Driven conformal field theory and circuit complexity

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Based on

2409.08319 with Johanna Erdmenger and Tim Schuhmann

2306.00099 with Jan de Boer, Victor Godet and Esko Keski-Vakkuri

Introduction

Driven quantum systems

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- They are characterized by a time-dependent Hamiltonian (in the Schrödinger picture)

$$H(t) = \sum_n \lambda_n(t)\, \mathcal{O}_n \tag{1}$$

with a set of operators $\mathcal{O}_n$ and time-dependent parameters $\lambda_n(t)$ controlled externally

Motivation

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- The driving generates novel phases of matter that are inaccessible in equilibrium
- A lot of focus has been on periodically driven quantum many-body systems (Floquet systems)

$$\lambda_n(t+T) = \lambda_n(t) \tag{2}$$

where such phases have topological characterization

[Kitagawa–Berg–Rudner–Demler '10]

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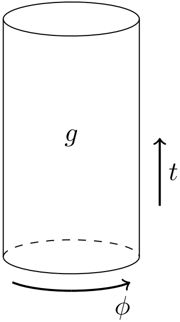
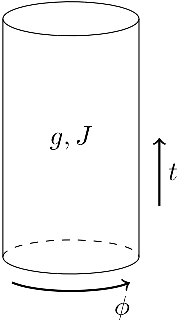
- Conformal field theory driven by classical background fields is a tractable example
- Related to quantum gravity via the AdS/CFT correspondence
- Connection to black holes and holographic complexity

2D CFTs driven by classical background fields

- We will focus on 1+1-dimensional CFTs driven by classical background fields on $S^1 \times \mathbb{R}$





[De Boer–Godet–JK–Keski-Vakkuri '23]

[Erdmenger–JK–Schuhmann '24 and on-going work]

Outline

1. **Driven systems as quantum circuits**

2. **CFTs driven by a background metric**

3. **CFTs driven by a scalar source**

1. Driven systems as quantum circuits

State evolution in a driven system

- In a closed driven system, state evolution is governed by the Schrödinger equation

$$i\,\partial_t\,|\Psi(t)\rangle = H(t)\,|\Psi(t)\rangle\ . \tag{3}$$

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- Here the time-ordered exponential

$$\overleftarrow{\mathcal{T}}\exp\left(-i\int_0^t ds\,H(s)\right) \equiv \lim_{N\to\infty} e^{-i\delta s_N H(s_{N-1})}\dots e^{-i\delta s_1 H(s_0)} \tag{5}$$

with $s_{N-1} \equiv t$ and $s_0 \equiv 0$

Interpretation as a quantum circuit

- Let $t \in [0, t_f]$ and fix the final state

$$|\Psi(t_f)\rangle \equiv |\Psi_f\rangle \ , \quad |\Psi(0)\rangle = |R\rangle \tag{6}$$

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- Final state constructed from the reference state by application of unitary quantum gates

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[Helstrom '69, Uhlmann '93, ...]

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- Quantum circuit $|\Psi(t)\rangle$ traces a curve on $\mathcal{D}$ between the points $|R\rangle$ and $|\Psi_f\rangle$

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[Nielsen '06]

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- The geodesic distance of $|\Psi_f\rangle$ from $|R\rangle$ is then the *circuit complexity* of $|\Psi_f\rangle$

[Nielsen '06]

- It measures how difficult it is to construct $|\Psi_f\rangle$ from $|R\rangle$ using a set of quantum gates

Fubini–Study cost and complexity

- The Fubini–Study metric on $\mathcal{D}$ is defined as

$$ds^2 = \langle d\Psi | d\Psi \rangle - \langle \Psi | d\Psi \rangle \langle d\Psi | \Psi \rangle \tag{9}$$

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- Geodesic distance = Fubini–Study circuit complexity

[Caputa–Magan '18, Flory–Heller '20, Flory–Heller '20]

2. CFTs driven by a background metric

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$$(\psi \Phi)_{a_1 \dots a_n}(x) = e^{-\Delta_\Phi \omega(D(x))} \frac{\partial D^{b_1}}{\partial x^{a_1}} \cdots \frac{\partial D^{b_n}}{\partial x^{a_n}} \Phi_{b_1 \dots b_n}(D(x)) \tag{12}$$

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- The action of the CFT is Diff $\ltimes$ Weyl invariant

$$I[\psi\Phi, \psi g] = I[\Phi, g] \tag{13}$$

CFT driven by a background metric

- We will endow the cylinder $(\phi, t) \in S^1 \times \mathbb{R}$ with a general (curved) metric g

$$g_{ab}(x)\, dx^a dx^b = e^{\omega}\,(d\phi + \nu dt)(d\phi + \bar{\nu} dt) \tag{14}$$

with three arbitrary functions $\omega(\phi, t)$, $\nu(\phi, t)$ and $\bar{\nu}(\phi, t)$

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[Erdmenger–JK–Schuhmann on-going work]

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- Expand in Fourier modes

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$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12} n^3\, \delta_{n,-m} \tag{20}$$

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- The CFT is an example of a driven quantum system

$$H(t) = \sum_{n=-\infty}^{\infty} \nu_n(t)\, (L_n \otimes \mathbf{1}) - \sum_{n=-\infty}^{\infty} \bar{\nu}_n(t)\, (\mathbf{1} \otimes L_n) \tag{21}$$

State evolution in the background metric

- Consider the Lie group of orientation-preserving diffeomorphisms of the circle

$$\mathrm{Diff}_+ S^1 \equiv \{ f : \mathbb{R} \to \mathbb{R} \,|\, f(\phi + 2\pi) = f(\phi) + 2\pi,\ f'(\phi) > 0 \} \tag{22}$$

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$$\nu(\phi, t) \equiv -(\dot{f}_t \circ f_t^{-1})(\phi)\,, \qquad \bar{\nu}(\phi, t) \equiv -(\dot{\bar{f}}_t \circ \bar{f}_t^{-1})(\phi)\,. \tag{23}$$

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[De Boer–Godet–JK–Keski-Vakkuri '23]

- $V_{f_t} \otimes \overline{V}_{\bar{f}_t}$ is a projective unitary representation of the conformal transformation $(f_t, \bar{f}_t)$

[Fewster–Hollands '04, Oblak '16]

Conformal group in $1+1$ dimensions

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$$(f, \bar{f}) \in \mathrm{Diff}_+ S^1 \times \mathrm{Diff}_+ S^1 \tag{27}$$

[Kong–Runkel '09, Schottenloher '08]

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- $\mathrm{Diff}_+ S^1 \times \mathrm{Diff}_+ S^1$ is the classical symmetry group of a 2D CFT on a flat cylinder

Unitary projective representations

- On the Hilbert space of the CFT, a conformal diffeo $(f, \bar{f})$ is represented by a unitary operator

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- Non-trivial Thurston–Bott 2-cocycle

$$B(f_1, f_2) = \frac{c}{48\pi} \int_0^{2\pi} d\phi \, \frac{f_2''(\phi)}{f_2'(\phi)} \, \log f_1'(f_2(\phi)) \tag{30}$$

[Fewster–Hollands '04, Oblak '16]

Virasoro circuits

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- A curve in the infinite-dimensional space of *Virasoro pure states*

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- Fubini–Study metric becomes the Kähler metric on $\mathrm{Diff}_+ S^1/U(1)$ (a Virasoro coadjoint orbit)

[Kirillov–Juriev '87, Erdmenger–JK–Schuhmann '24]

3. CFTs driven by a scalar source

Deformations of Virasoro circuits

- So far the discussion has been about Virasoro circuits generated by

$$H(t) = \int_0^{2\pi} d\phi\, \nu(\phi, t)\, T_{--}(\phi) - \int_0^{2\pi} d\phi\, \bar{\nu}(\phi, t)\, T_{++}(\phi) \tag{34}$$

- Unitary state evolution is restricted to a single Verma module $\mathcal{H}_h \otimes \mathcal{H}_{h'}$ (irrep of the Virasoro algebra)

$$\mathcal{H}_{\text{CFT}} = \bigoplus_{h,h'} \mathcal{H}_h \otimes \mathcal{H}_{h'} \tag{35}$$

Deformations of Virasoro circuits

- Generalize to *primary-deformed Virasoro circuits* generated by

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- Mode expansion

$$\mathcal{O}_h(\phi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \mathcal{O}_{h,n}\, e^{in\phi}\,, \quad [L_n, \mathcal{O}_{h,m}] = [(h-1)\,n - m]\, \mathcal{O}_{h,n+m} \quad (38)$$

Deformations of Virasoro circuits

- Generalize to *primary-deformed Virasoro circuits* generated by

$$H(t) = \int_0^{2\pi} d\phi\, \nu(\phi,t)\, T_{--}(\phi) - \int_0^{2\pi} d\phi\, \bar{\nu}(\phi,t)\, T_{++}(\phi) + \int_0^{2\pi} d\phi\, J(\phi,t)\, \mathcal{O}_h(\phi) \otimes \mathcal{O}_{\bar{h}}(-\phi) \quad (36)$$

[Erdmenger–JK–Schuhmann '24]

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- Commutation relations

$$[\mathcal{O}_{h,n}, \mathcal{O}_{h,m}] = \binom{n+h-1}{2h-1} \delta_{n,-m} + \sum_k D_k(n,m)\, \mathcal{O}_{h_k,\, n+m}\,, \quad (39)$$

Realization in a driven CFT

- When $h = \bar{h} = 1$, the deformed Hamiltonian $H(t)$ arises from the action

$$S[\Phi, g, J] = I[\Phi, g] + \int d^2x \sqrt{-g}\, J(x)\, \mathcal{O}(x) \tag{40}$$

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- More generally, renormalization group flow modifies the Hamiltonian $H(t)$

[Erdmenger–JK–Schuhmann on-going]

Fubini–Study circuit complexity

- State along the primary-deformed Virasoro circuit

$$|\Psi(t)\rangle = U(t)\,|h'\rangle \ , \quad U(t) = \overleftarrow{\mathcal{T}}\exp\left(-i\int_0^t ds\, H(s)\right) \tag{42}$$

where the deformed Hamiltonian

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- Goal: calculate the accumulated Fubini–Study cost

$$\mathcal{L}(t) = \int_0^t ds\, \sqrt{\mathcal{F}(s)} = \int_0^t ds\, \sqrt{\langle\Psi(s)|H(s)^2|\Psi(s)\rangle - \langle\Psi(s)|H(s)|\Psi(s)\rangle^2} \tag{44}$$

State evolution in the deformed circuit

- We will decompose the Hamiltonian as

$$H(t) = C(t) + \lambda P(t)\,, \quad P(t) \equiv \int_0^{2\pi} d\phi\, J(\phi, t)\, \mathcal{O}_h(\phi) \otimes \mathcal{O}_{\bar{h}}(-\phi) \tag{45}$$

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- The factors are given by

$$V(t) = \overleftarrow{\mathcal{T}}\exp\left(-i\int_0^t ds\, C(s)\right) = V_{f_t} \otimes \overline{V}_{\bar{f}_t}\,, \quad U_P(t) = \overleftarrow{\mathcal{T}}\exp\left(-i\int_0^t ds\, V(s)^\dagger\, \lambda P(s)\, V(s)\right)$$

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[Erdmenger–JK–Schuhmann ’24]

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$$C(t) = L_0 \otimes \mathbf{1} + \mathbf{1} \otimes L_0\,, \quad J(\phi, t) = S(\phi)\,j(t)\,, \quad |R\rangle = |0\rangle \tag{48}$$

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Expansion of the Fubini–Study cost

- We derive the explicit formula (assuming $\bar{h} = h$ for simplicity)

$$\mathcal{F}^{(2)}(t) = j(0)^2 \,\mathrm{Re}\,\langle \mathcal{R}_{hh}(0)^2 \rangle + 2\, j(0) \int_0^t ds\, \partial_s j(s)\, \mathrm{Re}\,\langle \mathcal{R}_{hh}(t)\, \mathcal{R}_{hh}(s) \rangle \qquad (50)$$
$$+ \int_0^t ds_1 \int_0^t ds_2\, \partial_{s_1} j(s_1)\, \partial_{s_2} j(s_2)\, \mathrm{Re}\,\langle \mathcal{R}_{hh}(s_1)\, \mathcal{R}_{hh}(s_2) \rangle \,,$$

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- Vacuum 2-point function of a *ring operator*

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- Important point: only the real part contributes (imaginary part is UV divergent)

[Erdmenger–JK–Schuhmann '24]

Expansion of the Fubini–Study cost

- Primary operator vacuum 2-point function

$$\langle 0| \mathcal{O}_h(\phi_1)\, \mathcal{O}_h(\phi_2)\, |0\rangle = \lim_{\varepsilon \to 0^+} \frac{1}{(2\pi)^2} \frac{1}{\left[2i \sin\left(\frac{\phi_1 - \phi_2 + i\varepsilon}{2}\right)\right]^{2h}} \tag{52}$$

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- Real part of the ring operator 2-point function

$$\begin{aligned} &\mathrm{Re}\, \langle 0| \mathcal{R}_{hh}(t_1)\, \mathcal{R}_{hh}(t_2) |0\rangle \\ &= \frac{1}{(2\pi)^2} \sum_{m=h}^{\infty} \frac{(-1)^{h+m}\,(m+h)!}{(2h-1)!\,(m-h)!} \left[\frac{|S_{2m}|^2}{2\,(m+h)}\, {}_2F_1\left(h-m, h-m; \frac{1}{2}; \cos^2(\Delta t)\right)\right. \\ &\qquad \left. + |S_{2m+1}|^2 \cos(\Delta t)\, {}_2F_1\left(h-m+1, h-m; \frac{3}{2}; \cos^2(\Delta t)\right)\right] \end{aligned} \tag{53}$$

Source profiles

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$$S(\phi) = \cos(n\phi)\,, \quad n = 0, 1, 2, \ldots \tag{54}$$

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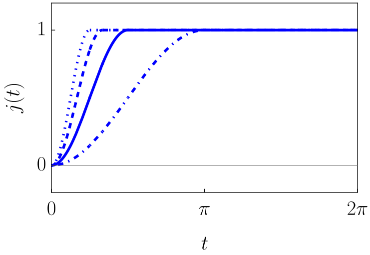
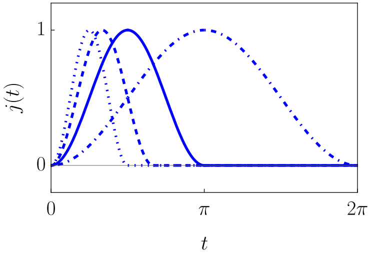
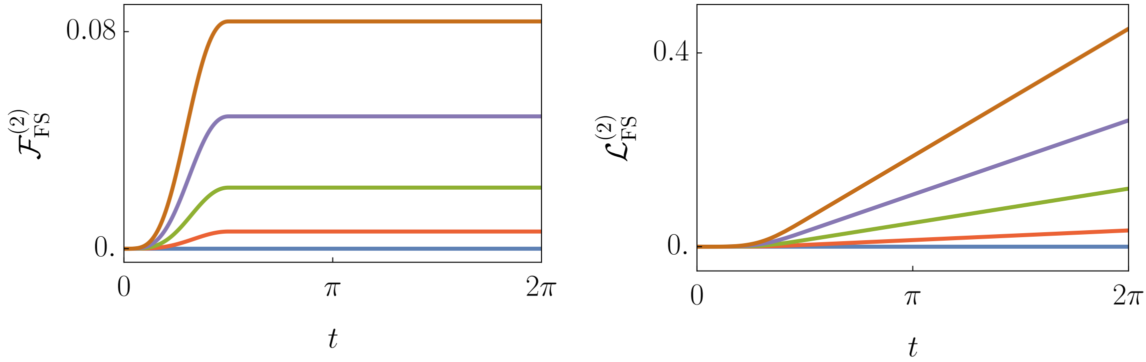
- And temporal profiles $j(t)$ of the form





Behavior of Fubini–Study cost and accumulated cost

- Switch on profile for $n = 1$ (blue), $n = 2$ (red), $n = 3$ (green), $n = 4$ (purple), $n = 5$ (brown)



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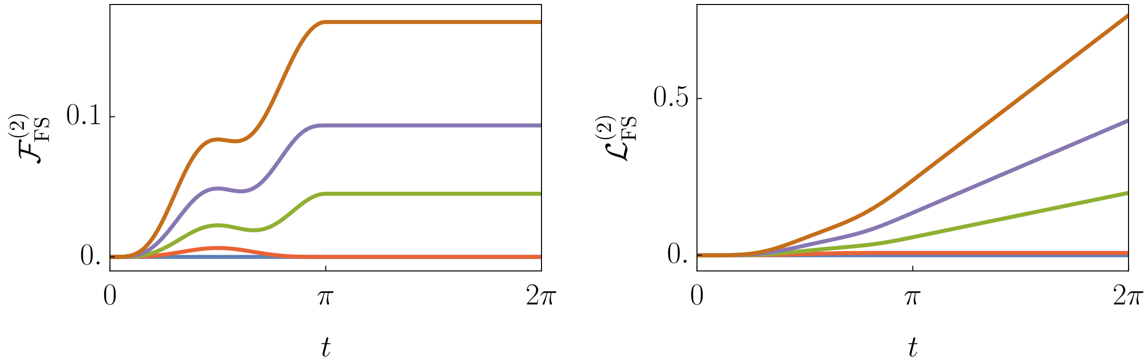

Conclusions

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- May be realized as physical time evolution in a CFT driven by background fields
- Accumulated FS cost of a simple primary-deformed Virasoro circuit exhibits linear growth (at leading order in the deformation)

Future directions

- Understand better the information geometry explored by primary-deformed Virasoro circuits

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[Erdmenger–JK–Schuhmann on-going]

- Gravity interpretation as black hole formation using the AdS/CFT correspondence

Thank you

Details of UV divergences

- For $h = \bar{h} = 1$, the 2-point function is explicitly $(\Delta t = t_1 - t_2)$

$$\langle 0|\, \mathcal{R}_{11}(t_1)\, \mathcal{R}_{11}(t_2)\, |0\rangle = \frac{1}{4}\frac{1}{(2\pi)^2}\sum_{n=-\infty}^{\infty} |S_n|^2\, e^{-i|n|\Delta t}\,(-|n| + i\cot\Delta t)\csc^2\Delta t\,, \tag{55}$$

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- The imaginary part diverges in the coincidence limit