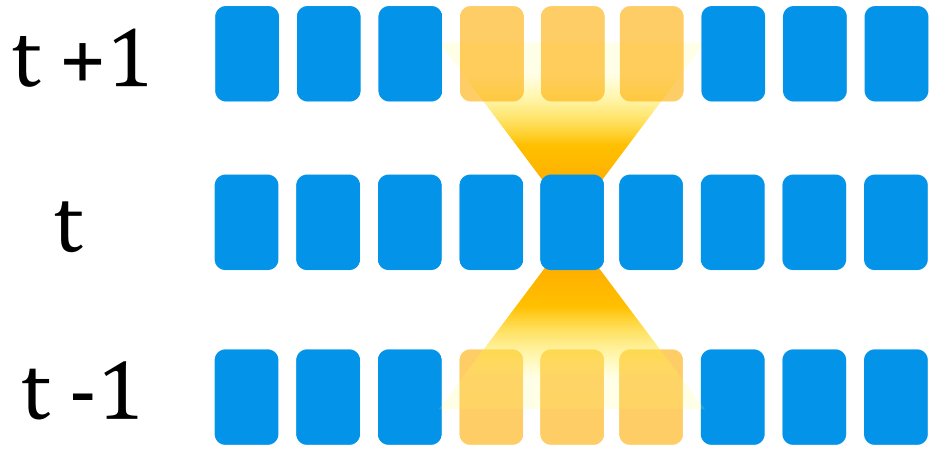




Cellular Automata: modern perspective

R. Feynman,
Simulating physics with computers,
Int. J. Theor. Phys. 21, 1982: pp. 467–488

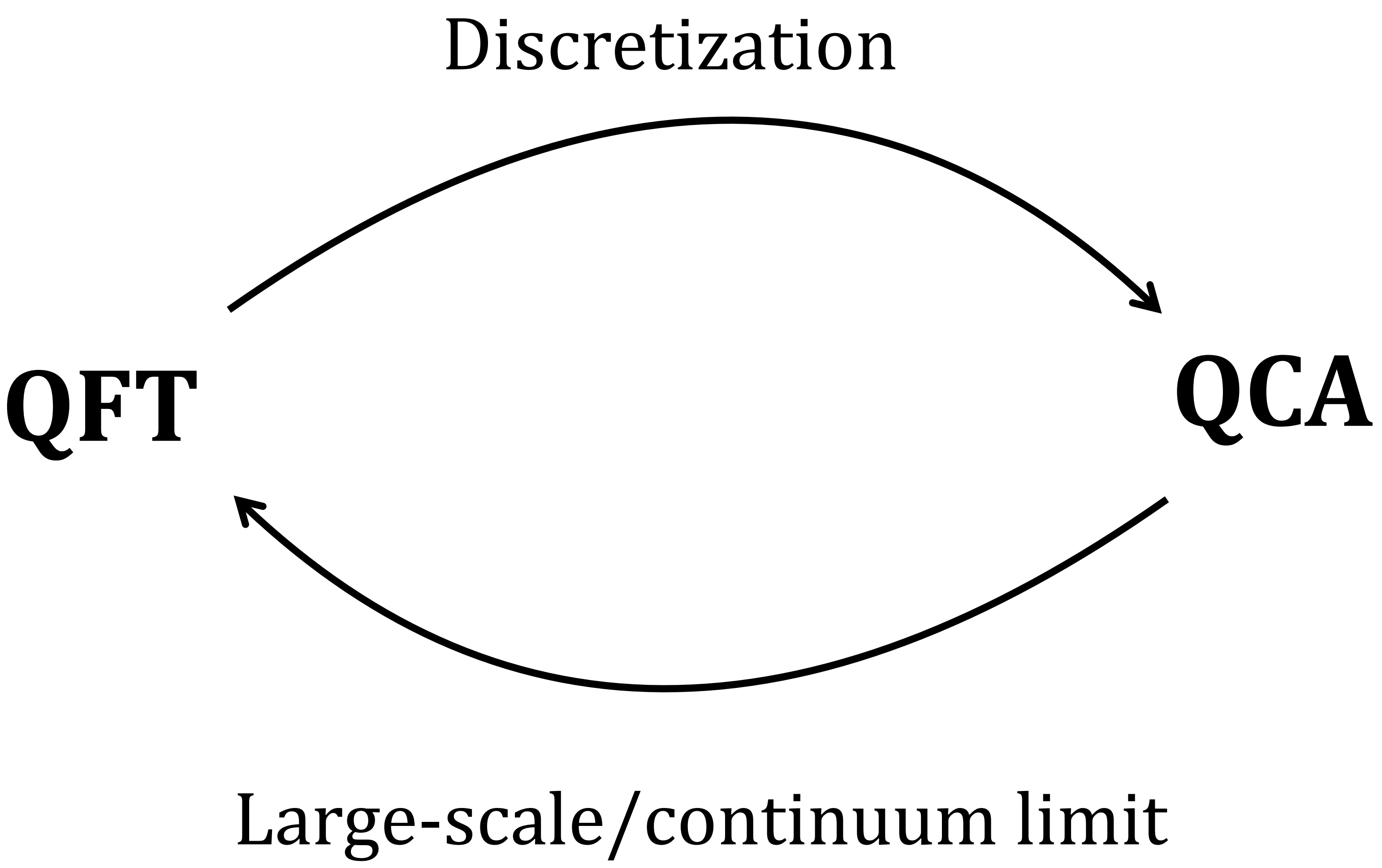
A Quantum Cellular Automaton (QCA) is the most general **discrete-time**, **local**, **reversible** dynamics of a lattice of quantum systems



Several applications:

- computation
- simulation
- topological phases of matter

Physical Motivation



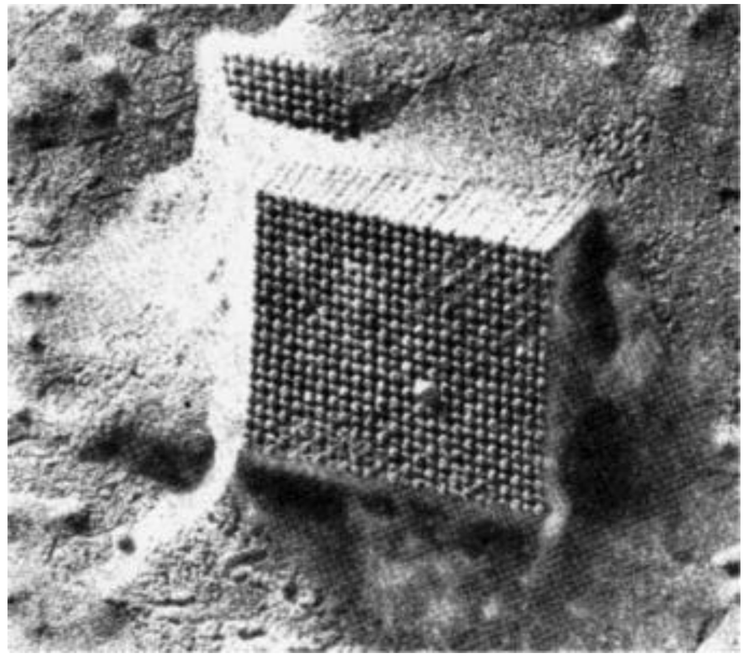
Discretization : Trotterization, Lattice QFT…

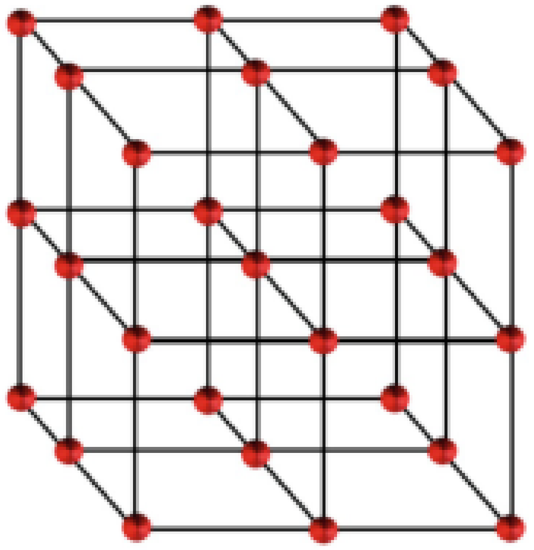
Large-scale/continuum limit: e.g. Dirac QCA, Thirring QCA…

A. Bisio, G. M. D'Ariano, A. Tosini, AOP 354, 244 (2015)
G. M. D'Ariano, P. Perinotti PRA 90, 062106 (2014)
F. Sciarrino, A. Bisio, P. Perinotti, et al. PRR 6, 033136 (2024)
A. Bisio, N. Mosco, P. Perinotti, PRL 126, 250503 (2021)

Natural Playground: many-body physics

Condensed matter models are «naturally» implemented as QCAs





What's new?

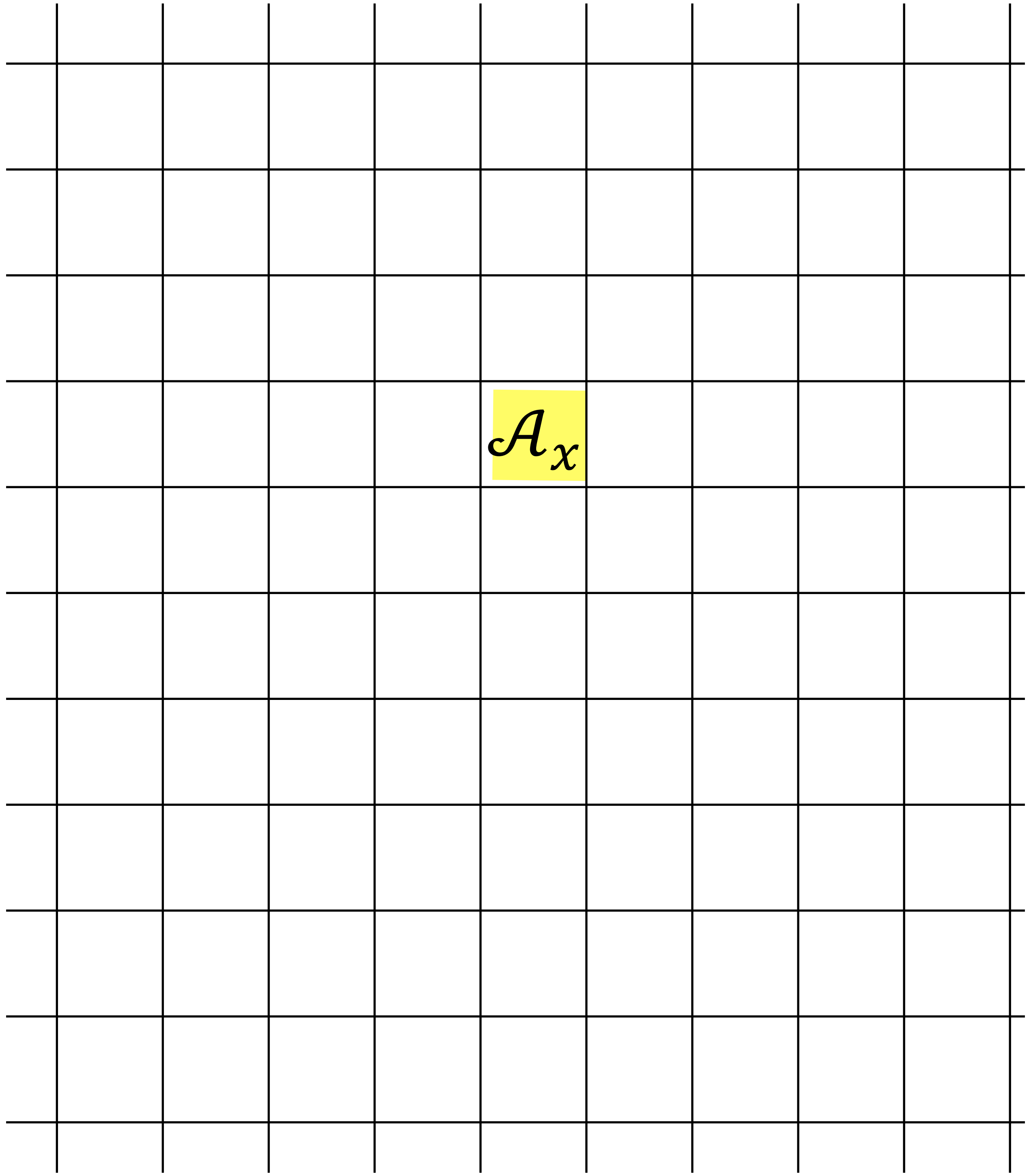
Topological phases classification

QCA: definition

B. Schumacher, R. F. Werner, arXiv:quant-ph/0405174 (2004)

- Lattice of cells, $x \in \mathbb{Z}^n$
- $\mathcal{A}_x$ **finite dim.** C*-algebra at site x

  e.g. qubits $\mathcal{A}_x = \mathcal{L}(\mathbb{C}^2)$



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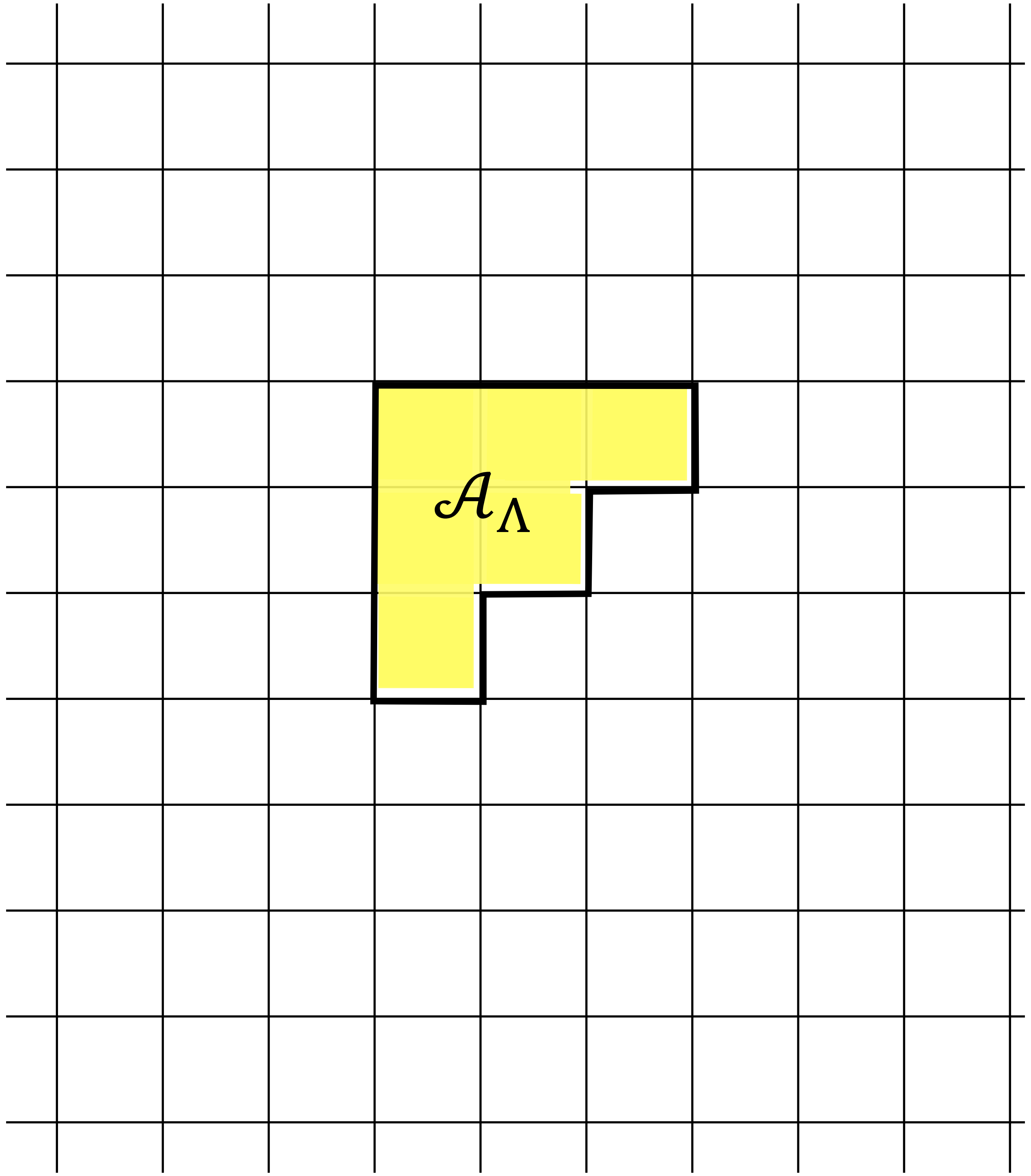
- Local algebras $\mathcal{A}_\Lambda \coloneqq \bigotimes_{x \in \Lambda} \mathcal{A}_x$

  $\Lambda \subset \mathbb{Z}^n$ is a **finite** region



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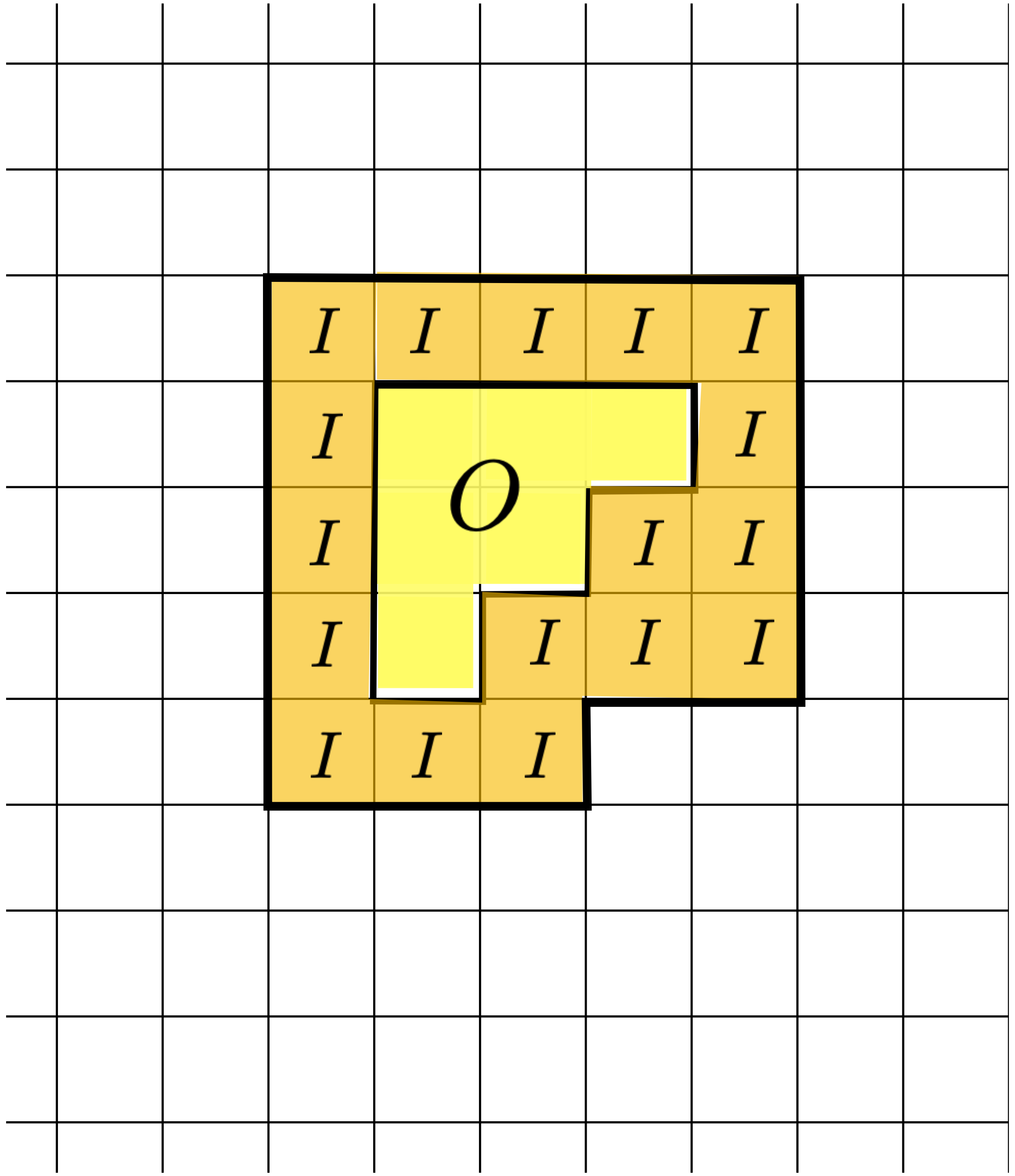
  $\Lambda \subset \mathbb{Z}^n$ is a **finite** region

  for $\Lambda \subset \Lambda'$ $\forall O \in \mathcal{A}_\Lambda$

  $O \otimes I_{\Lambda' \setminus \Lambda} \in \mathcal{A}_{\Lambda'}$



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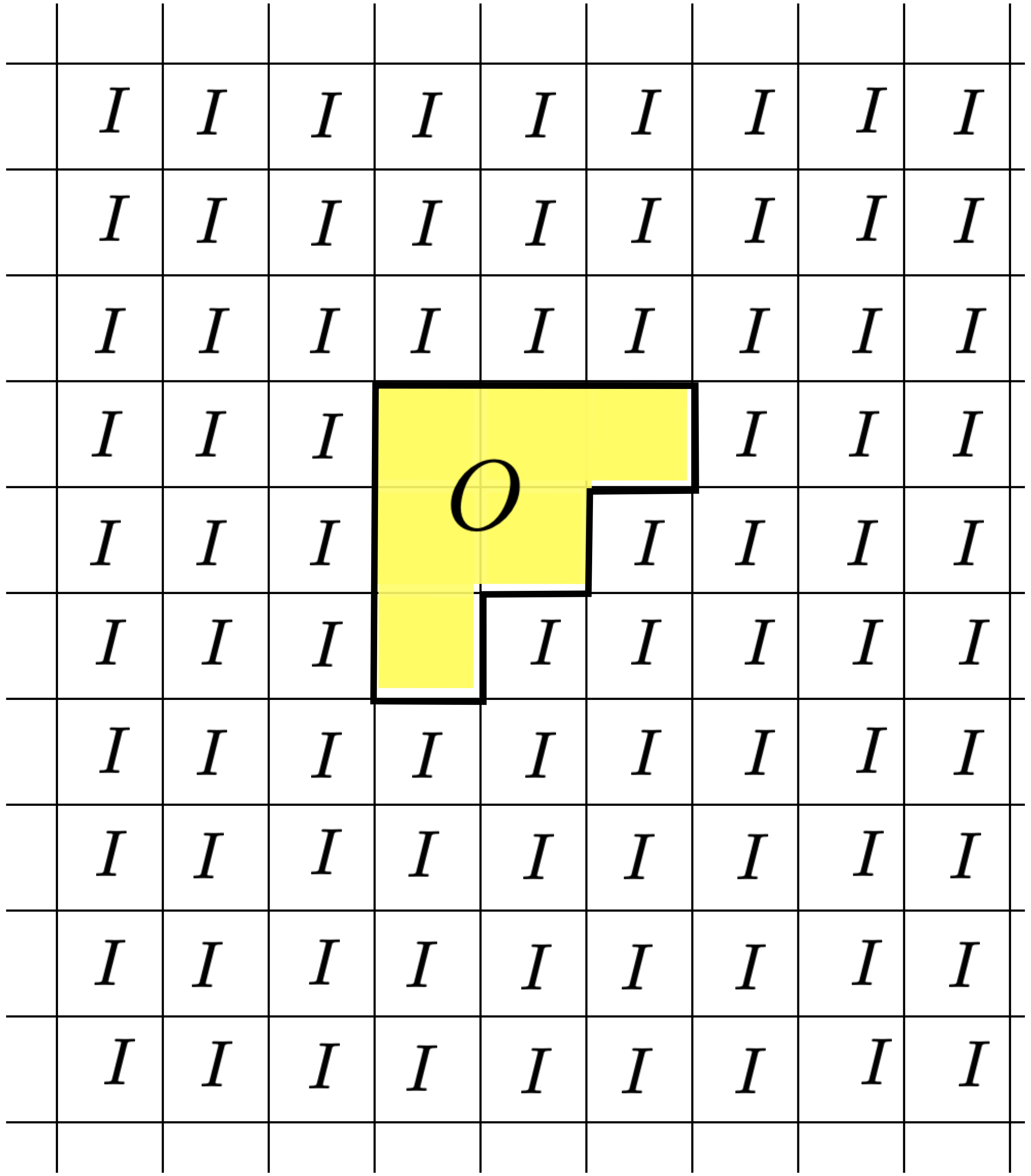
  for $\Lambda \subset \Lambda'$ $\forall O \in \mathcal{A}_\Lambda$

  $$O \otimes I_{\Lambda'\setminus\Lambda} \in \mathcal{A}_{\Lambda'}$$
- **Quasi-Local algebra**

  $$\mathcal{A}_{\mathbb{Z}^n} := \overline{\bigcup_{\Lambda\subset\mathbb{Z}^n} \mathcal{A}_\Lambda}^{\|\cdot\|_\infty}$$



QCA: definition

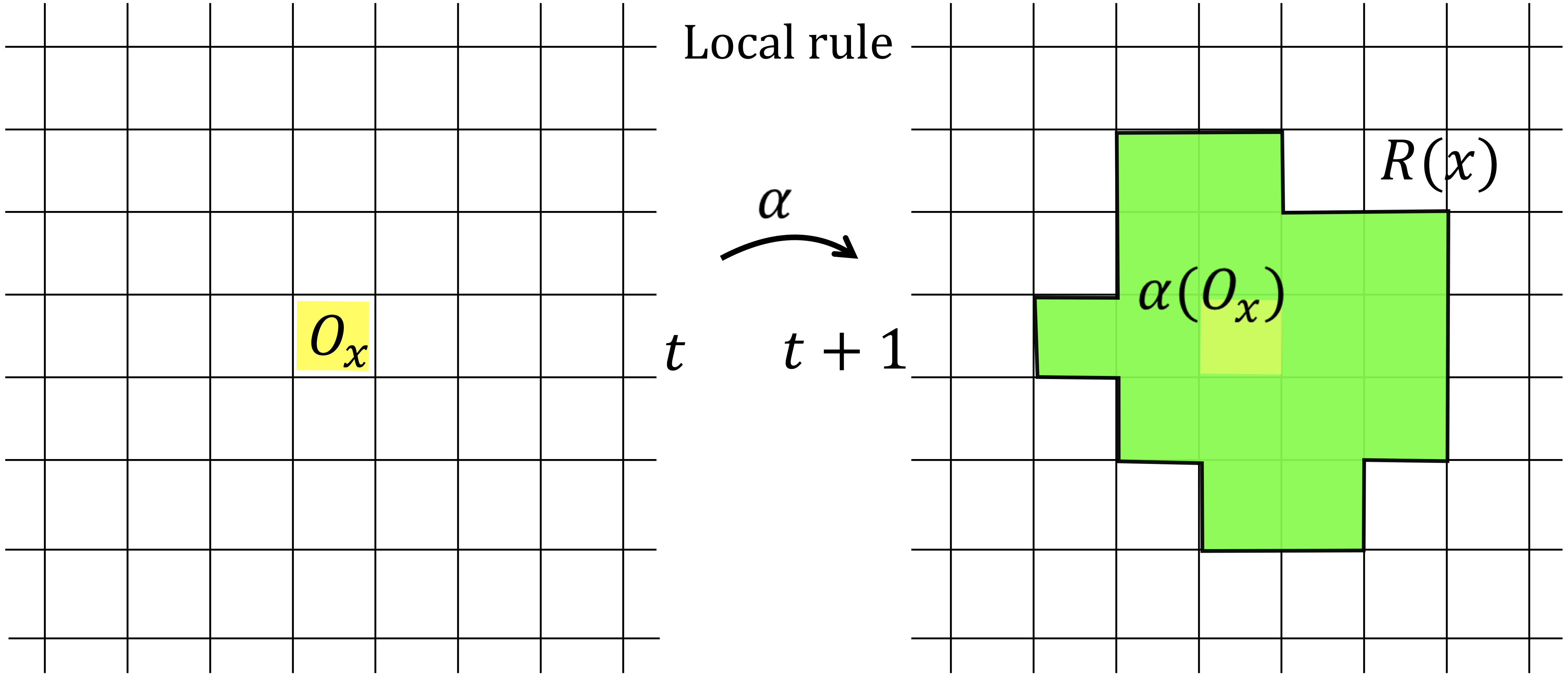
B. Schumacher, R. F. Werner, arXiv:quant-ph/0405174 (2004)

Def. A QCA α is a locality preserving $*$-automorphism of the quasi-local algebra

$$\alpha: \mathcal{A}_{\mathbb{Z}^n} \to \mathcal{A}_{\mathbb{Z}^n} \qquad \alpha(\otimes_{x\in\Lambda} O_x) = \prod_{x\in\Lambda} \alpha(O_x)$$

$$O_x \in \mathcal{A}_x \implies \alpha(O_x) \in \mathcal{A}_{R(x)}$$

uniform bound: same finite R for all x



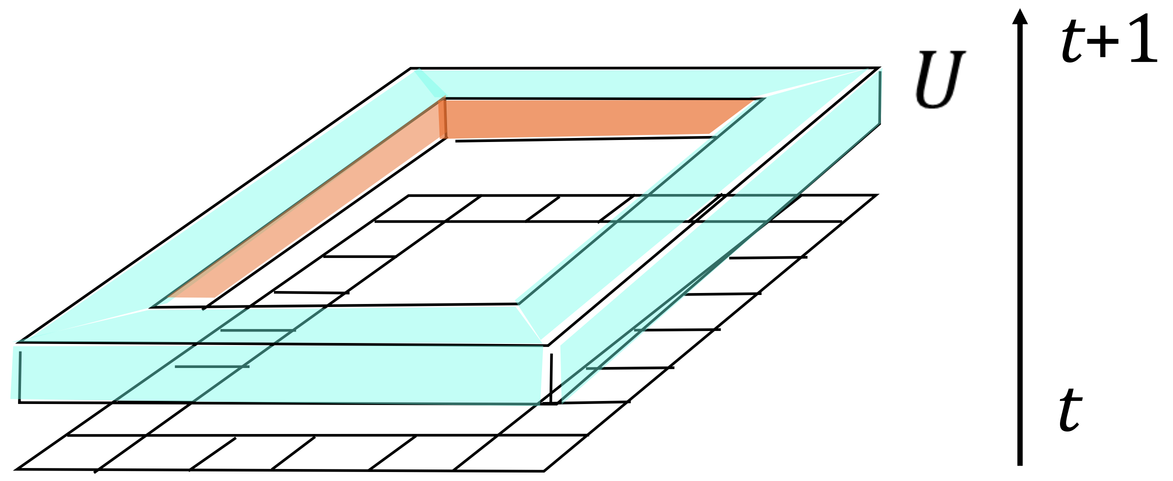
QCA: examples

- Finite lattice $\mathcal{L}$, $\alpha: \mathcal{A}_{\mathcal{L}} \to \mathcal{A}_{\mathcal{L}}$

$$\Downarrow$$

$$\alpha(O) = UOU^* \quad O \in \mathcal{A}_{\mathcal{L}}$$

for some unitary matrix $U \in \mathcal{A}_{\mathcal{L}}$



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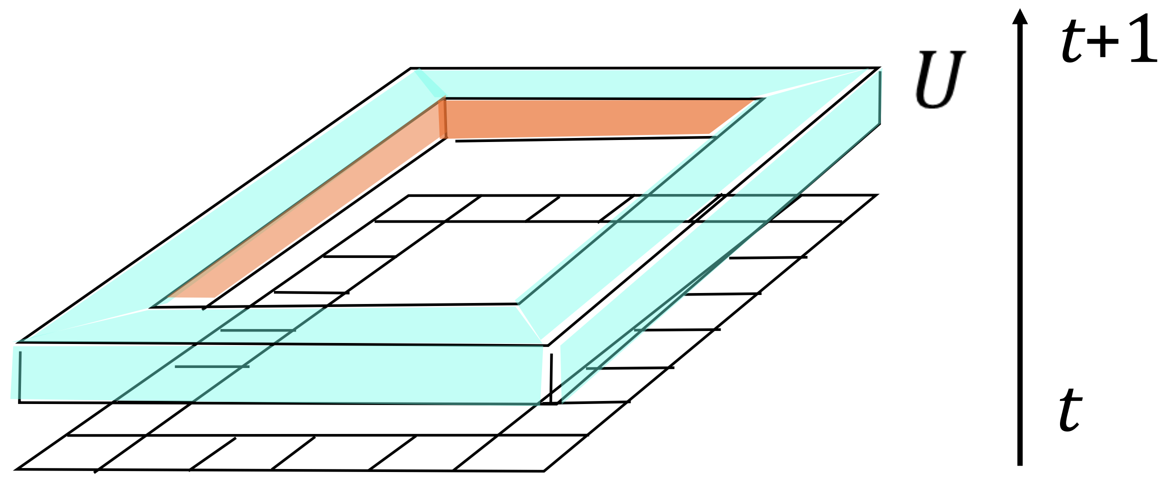
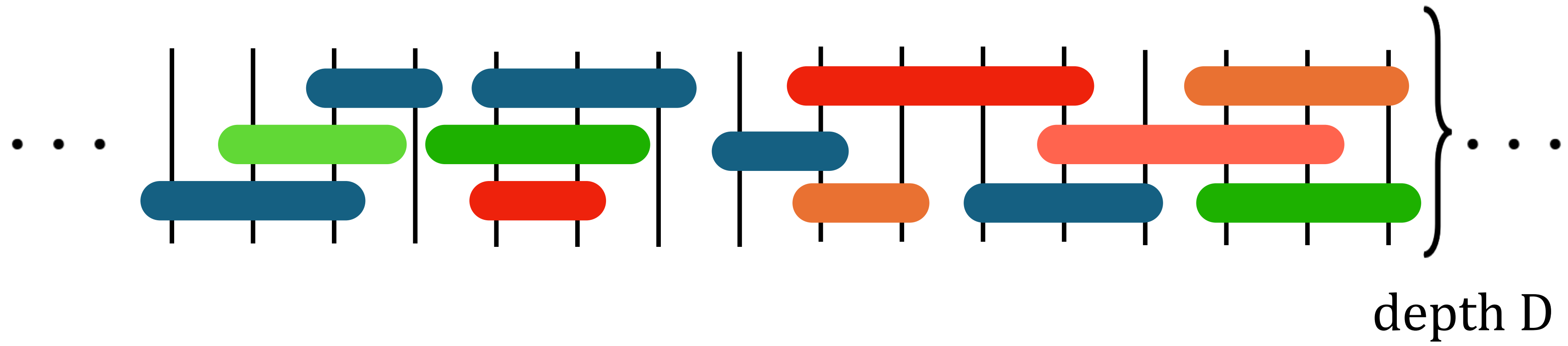


- A Finite-Depth Quantum Circuit (FDQC) $\phi: \mathcal{A}_{\mathbb{Z}^n} \to \mathcal{A}_{\mathbb{Z}^n}$ is a QCA

$$\phi(O) = \prod_{k=1}^{D<\infty} \prod_{\Lambda \in \mathcal{P}_k(\mathbb{Z}^n)} U_\Lambda \, O \, U_\Lambda^* \qquad U_\Lambda \in \mathcal{A}_{\mathbb{Z}^n}$$



FDQC

Finite-Depth Quantum Circuit

QCA: examples

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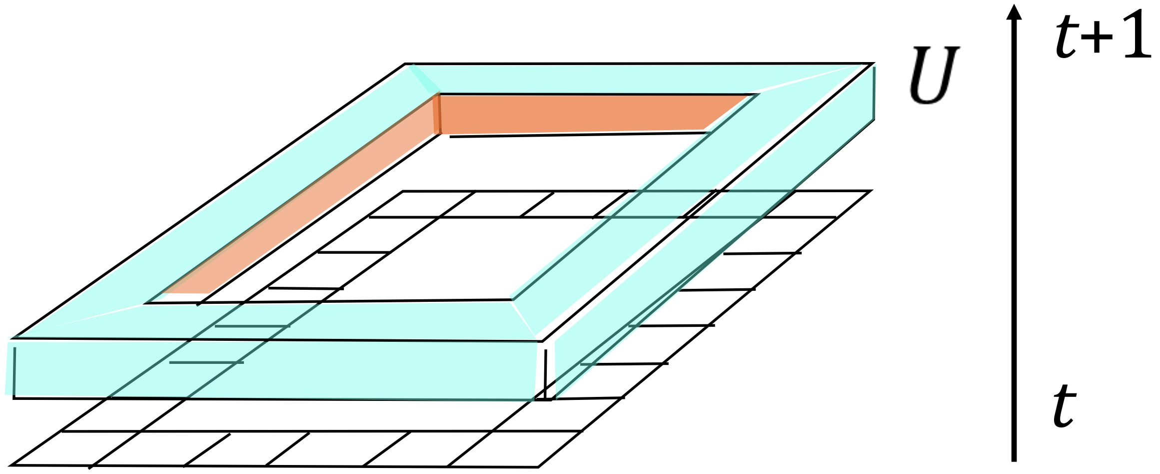
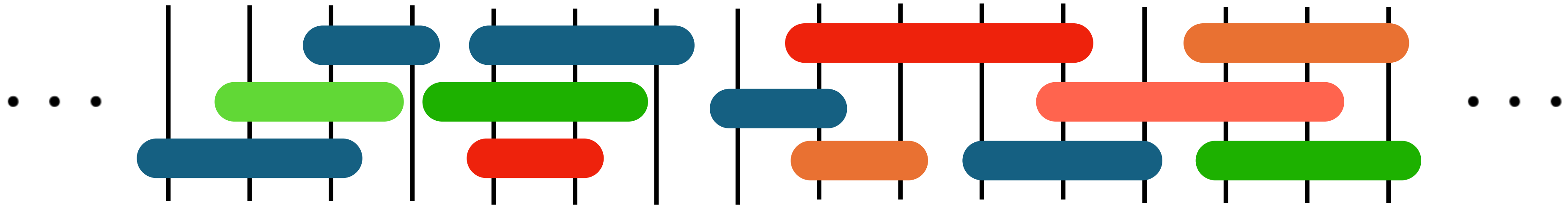


- A Finite-Depth Quantum Circuit is a QCA

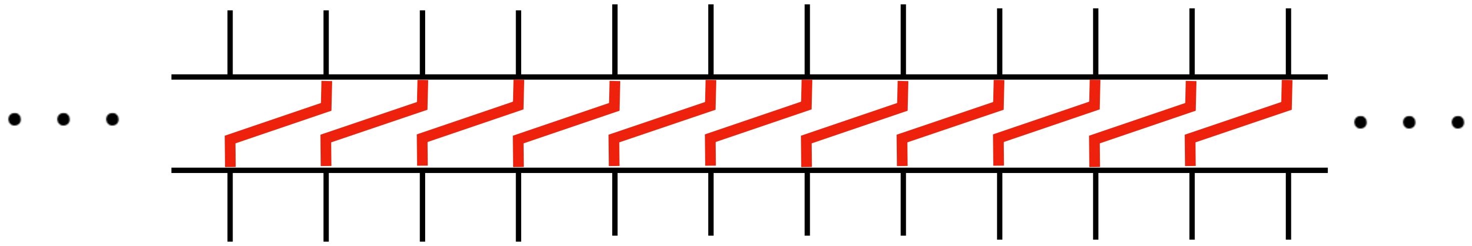


FDQC
Finite-Depth
Quantum Circuit

- But not every QCA is a FDQC!

The shift $\tau^{\pm}: \mathcal{A}_{\mathbb{Z}} \to \mathcal{A}_{\mathbb{Z}}$

$$\tau^{\pm}(O_x) = O_{x\pm 1}$$

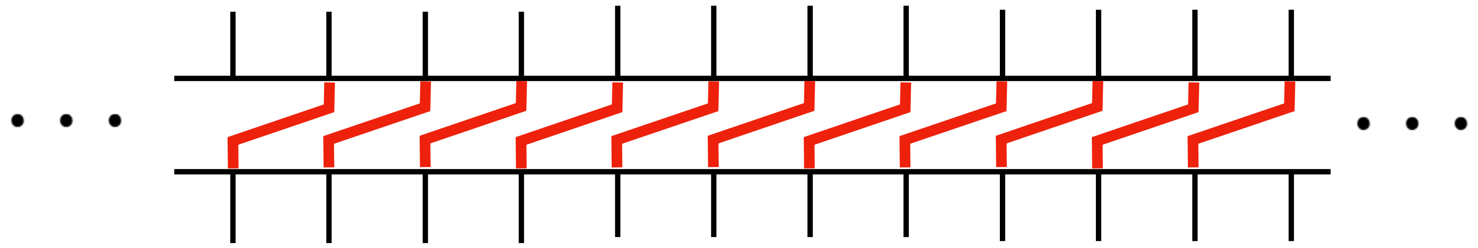


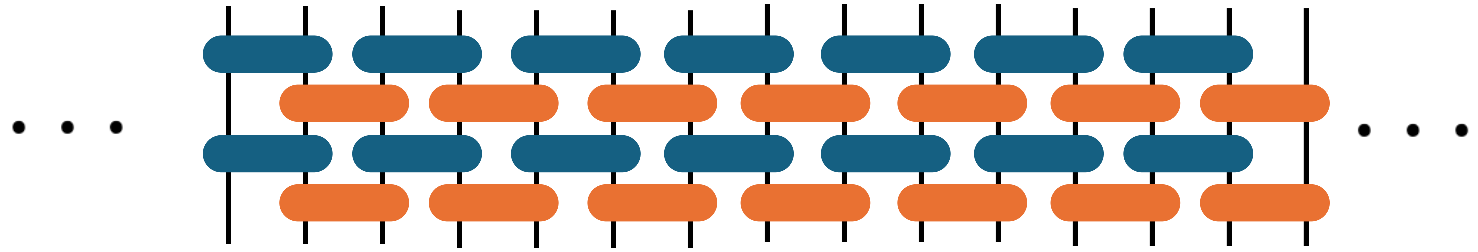
QCA: examples

- If the QCA acts in the same way everywhere on the lattice, we say that the QCA is **translation invariant**

$$\forall x \in \mathbb{Z}^n \;\; [\alpha, \tau^x] = 0$$

with $\tau^x : \mathcal{A}_{\mathbb{Z}^n} \to \mathcal{A}_{\mathbb{Z}^n}$
the shift by x





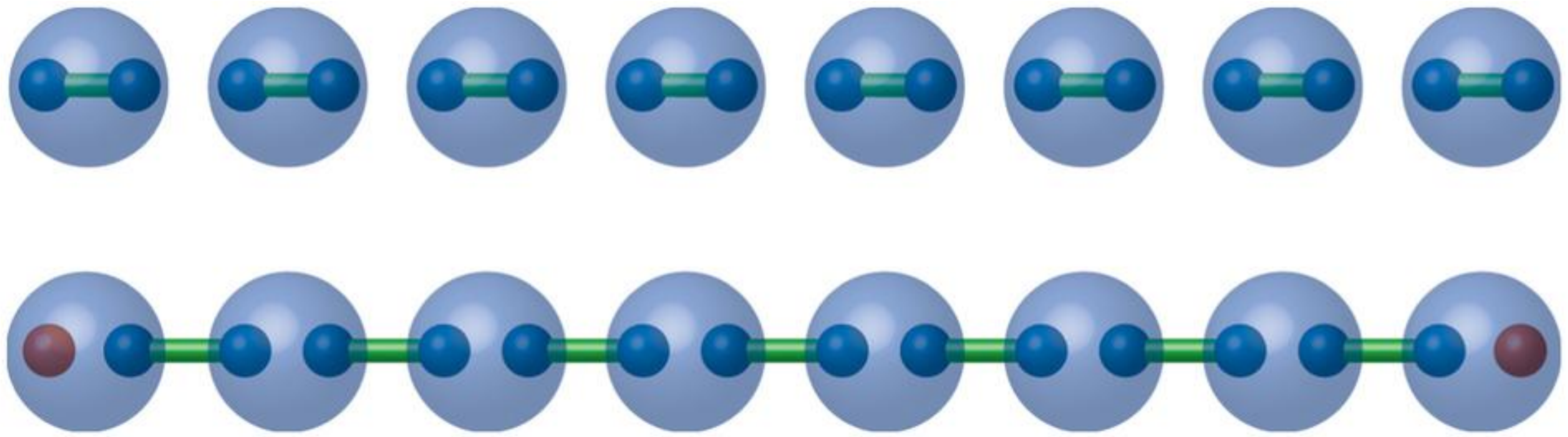
Natural Playground: many-body physics

Topological Phases of Matter

Equivalence classes of dynamics robust against local deformations

FDQCs

Example: **Kitaev chain**



Trivial phase

No FDQC
conneting them

Supercond. phase

A. Y. Kitaev, Phys.-Usp. 44 131 (2001)

Classification of QCAs

Interest:

1. Local implementability of QCAs and resources for simulation

   When a QCA is a FDQC?

2. Topological phases classification

   When two QCAs can be (continously) deformed into each other?

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$\Downarrow$

(QCA,∘) is a group

FDQC $\lhd$ QCA

QCA/FDQC

$\alpha \sim \beta$ if $\alpha = \beta \circ \phi$

with $\alpha, \beta \in QCA$ and $\phi \in FDQC$

Classification of QCAs

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$\alpha \sim \beta$ if $\alpha = \beta \circ \phi$

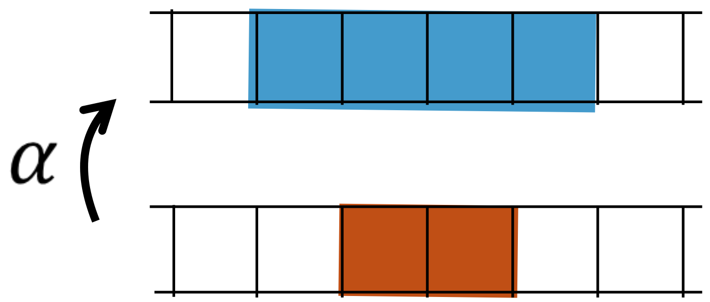
with $\alpha, \beta \in QCA$ and $\phi \in FDQC$

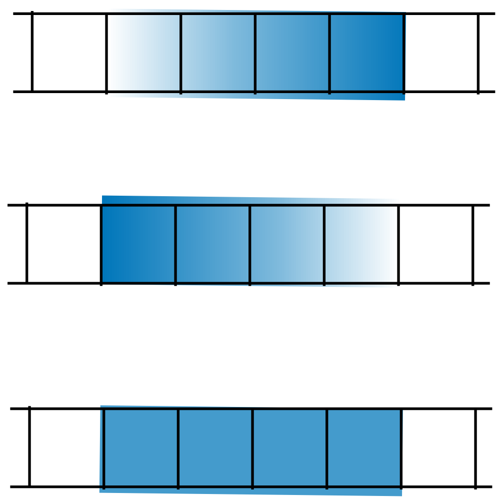
$\Rightarrow$ **How to classify these equivalence classes?**

For **1d QCAs** we know the answer: index theory.

Classification of QCAs: index theory (1+1d)

$\alpha \colon \mathcal{A}_{\mathbb{Z}} \to \mathcal{A}_{\mathbb{Z}}$





$i(\alpha) > 0$, $i(\alpha) < 0$: not a FDQC

$i(\alpha) = 0$: is a FDQC

$$i \colon QCA \to QCA \setminus FDQC \cong (\log \mathbb{Q}^{+}, +)$$

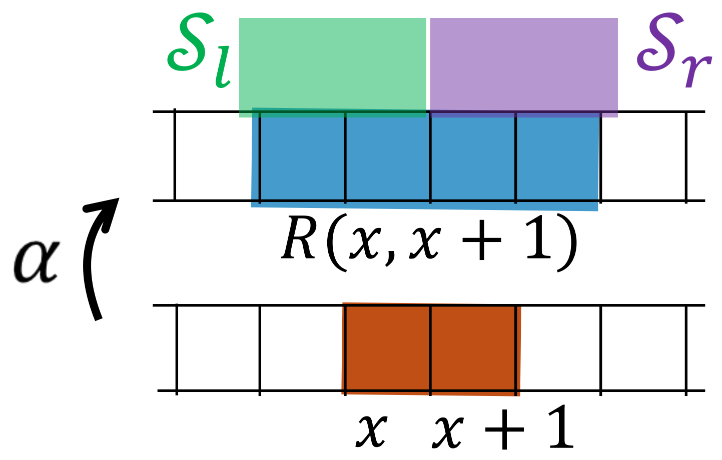
↓

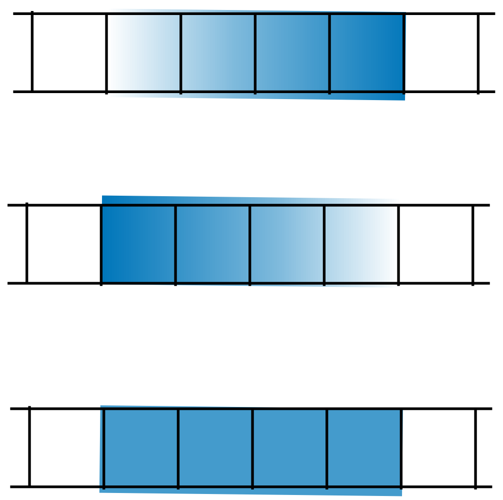
shifts

D. Gross, V. Nesme, H. Vogts, R.F. Werner, CMP 310, 419 (2012)

Classification of QCAs: index theory (1+1d)

$\alpha: \mathcal{A}_{\mathbb{Z}} \to \mathcal{A}_{\mathbb{Z}}$





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In general:

$$\alpha(\mathcal{A}_x \otimes \mathcal{A}_{x+1}) \subset \mathcal{A}_{R(x,\, x+1)}\ .$$

The **support algebra** $\mathcal{S}_\Lambda$ is the smallest C*-algebra such that:

$$\alpha(\mathcal{A}_x \otimes \mathcal{A}_{x+1}) \subseteq \mathcal{S}_\Lambda \otimes \mathcal{A}_{R(x,\, x+1)\setminus\Lambda} \subset \mathcal{A}_{R(x,\, x+1)}$$

$$\Rightarrow \alpha(\mathcal{A}_x \otimes \mathcal{A}_{x+1}) \subseteq \mathcal{S}_l \otimes \mathcal{S}_r \subset \mathcal{A}_{R(x,\, x+1)}\ .$$

D. Gross, V. Nesme, H. Vogts, R.F. Werner, CMP 310, 419 (2012)

Classification of QCAs: index theory (1+1d)

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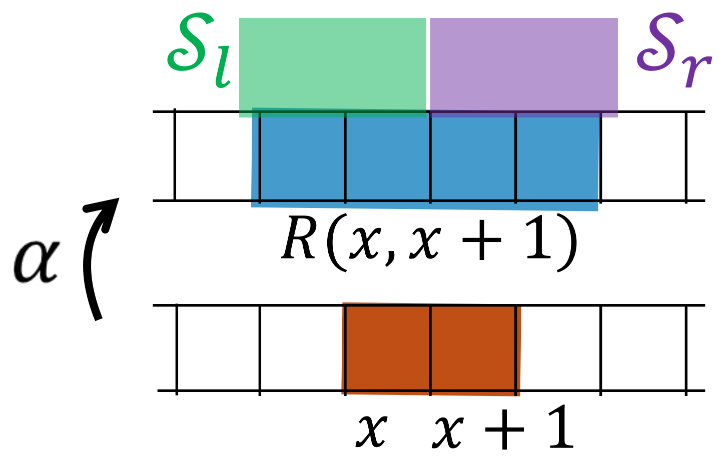

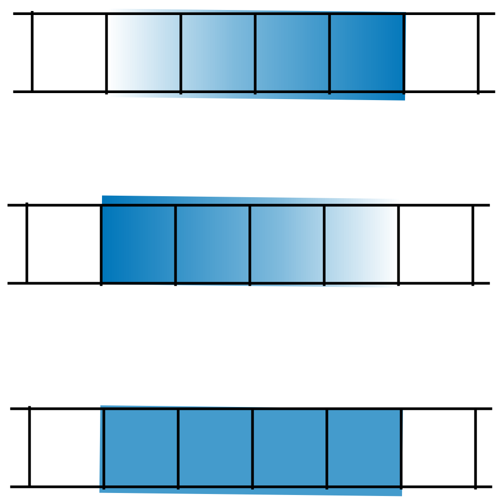

$i(\alpha) > 0$, $i(\alpha) < 0$ } not a FDQC

$i(\alpha) = 0$ is a FDQC

<u>Theorem GNVW</u>

Let $\alpha : \mathcal{A}_{\mathbb{Z}} \to \mathcal{A}_{\mathbb{Z}}$ be a QCA with $\mathcal{A}_x = \mathcal{L}\left(\mathbb{C}^{d(x)}\right)$, then

$$\alpha(\mathcal{A}_x \otimes \mathcal{A}_{x+1}) = \mathcal{S}_l \otimes \mathcal{S}_r = \mathcal{L}\left(\mathbb{C}^l\right) \otimes \mathcal{L}\left(\mathbb{C}^r\right)$$

$$i(\alpha) := \frac{1}{2} \log \frac{\dim \mathcal{S}_r}{\dim \mathcal{A}_{x+1}}$$ independent of x

$$i : QCA \to QCA \setminus FDQC \cong (\log \mathbb{Q}^+, +)$$

D. Gross, V. Nesme, H. Vogts, R.F. Werner, CMP 310, 419 (2012)

Results: fermions

L. Fidkowski, H. C. Po, A. C. Potter, and A. Vishwanath PRB 99, 085115 (2019)

Fermions $\longrightarrow$ **Grading over the algebra:**

$$\mathcal{A}(\mathbb{Z}_2) \coloneqq \mathcal{A}_{even} \oplus \mathcal{A}_{odd}$$

$$\mathcal{P}\mathcal{O}_{even} = \mathcal{O}_{even}$$

$$\mathcal{P}\mathcal{O}_{odd} = -\mathcal{O}_{odd}$$

$$\mathcal{O}_{even/odd} \in \mathcal{A}_{even/odd}$$

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$$\mathcal{P}\mathcal{O}_{even} = \mathcal{O}_{even}$$

$$\mathcal{P}\mathcal{O}_{odd} = -\mathcal{O}_{odd}$$

$$\mathcal{O}_{even/odd} \in \mathcal{A}_{even/odd}$$

Theorem

Let $\alpha: \mathcal{A}(\mathbb{Z}_2) \to \mathcal{A}(\mathbb{Z}_2)$ be a Fermionic QCA, then

$$i: Fermionic\ QCA \to Fermionic\ QCA \setminus FDQC \simeq \left(\frac{\log(\mathbb{Q}^+)}{2}, +\right)$$

L. S. T., A. Bisio, P. Perinotti, P. Meda, A. Tosini, M. Lugli, in preparation (2024)
L. S. T., A. Bisio, P. Perinotti, P. Meda, ongoing (2024)

Results: higher spatial dimensions

M. Freedman and M. B. Hastings, CMP 376, 1171 (2020)
J. Haah, L. Fidkowski, and M. B. Hastings, CMP 398, 469 (2023)
J. Haah, arXiv:2205.09141 (2024)

} existing results

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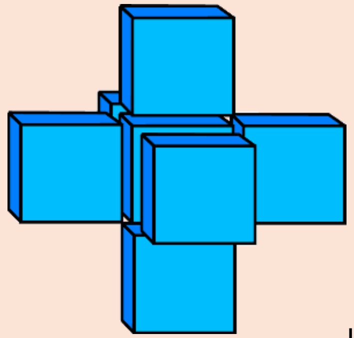
J. Haah, arXiv:2205.09141 (2024)

} existing results

Theorem

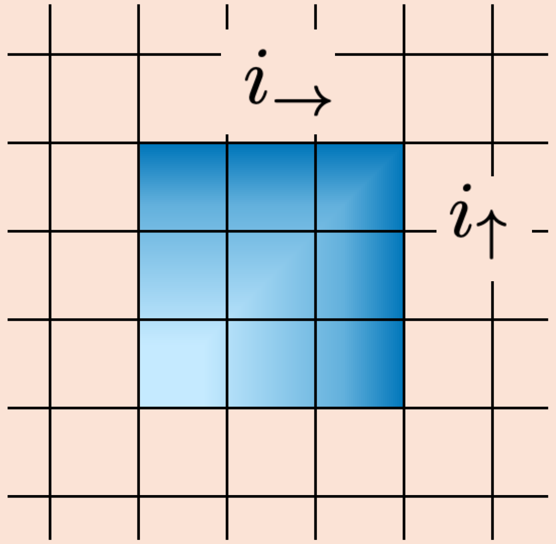
Assuming:

- **Hypercubic lattices of qubits** $\mathcal{A}_x = \mathcal{L}(\mathbb{C}^2) \quad x \in \mathbb{Z}^n$
- **Translation invariance**
- **von Neumann neighborhood**



(e.g. n=3)

Then: **one of such QCAs α is a FDQC iff $\vec{\imath}(\alpha) = \vec{0}$, otherwise is a shift.**



(e.g. n=2)

$\vec{\imath}(\alpha) = (i_{\rightarrow}, i_{\uparrow})$

A. P., A. Bisio, P. Perinotti, arXiv:2408.04493 (2024)

Conclusions & Outlooks

- Quantum Cellular Automata are the most general local, discrete dynamics
- Finite-Depth Quantum Circuits are QCAs that preserve topological phase of matter
- The index provides a topological classification of QCA group in 1d (and beyond!)

- Renormalization of (Fermionic) Quantum Cellular Automata
- Explicit classification of Fermionic Quantum Cellular Automata in higher dimensions
- ...

Thank you for the attention!

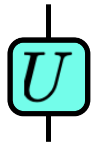
Backup Slides

Classification of qubit QCA

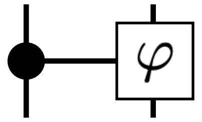
assuming:

- **Translation invariance**
- **qubit cells** $\mathcal{A}_x = \mathcal{L}(\mathbb{C}^2)$
- **von Neumann neighborhood**

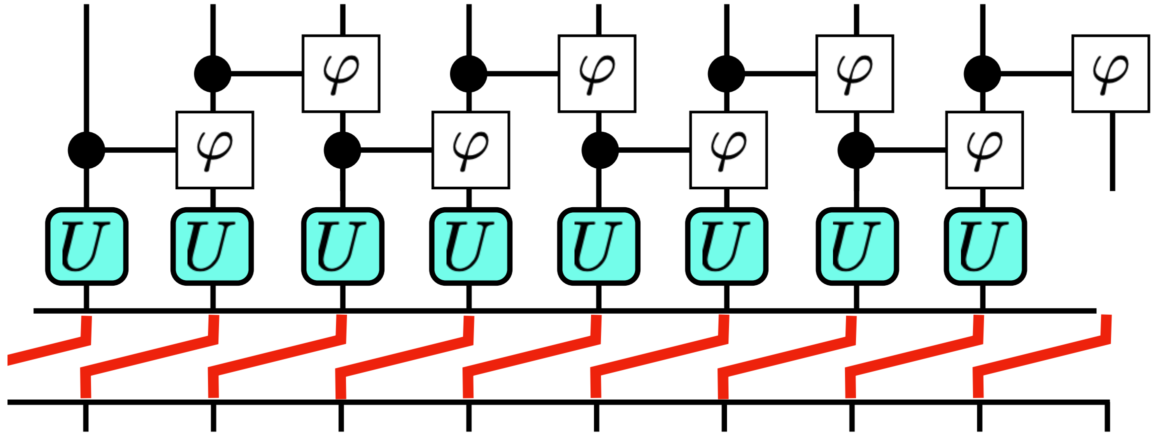
1 dimensional qubit QCA



arbitrary unitary gate



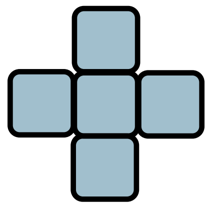
c-phase gate



B. Schumacher, R.F. Werner
e-print arXiv:0405174.

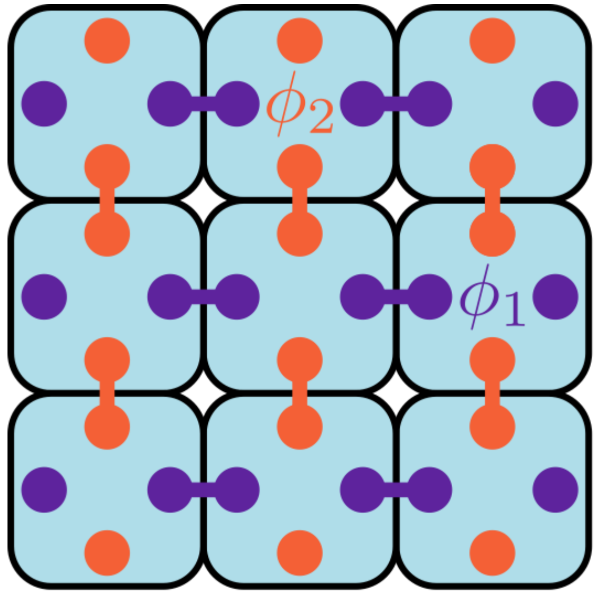
Classification of qubit QCA

2 dimensional qubit QCA

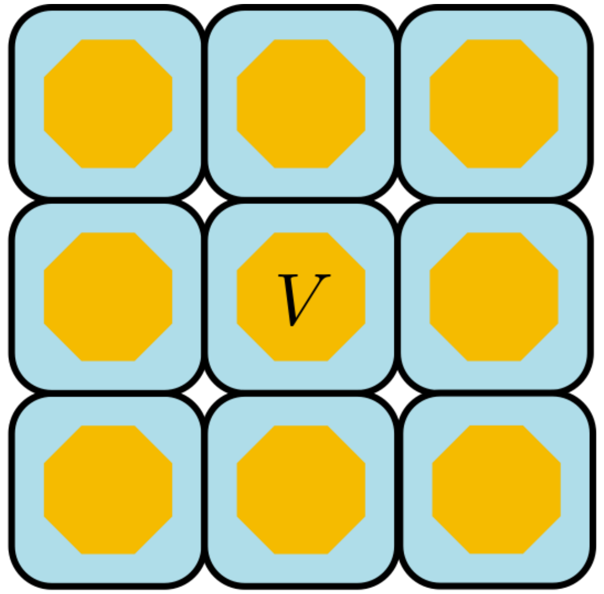


von Neumann
neighborhood in 2D

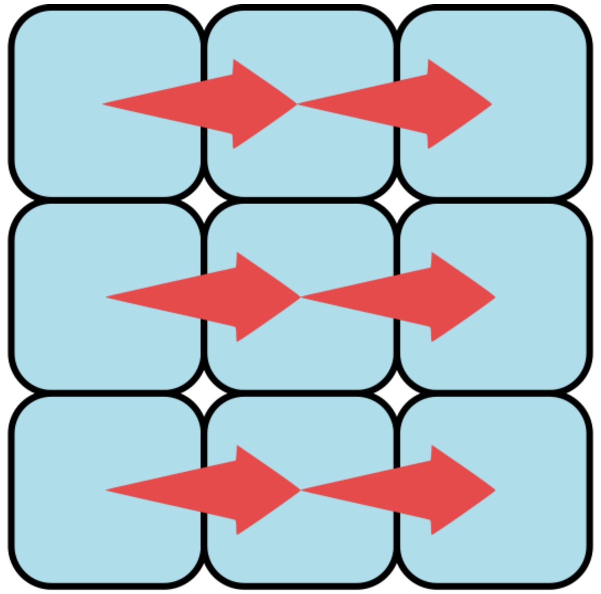
c-phase gates



local unitary gate

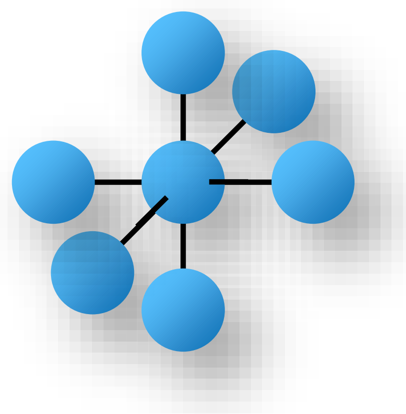


shift



N dimensional qubit QCA

(as in the 2D case)



von Neumann
neighborhood in 3D

A. P., A. Bisio, P. Perinotti, arXiv:2408.04493 (2024)

Coarse graining of CA

B is a coarse graining of A

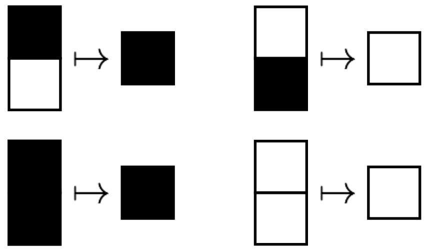
$$P \circ f_A^T = f_B \circ P$$

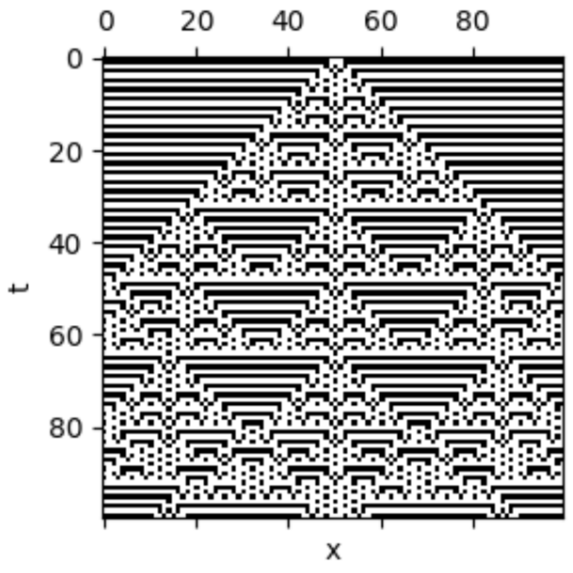
update rule

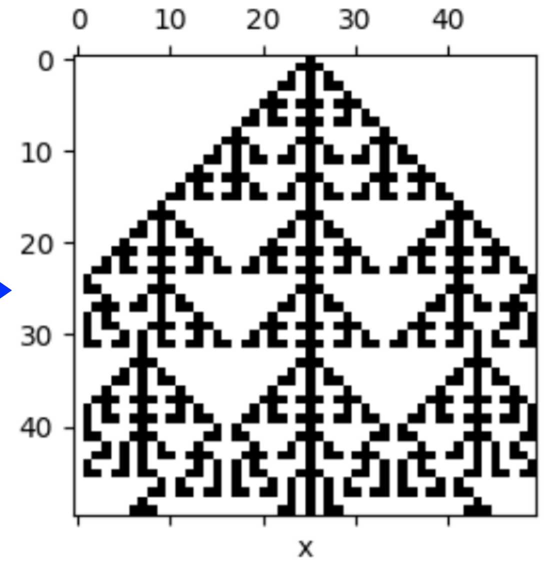
N cells → 1 cell

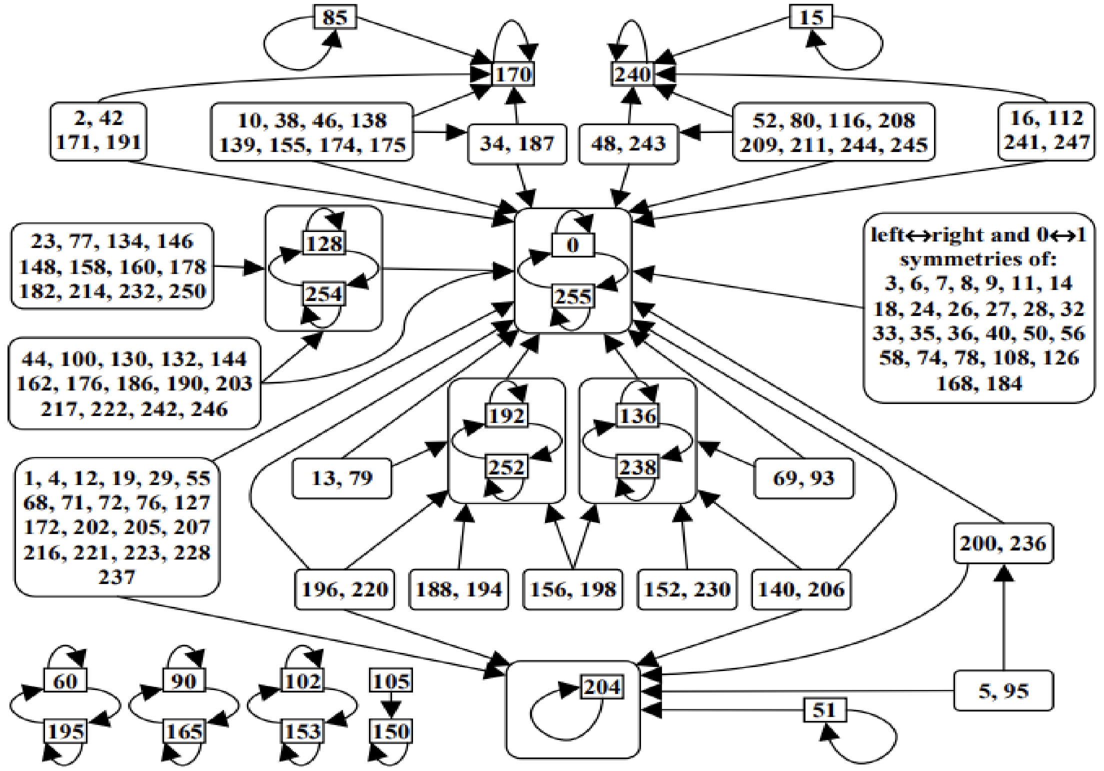
T steps → 1 step

$P:$







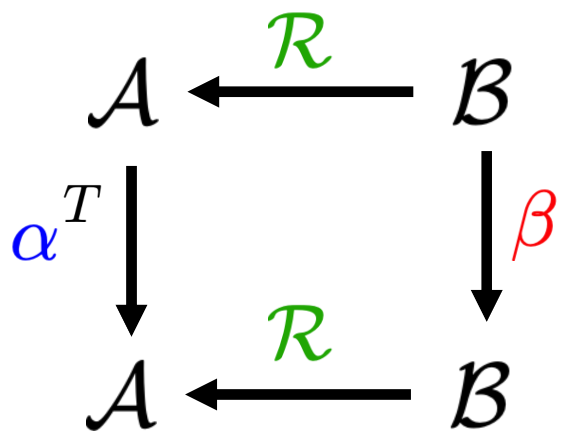


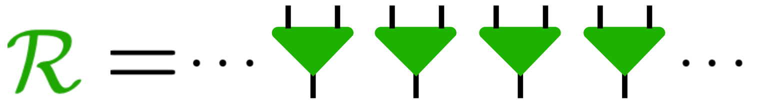
N. Israeli, N. Goldenfeld PRE 73, 026203 (2006)

Coarse graining of translation invariant QCA

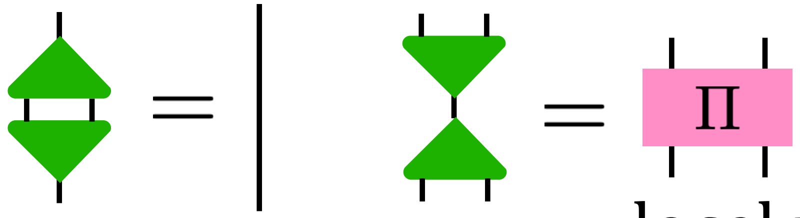
β **is a coarse graining of** α

$$\alpha^T \circ \mathcal{R} = \mathcal{R} \circ \beta$$

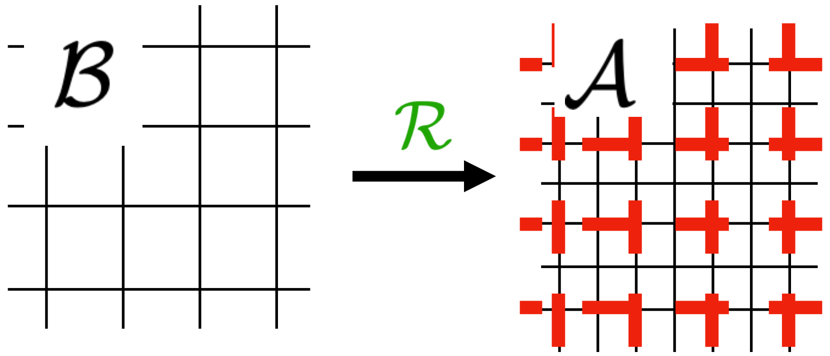


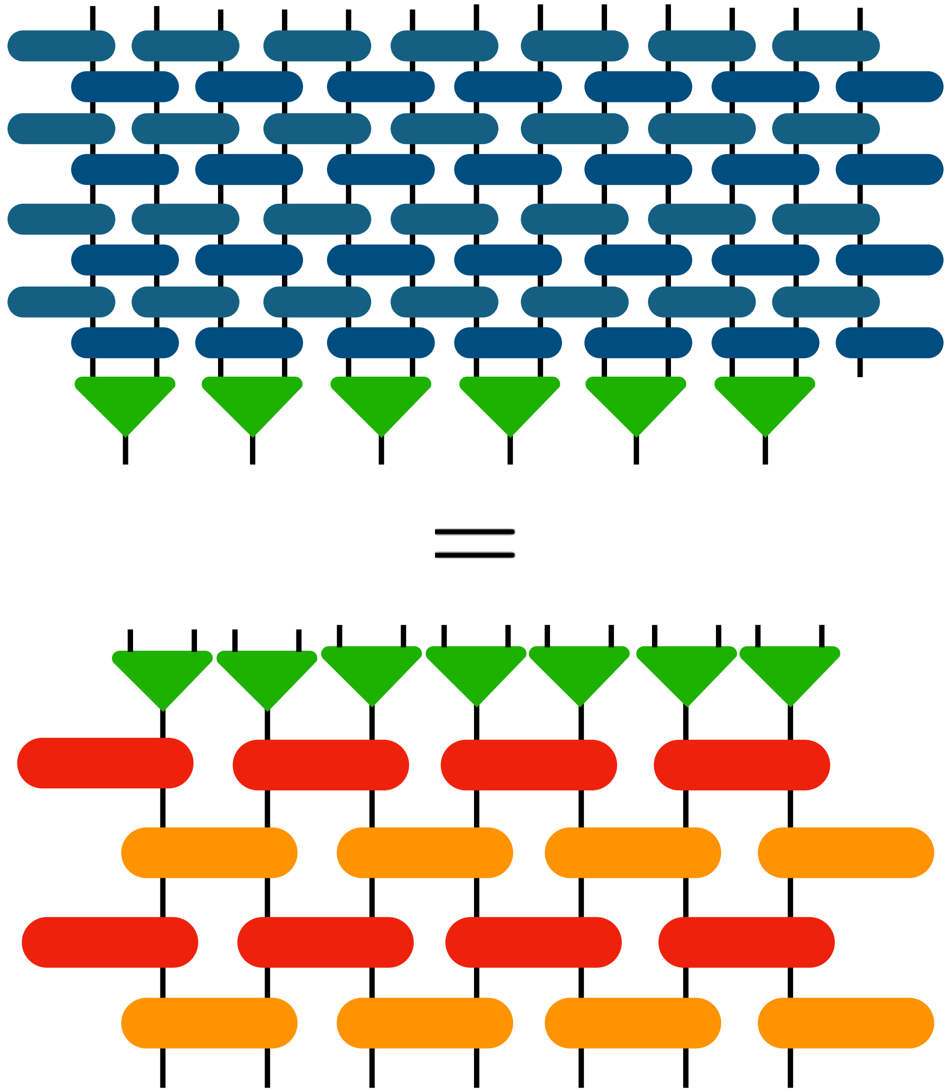


is an isometric homomorphism



local projector

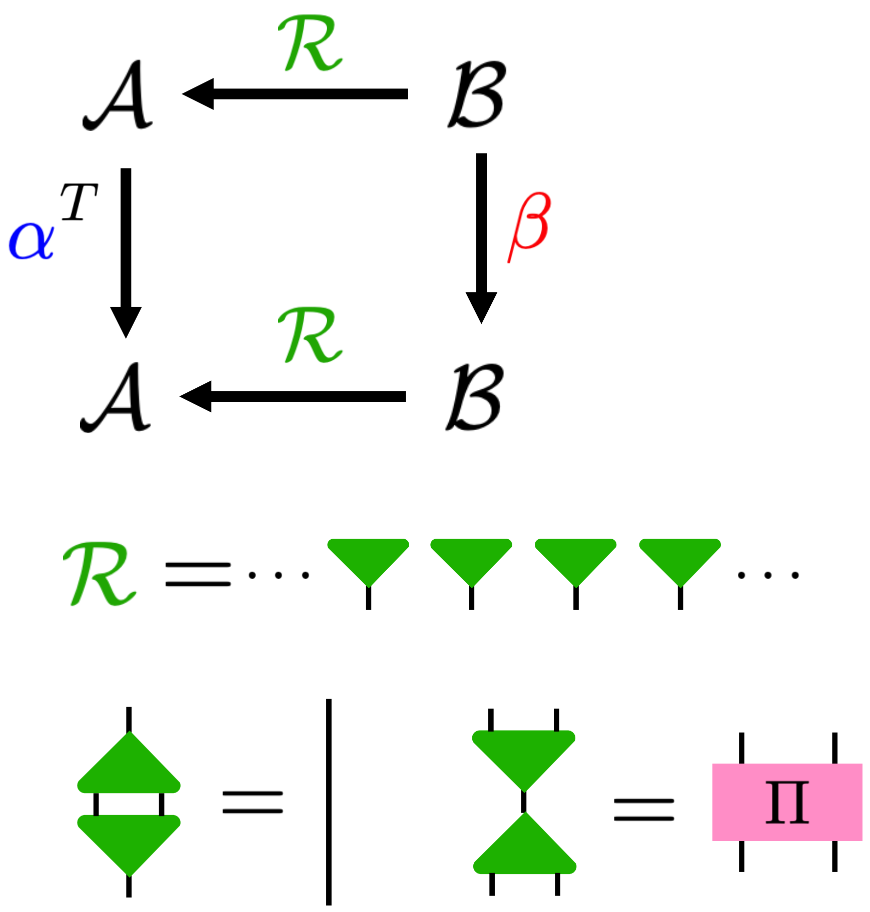




L. S. T., A. Bisio, P. Perinotti, arXiv:2407.12652 (2024)

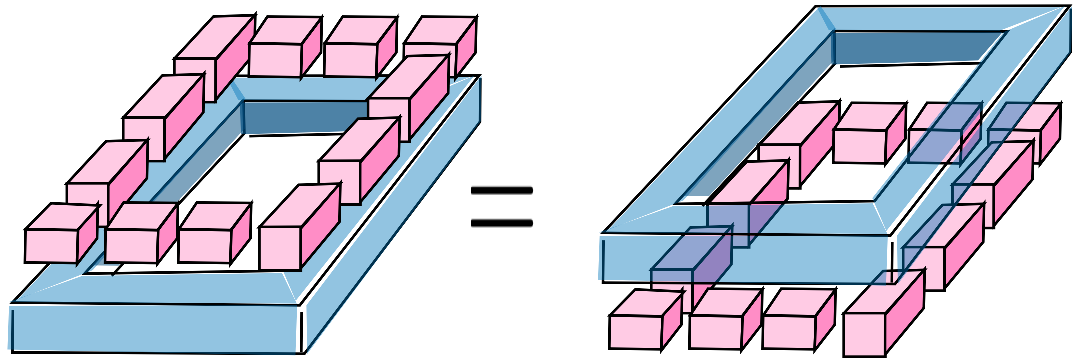
Coarse graining of QCA

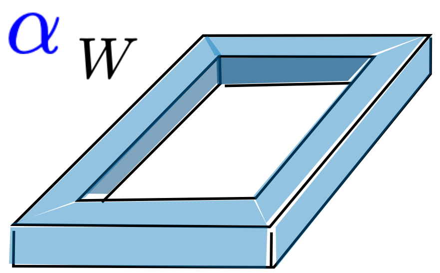
When does a QCA α admit a coarse graining β ?



$\Longleftrightarrow$

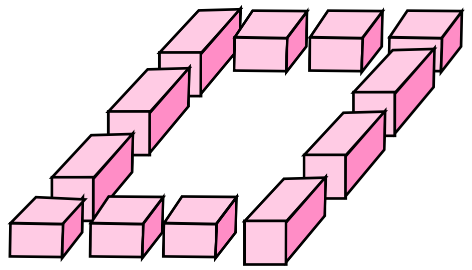
$$\alpha^T_W(\Pi_W) = \Pi_W$$





QCA on an arbitrary finite lattice

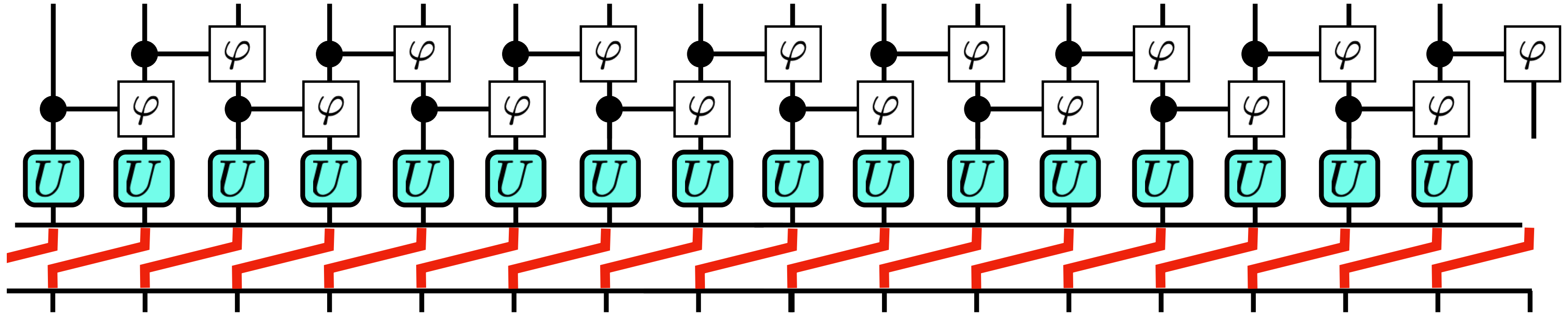
$$\Pi_W := \bigotimes_{i \in W} \Pi_i$$



L. S. T., A. Bisio, P. Perinotti, arXiv:2407.12652 (2024)

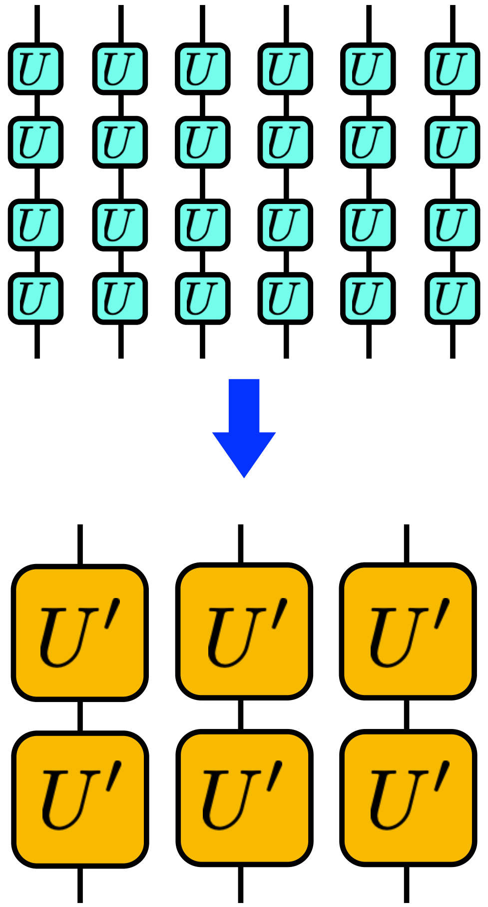
Coarse graining of qubit 1D QCA (2 steps, 2 cells)

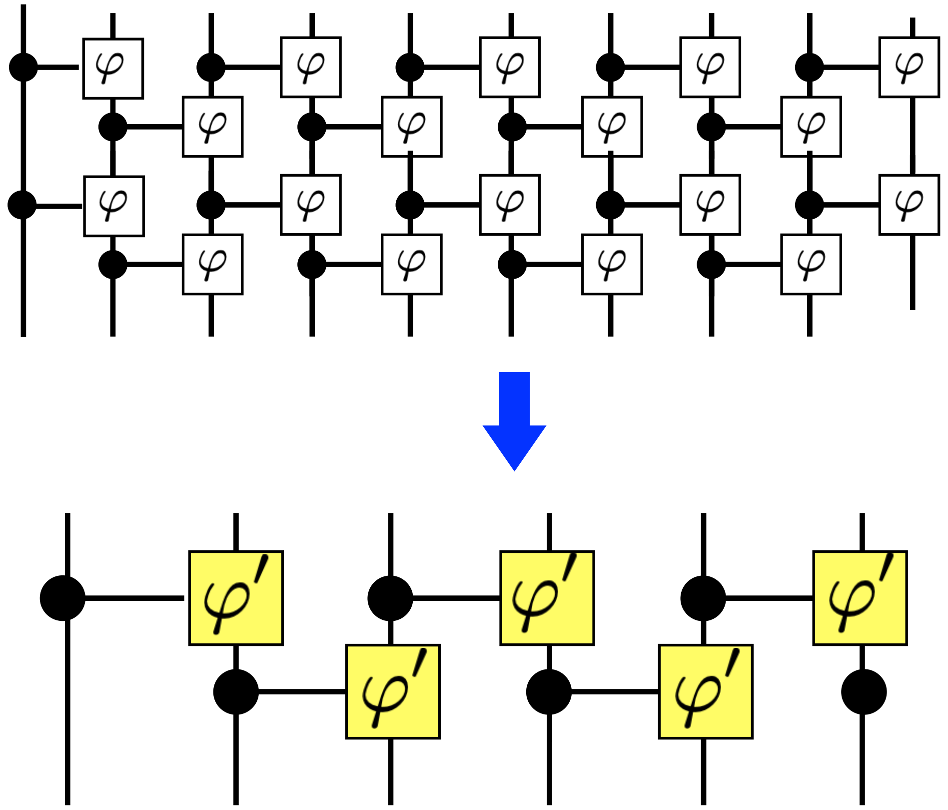
The qubit 1D QCA are classified



Results:

- the index cannot change
- the only 1D qubit QCA that admit a 2 cells coarse graining are “trivial” (no propagation of information)





L. S. T., A. Bisio, P. Perinotti, arXiv:2407.12652 (2024)