Quantum energy inequalities in the thermal sector

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Outline

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$\Longrightarrow$ Necessity to find lower bounds on the expectation value of the (time averaged) quantised version of the energy density.

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In the real scalar case ($P\phi = (\Box + m^2)\phi = 0$), the fundamental object is the abstract $*$-**algebra** $\boldsymbol{\mathcal{A}}$, polynomially generated by the smeared fields $\phi(f), f \in C_0^\infty(\mathbb{M})$, that satisfy:

- linearity, $\phi(\lambda f + g) = \lambda\phi(f) + \phi(g), \lambda \in \mathbb{C}$.
- hermiticity, $\phi(f)^* = \phi(\overline{f})$.
- weak solution of field equation, $\phi(Pf) = 0$.
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States ω are positive, normalised linear functionals over $\mathcal{A}$.

Thermal field theory

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Definition

A state ω^β satisfies the KMS condition with respect to the time evolution τ_t if:

$$\omega^\beta((\tau_t A)B) = \omega^\beta(B(\tau_{t+i\beta}A)),$$

and the function $z \in \mathbb{C} \to \omega^\beta(B\tau_z(A))$ is analytic inside the strip $\Im z \in [0, \beta]$ and continuous on the border.

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- We construct the representation induced by the KMS state (**purification procedure**).
- We identify therein the energy density operator.
- We study the expectation value of this operator in this representation. $\implies$ necessity to introduce mathematical tools as **modular theory** and **non-commutative** L^p **spaces**.

Outline

One particle structure

Given a $*$-algebra $\mathcal{A}$ (e.g. associated to the symplectic space $(\mathcal{C}_0^\infty(\mathbb{M}), \Delta)$) and a state ω over $\mathcal{A} \implies$ representation $\pi^\omega(\mathcal{A})$ via **GNS construction**.

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The GNS representation induced by ω^β is the Fock representation over the symmetrised Fock space $\mathcal{F}^s$ (**purification procedure**):

$$\mathcal{F}^s(L^2 \oplus L^2) \simeq \mathcal{F}^s(L^2) \otimes \mathcal{F}^s(L^2),$$

where $\mathcal{F}^s(L^2)$ is the usual bosonic Fock space over the Hilbert space of L^2 functions on the mass hyperboloid.

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where $\mathcal{F}^s(L^2)$ is the usual bosonic Fock space over the Hilbert space of L^2 functions on the mass hyperboloid.
Real valued test functions are mapped into the Hilbert space via the map K:

$$K(f)(\mathbf{k}) = \frac{\overline{\hat{f}|_{H_m^+}(\mathbf{k})}}{\sqrt{e^{\beta\omega_\mathbf{k}} - 1}} \oplus \frac{\hat{f}|_{H_m^+}(\mathbf{k})}{\sqrt{1 - e^{-\beta\omega_\mathbf{k}}}} =: \mathscr{B}_\mathbf{k}^- \overline{\hat{f}|_{H_m^+}}(\mathbf{k}) \oplus \mathscr{B}_\mathbf{k}^+ \hat{f}|_{H_m^+}(\mathbf{k}).$$

Thermal representation

We get explicit expression for the smeared **field** $\phi(f)$ (and for its commuting $\tilde{\phi}(g)$) in terms of the usual particles (and holes) creation and annihilation "operators" $b^{\#}$ $(a^{\#})$:

$$[a_{\mathbf{k}}, a^{\dagger}_{\mathbf{p}}] = (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{k}); \quad [b_{\mathbf{k}}, b^{\dagger}_{\mathbf{p}}] = (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{k});$$

$$\phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[\mathscr{B}^-_{\mathbf{k}} a_{\mathbf{k}} e^{ikx} + \mathscr{B}^-_{\mathbf{k}} a^{\dagger}_{\mathbf{k}} e^{-ikx} + \mathscr{B}^+_{\mathbf{k}} b_{\mathbf{k}} e^{-ikx} + \mathscr{B}^+_{\mathbf{k}} b^{\dagger}_{\mathbf{k}} e^{ikx} \right];$$

$$\tilde{\phi}(y) = \phi(y)|_{a \leftrightarrow b, a^{\dagger} \leftrightarrow b^{\dagger}}; \quad [\phi(x), \tilde{\phi}(y)] \equiv 0.$$

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In addition, the generator of time evolution $:\hat{H}:$ (thermal Hamiltonian or **Liouvillian**) and its **space density** $:\widehat{T_{00}}:(x)$ are given by:

$$:\hat{H}: = \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \left(b_{\mathbf{p}}^{\dagger} b_{\mathbf{p}} - a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} \right); \quad :\widehat{T_{00}}:(x) = :T_{00}:(x) - :\tilde{T}_{00}:(x),$$

where (and accordingly for $:\tilde{T}_{00}:(x)$):

$$:T_{00}: = \frac{1}{2} :(\partial_0 \phi)^2: + \frac{1}{2} \sum_{i=1}^{3} :(\partial_i \phi)^2: + \frac{1}{2} m^2 :(\phi)^2:.$$

Energy density quantum inequalities

Study the expectation value of $:\widehat{T_{00}}:(f) = :T_{00}:(f) - :\tilde{T}_{00}:(f)$, i.e. the Liouvillian density smeared in space *and* time with a positive test function $f \in \mathcal{C}_0^\infty(\mathbb{M})$:

$$\left(\Psi, :\widehat{T_{00}}:(f)\Psi\right), \quad \text{with} \quad \Psi \in \mathcal{F}^s(L^2) \otimes \mathcal{F}^s(L^2), (\Psi, \Psi) = N_\Psi^2.$$

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- State independent QEI for the term $:T_{00}:(f)$ (analogous to (Fewster 2012),(Fewster and Eveson 1998)). Smearing in time with a test function $|g(t)|^2$ (at $\mathbf{x} = \mathbf{0}$) we get:

$$\left(\Psi, :T_{00}:(|g|^2)\Psi\right) \geq$$
$$-\int_0^\infty \frac{\mathrm{d}\omega}{\pi} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{\omega_\mathbf{k}}{2} \left[|\hat{g}(\omega + \omega_\mathbf{k})|^2 (\mathscr{B}_\mathbf{k}^+)^2 + |\hat{g}(\omega - \omega_\mathbf{k})|^2 (\mathscr{B}_\mathbf{k}^-)^2\right] N_\psi^2$$

  where the integrals are convergent for every $\beta \in \mathbb{R}$ and $g \in \mathcal{C}_0^\infty(\mathbb{R})$ (**Hadamard property** of the KMS state ω^β).

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We look for a **state dependent QEI** for $-{:}\tilde{T}_{00}{:}(f)$. We restrict our attention to the set of vectors F, obtained perturbing the vacuum vector Ω (representing the KMS state ω^{β}) with operators that belong to the $*$-algebra $\pi^{\beta}(\mathcal{A})$:

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Remark: If a state dependent inequality exists for $-{:}\tilde{T}_{00}{:}(f)$ in terms of $\| \cdot \|_{s}$, than it extends to the operator ${:}\widehat{T_{00}}{:}(f)$.

Outline

Modular theory in a nutshell

Let $\mathcal{H}$ be a **Hilbert space**, $\mathcal{M}$ a **von Neumann algebra** (with commutant $\mathcal{M}'$) and Ω be a **cyclic and separating vector** ($\mathcal{M}\Omega, \mathcal{M}'\Omega$ are dense in $\mathcal{H}$). $\implies$ **modular theory** can be constructed Review: (Borchers 2000):

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- Let $S = J\Delta^{1/2}$ be the polar decomposition of S. Δ (**modular operator**) defines an **automorphism** for $\mathcal{M}$ via the adjoint action of the unitary group $\Delta^{it}, t \in \mathbb{R}$:

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- The state defined by the vector Ω (via $\omega(\cdot) = (\Omega, \cdot\Omega)$) is KMS respect to the **modular evolution** implemented by $\mathrm{Ad}\Delta^{it}(\cdot)$. The opposite is also true.

Non commutative L^p spaces in a nutshell

Noncommutative $\mathbf{L^p}$ spaces generalize usual L_p spaces from integration theory (commutative v.N. algebras) to general v.N. algebras. Given $\mathcal{M}$ over an Hilbert space $\mathcal{H}$ with a cyclic and separating vector Ω, we can construct a family $L^p(\mathcal{M}), 1 \leq p \leq \infty$ of Banach spaces following different approaches ((Araki and Masuda 1982),(Haagerup n.d.),(Kosaki 1984)).

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- For all $1 \leq p' \leq p \leq \infty$

$$\mathcal{M} \subseteq L^p(\mathcal{M}, \Omega) \subseteq L^{p'}(\mathcal{M}, \Omega) \subseteq \mathcal{M}_* \text{ with } \mathcal{M} \text{ dense in each } L^p;$$
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- For $a \in \mathcal{M}$ (corresponding to $a\Omega \in L^2$), seen as an element of L^4, it holds:

$$\|a\|_4 := \|\Delta_\Omega^{1/4} a^* a\Omega\|^{1/2}.$$

Remark: $\Delta^{1/4} a^* a\Omega \in V_\Omega^{1/4}$, the positive cone defined in (Araki 1974). We have a self contained proof that $\|\cdot\|_4$ defines a norm and $\overline{(\mathcal{M}\Omega, \|\cdot\|_4)}^{\|\cdot\|_4} \subseteq \mathcal{H}$.

Affiliated operator

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Definition: Affiliated operator

A closed densely defined operator A is said to be affiliated to a v.N. algebra $\mathcal{M}$ ($A\eta\mathcal{M}$), if $\mathcal{M}'\mathcal{D}(A) \subseteq \mathcal{D}(A)$ and $Aa' \supseteq a'A$ for all $a' \in \mathcal{M}'$.

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We can prove the following two technical lemma:

Lemma (Bostelmann, Cadamuro, S.)

Let be $A\eta\mathcal{M}$ and $\Omega \in \mathcal{D}(A)$. If $A\Omega \in \mathcal{D}(A^*)$, A belongs to the non-commutative L^4 space and we have:

$$\|A\|_4^2 = \|\Delta_\Omega^{1/4} A^* A \Omega\|,$$

Outline

L^4 QEIs

We can now prove the main result of the work, in a general and abstract form:

Theorem (Bostelmann, Cadamuro, S.)

Let be $\tilde{T}$ symmetric and affiliated with the commutant $\mathcal{M}'$ of the v.N. algebra $\mathcal{M}$, and suppose $\Omega \in \mathcal{D}(\tilde{T})$. Then, for every operator $A\eta\mathcal{M}$ s.t. $\Omega \in \mathcal{D}(A)$, $\Omega \in \mathcal{D}(A^*A)$ and $A\Omega \in \mathcal{D}(\tilde{T})$, the following inequality is satisfied:

$$-\left(A\Omega, \tilde{T}A\Omega\right) \geq -C\|A\|_4^2,$$

where C is the finite positive constant $C = \|\Delta^{-\frac{1}{4}}\tilde{T}\Omega\|$.

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where C is the finite positive constant $C = \|\Delta^{-\frac{1}{4}} \tilde{T}\Omega\|$.

(Naive) Proof: We have:

$$\left|\left(A\Omega, \tilde{T} A\Omega\right)\right| = \left|\left(A\Omega, A\tilde{T}\Omega\right)\right| = \left|\left(A^* A\Omega, \tilde{T}\Omega\right)\right| = \left|\left(\Delta^{1/4} A^* A\Omega, \Delta^{-1/4} \tilde{T}\Omega\right)\right|.$$

Using Cauchy–Schwarz inequality:

$$\left|\left(A\Omega, \tilde{T} A\Omega\right)\right| \leq \left\|\Delta^{1/4} A^* A\Omega\right\| \left\|\Delta^{-1/4} \tilde{T}\Omega\right\| = C\|A\|_4^2.$$

This concludes the proof.

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We directly prove that $-{:}\tilde{T}_{00}{:}(f)$ (smeared also in time) is affiliated to $\mathcal{M}'$. $\implies$ We get a **trivial state dependent** inequality for $-{:}\tilde{T}_{00}{:}(f)$:

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The non triviality can be checked via direct examples and descends from the unboundesness of the operator ${:}T_{00}{:}(f)$.

Outline

1. Motivation and general framework
2. Thermal representation of a scalar field
3. Mathematical tools
4. Main result: L^4 QEIs
5. Conclusion and outlook

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- We obtained a state dependent, non trivial QEIs in terms of the L^4 norm:

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Possible future outlook: application of the abstract theorem to other situations in which a similar structure in terms operator affiliated to an algebra and to their commutant is manifest (e.g. double Schwarzschild wedge in Kruskal spacetime, entanglement aspects in information theory).

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L_4 non-commutative space

Let $L_2 \equiv \mathcal{H} = \overline{(\mathcal{M}\Omega, \|\cdot\|_2)}$. On the subspace $\mathcal{M}\Omega$ we can define the map:

$$\begin{aligned} \|\cdot\|_4 : \quad & \mathcal{M}\Omega \to \mathbb{R}, \\ & a\Omega, a \in \mathcal{M} \mapsto \|a\Omega\|_4 = \|\Delta^{1/4} a^* a\Omega\|_2^{1/2}. \end{aligned}$$

The map $\|\cdot\|_4$ is a well defined function (Ω **separating**) and is a **norm**. **Subadditivity** follows from the inequality:

$$\left(\Delta^{1/4} b^* a\Omega, \Delta^{1/4} b^* a\Omega\right) \leq \|\Delta^{1/4} a^* a\Omega\|_2 \|\Delta^{1/4} b^* b\Omega\|_2 = \|a\Omega\|_4^2 \|b\Omega\|_4^2.$$

Let us define the inclusion map:

$$\begin{aligned} \iota : \quad & (\mathcal{M}\Omega, \|\cdot\|_4) \to (L^2, \|\cdot\|_2) \\ & a\Omega \mapsto a\Omega. \end{aligned}$$

It is a continuous inclusion:

$$\|\iota(a\Omega)\|_2 = (a\Omega, a\Omega)^{1/2} = (a^* a\Omega, \Omega)^{1/2} = (\Delta^{1/4} a^* a\Omega, \Omega)^{1/2} \leq \|\Delta^{1/4} a^* a\Omega\|_2^{1/2} = \|a\Omega\|_4.$$

L_4 non-commutative space

We can consider the closure $L_4 := \overline{(\mathcal{M}\Omega, \|\cdot\|_4)}$ and the extension (by continuity) of the inclusion:

$$
\begin{aligned}
\hat{\iota}: \quad & L_4 \to L_2 \\
& \psi = L_4 - \lim_{n\to\infty} \psi_n, \ \psi_n \in \mathcal{M}\Omega \mapsto \hat{\iota}(\psi) := L^2 - \lim_{n\to\infty} \iota(\psi_n) = L^2 - \lim_{n\to\infty} \psi_n.
\end{aligned}
$$

Is $\hat{\iota}$ **injective**?
We can prove:

Proposition

Let be $a_n\Omega$ a Cauchy sequence in L_4. Then, the sequence of vectors $\Delta^{1/4} a_n^* a_n \Omega$ is Cauchy in L_2.

We now want to prove that, if $a_n\Omega \to 0$ in L_2 and $a_n\Omega$ is Cauchy in L_4, then $\Psi = L_4 - \lim a_n\Omega = 0$. By the proposition, this is equivalent to show that $\Psi' = L_2 - \lim \Delta^{1/4} a_n^* a_n \Omega = 0$.

L_4 non-commutative space

We show that the vector ψ' is orthogonal to the dense set of vectors generated acting on Ω with $\tilde{\mathcal{M}} \subset \mathcal{M}'$, the subset of analytic elements:

$$\begin{aligned}
|(b'\Omega, \Psi')| &= \lim_{n\to\infty} |(b'\Omega, \Delta^{1/4} a_n^* a_n \Omega)| \\
&= \lim_{n\to\infty} |(a_n \Delta^{1/4} b' \Delta^{-1/4} \Omega, a_n \Omega)| \\
&= \lim_{n\to\infty} |(\Delta^{1/4} b' \Delta^{-1/4} a_n \Omega, a_n \Omega)| \\
&\leq \lim_{n\to\infty} \|\Delta^{1/4} b' \Delta^{-1/4}\|_{op} \|a_n \Omega\|^2 = 0.
\end{aligned}$$

This concludes the proof.