New modular Hamiltonians

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Tomita–Takesaki modular theory

Tomita–Takesaki modular theory (1/2)

- Tomita–Takesaki theory gives information on structure of von Neumann algebra of operators $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ acting on Hilbert space $\mathcal{H}$, given a cyclic and separating vector $\Omega \in \mathcal{H}$
- Tomita operator S is the closure of the map $S_0 \colon a\Omega \to a^\dagger\Omega$ for $a \in \mathfrak{A}$
- Polar decomposition $S = J\Delta^{\frac{1}{2}}$ gives positive modular operator $\Delta = S^\dagger S \geq 0$ and antilinear modular conjugation J
- Modular flow $\sigma_s(a) = \Delta^{\mathrm{i}s} a \, \Delta^{-\mathrm{i}s} \in \mathfrak{A}$ for $a \in \mathfrak{A}$
- State ω defined by Ω is a thermal (KMS) state: $\omega(\sigma_s(a)b) = (\Omega, \sigma_s(a)b\Omega)$ satisfies $\omega(\sigma_{s-\mathrm{i}}(a)b) = \omega(b \, \sigma_s(a))$, with inverse temperature normalised to $\beta = 1$
- Modular flow is an internal "time evolution", with respect to which the state ω is thermal
- Both J and $\Delta^{\frac{1}{2}}$ map $\mathfrak{A}$ to commutant $\mathfrak{A}' = \{b \in \mathcal{B}(\mathcal{H}) \colon [b, a] = 0 \ \forall \ a \in \mathfrak{A}\}$

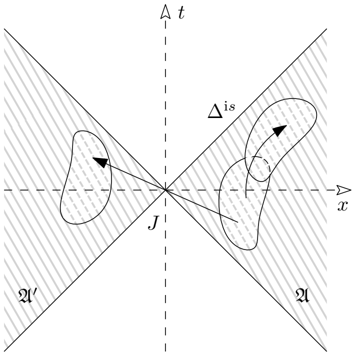
Tomita–Takesaki modular theory (2/2)

- Relative Tomita operator $S_{\Phi|\Psi}$ is closure of map $a\Phi \mapsto a^\dagger \Psi$ for $a \in \mathfrak{A}$ and cyclic and separating vectors Φ, $\Psi \in \mathcal{H}$, relative modular operator $\Delta_{\Phi|\Psi}$ and relative modular conjugation $J_{\Phi|\Psi}$ defined by polar decomposition $S_{\Phi|\Psi} = J_{\Phi|\Psi} \Delta_{\Phi|\Psi}^{1/2}$
- Application: Araki formula relates relative modular Hamiltonian $\ln \Delta_{\Phi|\Psi}$ to relative entropy: $S(\Phi \| \Psi) = -\left(\Phi, \ln \Delta_{\Phi|\Psi} \Phi \right)$ (well-defined and finite)
- Important case: $\Phi = uu'\Omega$ and $\Psi = vv'\Omega$ for unitary operators $u, v \in \mathfrak{A}$ and $u', v' \in \mathfrak{A}'$ commuting with u and v
- $\Rightarrow \Delta_{\Phi|\Psi} = u' v \Delta_\Omega v^\dagger (u')^\dagger$ and $S(\Phi \| \Psi) = -\left(v^\dagger u \Omega, \ln \Delta_\Omega v^\dagger u \Omega \right)$
- Relative entropy between two "excited" states relative to a "vacuum" state Ω can be computed using only the modular Hamiltonian $\ln \Delta_\Omega$ of the "vacuum" state, e.g., for coherent state with $u = u' = v' = \mathbb{1}$ and $v = \exp[\mathrm{i}\phi(f)]$

Known modular Hamiltonians

Known modular Hamiltonians (1/5)

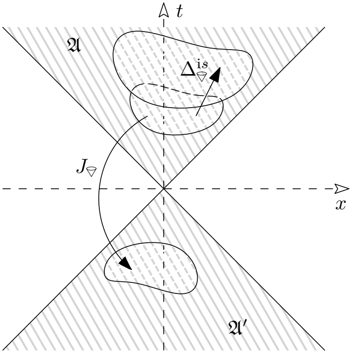
- (Relative) modular Hamiltonian $\ln \Delta_\Omega$ only known in special cases
- Minkowski vacuum state Ω and algebra $\mathfrak{A}$ generated by fields restricted to (right) Minkowski wedge $W_1 = \{x^1 \geq |x^0|\}$: $\ln \Delta_\Omega = \mathrm{i} M_{01}$, the generator of boosts
- Modular conjugation maps fields between left and right wedge
- Result for arbitrary (Wightman) quantum fields, including interacting ones



(Bisognano/Wichmann, On the duality condition for a Hermitian scalar field 1975, On the duality condition for quantum fields 1976)

Known modular Hamiltonians (2/5)

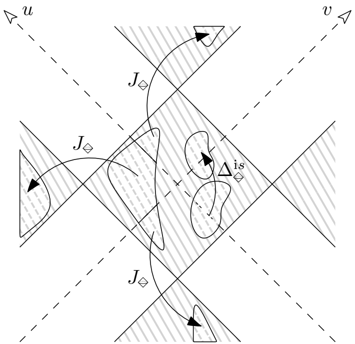
- Minkowski vacuum state Ω and algebra $\mathfrak{A}$ generated by free massless scalar fields restricted to future lightcone with tip $(\tau, \mathbf{0})$: $\ln \Delta_\Omega = 2\pi(D - \tau H)$, a linear combination of time translations and dilations
- Modular conjugation maps to past lightcone



(Buchholz, On the structure of local quantum fields with non-trivial interaction 1977, for $\tau = 0$)

Known modular Hamiltonians (3/5)

- Minkowski vacuum state Ω and algebra $\mathfrak{A}$ generated by free massless scalar fields restricted to diamond of size ℓ with center $(\tau, \mathbf{0})$: $\ln \Delta_\Omega = \frac{\pi}{\ell}\left[(\ell^2 - \tau^2)H + 2\tau D + K\right]$, a linear combination of time translations, dilations and special conformal transformations
- Modular conjugation maps in future/past lightcone and spacelike separated region



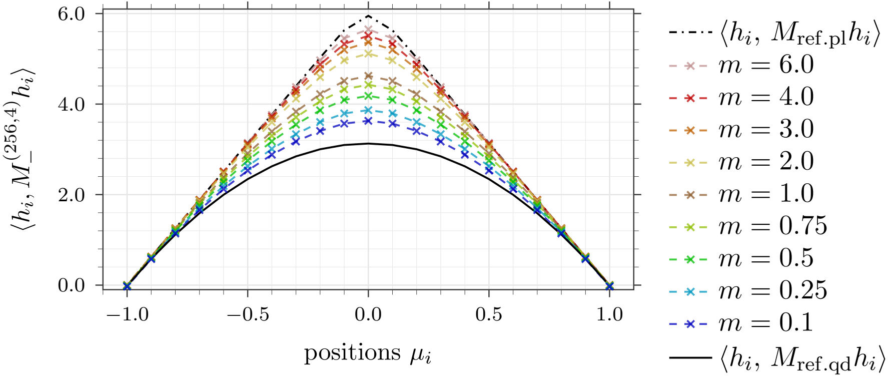
(Hislop/Longo, Modular structure of the local algebras associated with the free massless scalar field theory 1982,
Hislop, Conformal Covariance, Modular Structure, and Duality for Local Algebras in Free Massless Quantum Field Theories 1988, for $\tau = 0$)

Known modular Hamiltonians (4/5)

- Many examples for free, massless fermions and CFTs in 1+1 dimensions (Casini/Huerta/Rehren/Hollands/Tonni/Peschel/...), Schwarzschild (Kay/Sewell/Wald), de Sitter, see Fröb arXiv:2308.14797 for list
- Massive fields (even free) much more complicated, various (unsuccessful) attempts:
- Brunetti/Moretti arXiv:1009.4990: free massive bosons, approached the problem by investigating the algebra on the boundary of the diamond (issue: leading term of the near-boundary expansion is mass-independent, but subleading terms are not)
- Longo/Morsella arXiv:2012.00565: free massive bosons, approached the problem by generalizing the free-field modular Hamiltonian expressed using the stress tensor (issue: candidate expression not symmetric with respect to certain scalar product)

Known modular Hamiltonians (5/5)

- Numerical approach (Bostelmann/Cadamuro/Minz arXiv:2209.04681): one component of $\ln \Delta_\Omega$ (acting on Cauchy data) is quite probably a mass-dependent multiplication operator, which with increasing mass interpolates between the massless modular Hamiltonian for diamonds and the mass-independent one for wedges



Modular Hamiltonian for fermions of small mass

Modular Hamiltonian for fermions of small mass (1/5)

- Cadamuro/Fröb/Minz arXiv:2312.04629: algebra $\mathfrak{A}$ of free fermions of small mass in 1+1 dimensions inside diamond of size ℓ, Minkowski vacuum state Ω
- Simpler than bosons because fermion creation and annihilation operators are bounded
- $\ln \Delta_\Omega = \sum_{a,b=1}^{2} \iint H_{ab}(x,y) {:}\psi_a(x)\psi_b(y){:} \, dx \, dy$ (on Cauchy hypersurface $t = 0$)
- Massless fermions: conformal result $\ln \Delta_\Omega = \frac{\pi}{\ell}(\ell^2 H + K)$, evaluating at $t = 0$ gives $H_{11}(x,y) = -H_{22}(x,y) = \mathrm{i}\frac{\pi}{\ell}(\ell^2 - xy)\delta'(x-y)$, $H_{12}(x,y) = H_{21}(x,y) = 0$
- For free fermions with two-point function $\mathcal{G}$: general formula $H = -\ln(\mathcal{G}^{-1}|_V - \mathbb{1})$, understood as equality between integral kernels on the interval $V = [-\ell, \ell]$ (Araki, On Quasifree States of CAR and Bogoliubov Automorphisms 1970, Peschel, arXiv:cond-mat/0212631, Casini/Huerta arXiv:0903.5284)

Modular Hamiltonian for fermions of small mass (2/5)

- Two-point function of free fermions of mass m:
  $\mathcal{G}_{11}(x,y) = [\mathcal{G}_{22}(x,y)]^* = \frac{1}{2\pi \mathrm{i}}\left(\lim_{\epsilon\to 0^+} \frac{1}{x-y-\mathrm{i}\epsilon} + \frac{m|x-y|\,\mathrm{K}_1(m|x-y|)-1}{x-y}\right)$,
  $\mathcal{G}_{12}(x,y) = [\mathcal{G}_{21}(x,y)]^* = \frac{1}{2\pi \mathrm{i}} m\,\mathrm{K}_0(m|x-y|)$
- Massless case diagonalizes: $\mathcal{G}_{11}(x,y) = [\mathcal{G}_{22}(x,y)]^* = \frac{1}{2\pi \mathrm{i}}\mathcal{P}f\,\frac{1}{x-y} + \frac{1}{2}\delta(x-y)$,
  $\mathcal{G}_{12}(x,y) = [\mathcal{G}_{21}(x,y)]^* = 0$
- Explicit spectral decomposition of the massless case: $G_{ab}\Psi_b^{(k)}(s) = \lambda^{(k)}(s)\Psi_a^{(k)}(s)$ with generalized orthonormal eigenvectors $\Psi_a^{(k)}(s,x) = \delta_a^k\sqrt{\frac{\ell}{\pi}}(\ell+x)^{-\frac{1}{2}-\mathrm{i}s}(\ell-x)^{-\frac{1}{2}+\mathrm{i}s}$, $s\in\mathbb{R}$, eigenvalues $\lambda^{(1)}(s) = \lambda^{(2)}(-s) = (1+\mathrm{e}^{-2\pi s})^{-1} \in (0,1)$ (Koppelman/Pincus, Spectral representations for finite Hilbert transformations 1959)
- Kernel of modular Hamiltonian $H_{ab} = -\sum_{k=1}^{2}\int \ln\left(\frac{1}{\lambda^{(k)}(s)} - 1\right)\Psi_a^{(k)}(s)\Psi_b^{(k)*}(s)\,\mathrm{d}s$ gives the correct result

Modular Hamiltonian for fermions of small mass (3/5)

- First-order perturbation: use resolvent calculus
  $\ln(A^{-1} - \mathbb{1}) = \int_0^\infty \left[(A + \mu\mathbb{1})^{-1} - (\mathbb{1} - A + \mu\mathbb{1})^{-1} \right] \mathrm{d}\mu$ with $A = G|_V$
- $\delta \ln(A^{-1} - \mathbb{1}) = -\int_0^\infty \left[(A+\mu\mathbb{1})^{-1} \delta A (A+\mu\mathbb{1})^{-1} + (\mathbb{1}-A+\mu\mathbb{1})^{-1} \delta A (\mathbb{1}-A+\mu\mathbb{1})^{-1} \right] \mathrm{d}\mu$
- $\delta A_{11} = \delta A_{22} = \mathcal{O}(m^2 \ln m)$, $\delta A_{21} = -\delta A_{12} = \frac{m}{2\pi \mathrm{i}} \ln\left(\frac{m|x-y|}{2} \mathrm{e}^\gamma \right) + \mathcal{O}(m^3 \ln m)$
- $H_{11}(x,y) = [H_{22}(x,y)]^* = \mathrm{i}\pi \frac{\ell^2 - xy}{\ell} \delta'(x-y) + \mathcal{O}(m^2 \ln m)$
- $H_{12}(x,y) = [H_{21}(x,y)]^* = 2\pi \mathrm{i} m \ell K_{12}(x,y) + \mathcal{O}(m^2 \ln m)$
- $K_{12}(x,y) = \ln\left(m\ell \frac{\ell^2 - x^2}{2\ell} \mu \right) \frac{\ell^2 - x^2}{2\ell^2} \delta(x+y) - \frac{\ell^2 - x^2}{2\ell^2} \delta(x-y) + \frac{1}{8\ell^2} |x-y| - \frac{2\ell^2 - x^2 - y^2}{8\ell^2} Pf_\mu \frac{1}{|x+y|}$
- Generically a non-local operator, contrary to the wedge or massless fields
- Agrees with numerical results for small mass (Bostelmann/Cadamuro/Minz in preparation)

Modular Hamiltonian for fermions of small mass (4/5)

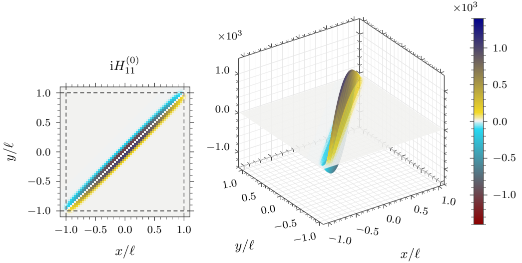


Figure: Massless integral kernel $\mathrm{i}H_{11}(x, y)$

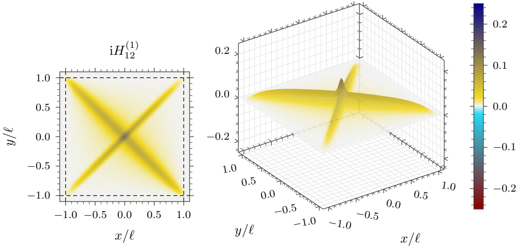


Figure: First-order contribution to massive integral kernel $\mathrm{i}H_{12}(x, y)$ for mass $m\ell = 0.02$

Modular Hamiltonian for fermions of small mass (5/5)

- Relation with standard subspaces (Figliolini/Guido, On the type of second quantization factors 1994, Longo arXiv:2111.11266)
- Complex structure ($I^2 = -\mathbb{1}$, $I^\dagger = -I$) on one-particle Hilbert space $\mathcal{H}$ of real-valued initial data (Majorana fermions) is $I = -\mathrm{i}(\mathbb{1} - 2\mathcal{G})$
- Orthogonal projector E on standard subspace $\mathcal{L} = E\mathcal{H}$ is multiplication by characteristic function of V, $\mathcal{L}$ is separating: $\mathcal{L} \cap I\mathcal{L} = \{0\}$ and cyclic: $\overline{\mathcal{L} + I\mathcal{L}} = \mathcal{H}$
- Relation $E = (\mathbb{1} + S)(\mathbb{1} + \Delta)^{-1}$ to Tomita operator S and modular operator $\Delta = S^\dagger S$ (recall: $S(h + Ik) = h - Ik$ for $h, k \in \mathcal{L}$) $\Rightarrow$ $\ln \Delta = 2 \operatorname{artanh}(\mathbb{1} - E + IEI)$
- Define $R = -I + IE + EI = I(\mathbb{1} - \Delta)(\mathbb{1} + \Delta)^{-1}$, with spectrum $\sigma(R) \subset [-\mathrm{i}, \mathrm{i}]$ $\Rightarrow$ $\ln \Delta = 2I \arctan(R)$ by spectral calculus and $\operatorname{artanh}(\mathrm{i}z) = \mathrm{i} \arctan(z)$ $\forall\, z \neq \pm \mathrm{i}$
- Restriction to subspace fulfills $(I \ln \Delta)|_V = -2E \arctan(R) E = -2 \arctan(EIE)$ and with complex structure $-2 \arctan(EIE) = 2 \arctan[\mathrm{i}(\mathbb{1} - 2\mathcal{G}|_V)] = -\mathrm{i} H_V$

Modular Hamiltonian for massless fermions on a cylinder

Modular Hamiltonian for massless fermions on a cylinder (1/10)

- Cadamuro/Fröb/Pérez-Nadal arXiv:2406.19360: algebra $\mathfrak{A}$ of free massless fermions in 1+1 dimensions inside diamond of size ℓ on cylinder of size $L > 2\ell$
- Antiperiodic (Neveu–Schwarz) boundary conditions $\psi(x+L) = -\psi(x)$: unique ground state $\omega_{\text{NS}}\left(\psi(f)[\psi(g)]^\dagger\right) = \frac{1}{2\mathrm{i}L} \lim_{\epsilon\to 0^+} \iint_0^L \left[\frac{f_1(x) g_1^*(y)}{\sin\left[\frac{\pi}{L}(x-y-\mathrm{i}\epsilon)\right]} - \frac{f_2(x) g_2^*(y)}{\sin\left[\frac{\pi}{L}(x-y+\mathrm{i}\epsilon)\right]} \right] \mathrm{d}x\, \mathrm{d}y$
- Periodic (Ramond) boundary conditions $\psi(x+L) = \psi(x)$: four-parameter family of quasi-free ground (zero-energy) states
$$\omega_{\text{R},h}\left(\psi(f)[\psi(g)]^\dagger\right) = \frac{1}{2\mathrm{i}L} \lim_{\epsilon\to 0^+} \iint_0^L \left[\frac{f_1(x) g_1^*(y)}{\tan\left[\frac{\pi}{L}(x-y-\mathrm{i}\epsilon)\right]} - \frac{f_2(x) g_2^*(y)}{\tan\left[\frac{\pi}{L}(x-y+\mathrm{i}\epsilon)\right]} \right] \mathrm{d}x\, \mathrm{d}y + \sum_{a,b=1}^2 h_{ab} \int_0^L f_a(x)\, \mathrm{d}x \int_0^L g_b^*(y)\, \mathrm{d}y$$
with $h = \frac{h_1+h_2}{2} \mathbb{1} + \frac{h_1-h_2}{2} \begin{pmatrix} \cos\psi & \sin\psi\, \mathrm{e}^{\mathrm{i}\phi} \\ \sin\psi\, \mathrm{e}^{-\mathrm{i}\phi} & -\cos\psi \end{pmatrix}$, $\lvert h_i \rvert \leq \frac{1}{2L}$ for $i \in \{1,2\}$, $\phi \in [0, 2\pi)$ and $\psi \in [0, \pi]$

Modular Hamiltonian for massless fermions on a cylinder (2/10)

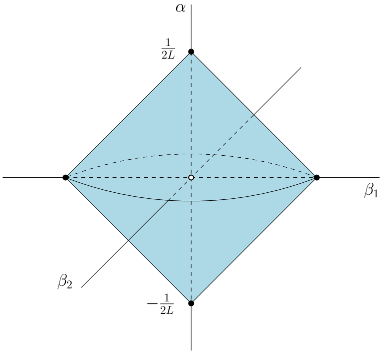


Figure: Quasi-free zero-energy states with $h = \alpha \mathbb{1} + \beta \cdot \sigma$, $|\alpha| + |\beta| \leq \frac{1}{2L}$.
h_1 and h_2 are eigenvalues of h.

- State is pure for $|h_1| = |h_2| = \frac{1}{2L}$, mixed otherwise
- Pure states correspond to extreme points of the double cone (two tips and rim), all other points correspond to mixed states
- Zero-temperature limit of thermal state (white dot) has $h_1 = h_2 = 0$
- Massless limit of massive vacuum (black dot) has $\phi = \psi = \frac{\pi}{2}$ and $h_2 = -h_1 = \frac{1}{2L}$
- Other black dots are zero-momentum excitations of massive vacuum (one or two chiralities)

Modular Hamiltonian for massless fermions on a cylinder (3/10)

- Computation of modular Hamiltonian and modular flow via spectral calculus
- Stone's formula for absolutely continuous spectrum:
  $\mathrm{d}E_A(\mu) = \frac{1}{2\pi\mathrm{i}} \lim_{\epsilon\to 0^+} \left[R_A(\mu + \mathrm{i}\epsilon) - R_A(\mu - \mathrm{i}\epsilon) \right] \mathrm{d}\mu$ with E_A spectral measure of A,
  $R_A(\mu) = (A - \mu\mathbb{1})^{-1}$ resolvent of A, limit taken in strong topology
- Araki–Peschel–Casini–Huerta formula for modular flow inside $V = [-\ell, \ell]$:
  $\Delta^{\mathrm{i}t}\psi(f)\Omega = \int_0^1 \left(\frac{\mu}{1-\mu} \right)^{\mathrm{i}t} \mathrm{d}E_{\mathcal{G}|_V}(\mu)\psi(f)\Omega = \psi(f_t)\Omega \;\Leftrightarrow\; \ln\Delta = -\ln(\mathcal{G}^{-1}|_V - \mathbb{1})$
- Modular Hamiltonian $\ln\Delta$ obtained from derivative at $t = 0$

Modular Hamiltonian for massless fermions on a cylinder (4/10)

- Computation of resolvent via Riemann–Hilbert problem (Carleman's method) (Carleman, Über die Abelsche Integralgl. mit konstanten Integrationsgrenzen 1922)
- Rewrite resolvent equation $\left[\mathcal{G}|_V - \mu\mathbb{1}\right] R(\mu) f = f$ as
  $\left[R(\mu)f\right]_1(x) = \frac{1}{\mu}\left[\lim_{y\to 0^+} S_1(x - \mathrm{i}y) - f_1(x)\right]$ and
  $\left[R(\mu)f\right]_2(x) = \frac{1}{\mu}\left[\lim_{y\to 0^+} S_2(x + \mathrm{i}y) - f_2(x)\right]$ with $S = HR(\mu)f$
- $(Hf)(z)$ defined on $\mathcal{C}_L \setminus [-\ell, \ell]$ with cylinder $\mathcal{C}_L = \mathbb{S}^1 \times \mathbb{R}$
- $\left(H^{\mathrm{NS}} f\right)_a(z) = \frac{1}{2\mathrm{i}L} \int_{-\ell}^{\ell} \frac{\delta_{a1} f_1(y) - \delta_{a2} f_2(y)}{\sin\left[\frac{\pi}{L}(z-y)\right]} \mathrm{d}y$
- $\left(H^{\mathrm{R}} f\right)_a(z) = \int_{-\ell}^{\ell} \left[\frac{1}{2\mathrm{i}L} \frac{\delta_{a1} f_1(y) - \delta_{a2} f_2(y)}{\tan\left[\frac{\pi}{L}(z-y)\right]} + \sum_{b=1}^{2} h_{ab} f_b(y)\right] \mathrm{d}y$
- Convolution with two-point function obtained as boundary value:
  $(\mathcal{G}|_V f)_1(x) = \lim_{y\to 0^+}(Hf)(x - \mathrm{i}y)$ and $(\mathcal{G}|_V f)_2(x) = \lim_{y\to 0^+}(Hf)(x + \mathrm{i}y)$

Modular Hamiltonian for massless fermions on a cylinder (5/10)

- Determine function F on cylinder $\mathcal{C}_L = \mathbb{S}^1 \times \mathbb{R}$ such that (Ramond boundary conditions):
- F is analytic on $\mathcal{C}_L \setminus [-\ell, \ell]$,
- $\lim\limits_{z \to \pm\ell} \left[(z \mp \ell) F(z) \right] = 0$, where the limit is taken in $\mathcal{C}_L \setminus M_{\ell,\alpha}$ for arbitrary but fixed angle $\alpha \in (0, \frac{\pi}{2})$ and $M_{\ell,\alpha} \equiv \{z \colon |\Re z| + |\Im z| \cot \alpha \leq \ell\}$ is a rhombus with angle 2α at the points $z = \pm\ell$,
- $\lim_{y \to 0^+} \left[F(x - \mathrm{i}y) - F(x + \mathrm{i}y) \right] = (f_1(x), -f_2(x))$ for $x \in [-\ell, \ell]$,
- $\lim\limits_{y \to \infty} \left[\hat{h}^+ F(x + \mathrm{i}y) - \hat{h}^- F(x - \mathrm{i}y) \right] = 0$, where $\hat{h}^\pm \equiv \begin{pmatrix} h_{11} \pm \frac{1}{2L} & -h_{12} \\ h_{21} & -h_{22} \pm \frac{1}{2L} \end{pmatrix}$.
- Solution is unique: $F(z) = (Hf)(z)$
- Similar characterization for Neveu–Schwarz boundary conditions

Modular Hamiltonian for massless fermions on a cylinder (6/10)

- For $S = HR(\mu)f$ same conditions except jump
- $\lim_{y\to 0^+}\left[S_1(x-\mathrm{i}y) - S_1(x+\mathrm{i}y)\right] = \left[R_G(\mu)f\right]_1(x) = \frac{1}{\mu}\left[\lim_{y\to 0^+} S_1(x-\mathrm{i}y) - f_1(x)\right]$
  $\lim_{y\to 0^+}\left[S_2(x-\mathrm{i}y) - S_2(x+\mathrm{i}y)\right] = -\left[R_G(\mu)f\right]_2(x) = -\frac{1}{\mu}\left[\lim_{y\to 0^+} S_2(x+\mathrm{i}y) - f_2(x)\right]$
- Non-standard RH problem, since function itself appears on RHS of jump condition
- Let $M(z)$ a fixed analytic function with jump $\lim_{y\to 0^+}\frac{M(x-\mathrm{i}y)}{M(x+\mathrm{i}y)} = 1-\mu^{-1}$, then $A_1 = S_1 M$ and $A_2 = S_2 M^{-1}$ satisfy standard RH problem with jump
  $\lim_{y\to 0^+}\left[A_1(x-\mathrm{i}y) - A_1(x+\mathrm{i}y)\right] = \frac{1}{1-\mu} f_1(x) \lim_{y\to 0^+} M(x-\mathrm{i}y) = a_1(x)$ and
  $\lim_{y\to 0^+}\left[A_2(x-\mathrm{i}y) - A_2(x+\mathrm{i}y)\right] = -\frac{1}{1-\mu} f_2(x) \lim_{y\to 0^+} M^{-1}(x+\mathrm{i}y) = -a_2(x)$
- Magic happens: we know that the unique solution is $A(z) = (Ha)(z)$!

Modular Hamiltonian for massless fermions on a cylinder (7/10)

- Explicit solution for resolvent (Ramond boundary conditions):
$$R^{\mathrm{R}}_{ab}(\mu;x,y) = -\frac{1-2\mu}{2\mu(1-\mu)}\delta_{ab}\delta(x-y) + \frac{1}{\mu(1-\mu)}\left[\frac{1}{2\mathrm{i}L}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_{ab}\mathcal{P}f\cot\left[\tfrac{\pi}{L}(x-y)\right] + g_{ab}(\mu)\right]\left(1-\frac{1}{\mu}\right)^{\frac{\mathrm{i}}{2\pi}[\Omega_a(x)-\Omega_b(y)]}$$
- $\Omega_1(x) = -\Omega_2(x) = \ln\left(\frac{\sin\left[\frac{\pi}{L}(\ell+x)\right]}{\sin\left[\frac{\pi}{L}(\ell-x)\right]}\right)$,
$$g(\mu) = \frac{2h - \mathrm{tr}\,h\,\mathbb{1} + \frac{(1+2Lh_1)(1+2Lh_2)}{4L}\left(1-\frac{1}{\mu}\right)^{2w}\mathbb{1} - \frac{(1-2Lh_1)(1-2Lh_2)}{4L}\left(1-\frac{1}{\mu}\right)^{-2w}\mathbb{1}}{1-4L^2h_1h_2+\frac{1}{2}(1+2Lh_1)(1+2Lh_2)\left(1-\frac{1}{\mu}\right)^{2w}+\frac{1}{2}(1-2Lh_1)(1-2Lh_2)\left(1-\frac{1}{\mu}\right)^{-2w}},\quad w = \ell/L$$
- Agrees with Neumann series solution $R(\mu) = -\mu^{-1}\mathbb{1} - \mu^{-2}\mathcal{G}|_V + \mathcal{O}(\mu^{-3})$
- Similar expression for Neveu–Schwarz boundary conditions

One Eternity later...

(four weeks of computations,
five coffees per day,
and 12% inflation in Argentina)

Modular Hamiltonian for massless fermions on a cylinder (8/10)

- Modular flow $\Delta^{\mathrm{i}t}\psi(f)\Omega = \psi(f_t)\Omega$ with $f_{t,a}(x) = \sum_{b=1}^{2} \int_{-\ell}^{\ell} K_{ab}(t,x,y) f_b(y)\,\mathrm{d}y$
- Neveu–Schwarz boundary conditions: local flow with integral kernel

$$K^{\mathrm{NS}}_{ab}(t,x,y) = \frac{2\pi}{L} \frac{\sinh(\pi t)}{\sin\left[\frac{\pi}{L}(x-y)\right]} \delta\left[2\pi t - \Omega_a(x) + \Omega_a(y)\right] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{ab}$$

- Ramond boundary conditions: generically non-local flow with integral kernel

$$K^{\mathrm{R},h}_{ab}(t,x,y) = K^{\mathrm{NS}}_{ab}(t,x,y) + \frac{1}{4\ell} \sinh(\pi t)\, \mathcal{P}f \frac{1}{\sinh\left[\frac{L}{4\ell}\left[2\pi t - \Omega_a(x) + \Omega_b(y)\right]\right]}$$

$$\times \left[\left(\frac{1+2Lh_1}{1-2Lh_1}\right)^{\frac{\mathrm{i}\left[2\pi t - \Omega_a(x) + \Omega_b(y)\right] L}{4\pi\ell}} \begin{pmatrix} 1+\cos\psi & \sin\psi\,\mathrm{e}^{\mathrm{i}\phi} \\ \sin\psi\,\mathrm{e}^{-\mathrm{i}\phi} & 1-\cos\psi \end{pmatrix}_{ab} \right.$$

$$\left. + \left(\frac{1+2Lh_2}{1-2Lh_2}\right)^{\frac{\mathrm{i}\left[2\pi t - \Omega_a(x) + \Omega_b(y)\right] L}{4\pi\ell}} \begin{pmatrix} 1-\cos\psi & -\sin\psi\,\mathrm{e}^{\mathrm{i}\phi} \\ -\sin\psi\,\mathrm{e}^{-\mathrm{i}\phi} & 1+\cos\psi \end{pmatrix}_{ab} \right]$$

Modular Hamiltonian for massless fermions on a cylinder (9/10)

- (Anti-)Local limit of flow for pure states also for Ramond boundary conditions
- States at the tip:
  $\lim_{h_1=h_2\to\pm\frac{1}{2L}} K^{\mathrm{R},h}_{ab}(t,x,y) = K^{\mathrm{NS}}_{ab}(t,x,y) \pm \frac{2\pi\mathrm{i}}{L}\sinh(\pi t)\,\delta\left[2\pi t - \Omega_a(x) + \Omega_a(y)\right]\delta_{ab}$
- States at the rim: $\lim_{h_1=-h_2\to\pm\frac{1}{2L}} K^{\mathrm{R},h}_{ab}(t,x,y) =$
  $K^{\mathrm{NS}}_{ab}(t,x,y) \pm \frac{2\pi\mathrm{i}}{L}\sinh(\pi t)\,\delta\left[2\pi t - \Omega_a(x) + \Omega_b(y)\right]\begin{pmatrix}\cos\psi & \sin\psi\,\mathrm{e}^{\mathrm{i}\phi}\\ \sin\psi\,\mathrm{e}^{-\mathrm{i}\phi} & -\cos\psi\end{pmatrix}_{ab}$
- Recall that $\Omega_1(x) = -\Omega_2(x) = \ln\left(\frac{\sin\left[\frac{\pi}{L}(\ell+x)\right]}{\sin\left[\frac{\pi}{L}(\ell-x)\right]}\right)$
- Need distributional limit:
  $\lim_{a\to 0}\left[a^{\mathrm{i}t}\,\mathcal{P}f\,\frac{1}{\sinh(\pi t)} - \mathrm{i}\frac{a-1}{a+1}\delta(t)\right] = \lim_{a\to\infty}\left[a^{\mathrm{i}t}\,\mathcal{P}f\,\frac{1}{\sinh(\pi t)} - \mathrm{i}\frac{a-1}{a+1}\delta(t)\right] = 0$

Modular Hamiltonian for massless fermions on a cylinder

(10/10)

- Modular Hamiltonian obtained by expanding for small t
- Neveu–Schwarz boundary conditions:

$$H^{\mathrm{NS}}_{ab}(x,y) = 2\mathrm{i}L\frac{\sin^2\left(\frac{\pi}{L}\ell\right)-\sin\left(\frac{\pi}{L}x\right)\sin\left(\frac{\pi}{L}y\right)}{\sin\left(\frac{2\pi\ell}{L}\right)}\delta'(x-y)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}_{ab}$$

- Ramond boundary conditions: $H^{\mathrm{R},h}_{ab}(x,y) = H^{\mathrm{NS}}_{ab}(x,y) + \frac{\mathrm{i}\pi}{4\ell}\mathcal{P}f\frac{1}{\sinh\left[\frac{L}{4\ell}\left[\Omega_a(x)-\Omega_b(y)\right]\right]}$

$$\times\left[\left(\frac{1+2Lh_1}{1-2Lh_1}\right)^{\frac{-\mathrm{i}\left[\Omega_a(x)-\Omega_b(y)\right]L}{4\pi\ell}}\begin{pmatrix}1+\cos\psi & \sin\psi\,\mathrm{e}^{\mathrm{i}\phi}\\ \sin\psi\,\mathrm{e}^{-\mathrm{i}\phi} & 1-\cos\psi\end{pmatrix}_{ab}\right.$$
$$\left.+\left(\frac{1+2Lh_2}{1-2Lh_2}\right)^{\frac{-\mathrm{i}\left[\Omega_a(x)-\Omega_b(y)\right]L}{4\pi\ell}}\begin{pmatrix}1-\cos\psi & -\sin\psi\,\mathrm{e}^{\mathrm{i}\phi}\\ -\sin\psi\,\mathrm{e}^{-\mathrm{i}\phi} & 1+\cos\psi\end{pmatrix}_{ab}\right]$$

- Again (anti-)local limit for pure states

Thank you for your attention

Questions?

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References: arXiv:2312.04629 and arXiv:2406.19360