Vertex algebras and non-unitary Wightman CFTs

Sebastiano Carpi

University of Rome "Tor Vergata"

LQP 49

Erlangen, 9 November 2024

Based on a joint work with C. Raymond, Y. Tanimoto and J. Tener
(arXiv:2409.08454 [math-ph])

Introduction

- Chiral conformal field theories in two space-time dimensions (chiral CFTs) are theories generated by local quantities (fields, observables, ...) depending only one of the light-ray cordinates $x_+ := t + x$ and $x_- := t - x$. They can be seen as QFT on the unit circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ (the compactified light-ray). Chiral CFTs can be considered the building blocks of two-dimensional CFTs.

- A remarkable aspect of conformal (quantum) field theory (CFT) in two space-time dimensions is that it provides a bridge between an impressive number of different areas of physics and mathematics

- Besides the various Euclidean approaches, the main direct mathematical axiomatizations of chiral Lorentzian CFTs are: Wightman chiral CFTs, conformal nets (the chiral CFT version of AQFT) and vertex algebras. A clear understanding of the mathematical relations between these three different axiomatizations appears to be desirable for various important reasons.

- In all cases the spacetime-symmetry group can be described through the full conformal group $\mathrm{Diff}^+(S^1)$ or its finite-dimensional subgoup Möb (or the corresponding Lie algebras). In this talk I will focus on the more general Möb covariant CFTs.
- The conformal net approach is intrinsically functional analytic and unitary. AQFT is historically deeply related to the Wightman QFT which also have important analytic aspects. In particular various specific conditions allowing the construction of an AQFT model from a Wightman QFT model or viceversa have been studied in the last sixty years.
- Although the vertex algebra approach is heuristically motivated by the Wightman axioms, it is mainly of algebraic nature.
- In order to understand the connection between conformal nets and unitary vertex algebras it is natural to look at (unitary) Wightman CFTs as a possible bridge.

- This strategy was used by Kawahigashi, Longo, Weiner and me [CKLW18] in order to study a general connection (i.e. model independent) between vertex operator algebras (VOAs : an important class of vertex algebras with full conformal invariance) and conformal nets. In [CKLW18] we assumed from the beginning that the fields satisfies polynomial energy bounds, and prove that with this assumption, unitary VOAs gives rise to Wightman CFTs. Conversely, Wightman CFTs with polynomial energy bounds, $\mathrm{Diff}^+(S^1)$ covariance, and finite-dimensional conformal energy eigenspaces gives rise to unitary VOAs (this is more or les explicit in [CKLW18]).
- Subsequently Raymond, Tener and Tanimoto [RTT22] improved the results in [CKLW18]. Assuming finite-dimensional conformal energy eigenspaces, they proved that Möbius vertex algebra are equivalent to Wightman CFTs satisfying the uniformly bounded order condition (a weaker version of polynomial energy bounds).
- In this talk I will describe some recent results by Raymond, Tanimoto, Tener and me [CRTT24] which give a remarkable generalization of the results in [RTT22] by avoiding most of the extra assumptions (including unitarity!)

- Note that there are a lot of very interesting examples of non-unitary vertex algebras. Some of them admit a direct physical interpretation. This is e.g.the case of vertex algebras associated to certain models of statistical physics such as the Yang-Lee model, critical dense polymers, and critical percolation.
- Other remarkable examples are the vertex algebras associated to four-dimensional $N = 2$ superconformal field theory. Moreover, non-unitary vertex algebras naturally occur by deforming unitary vertex algebra and can give important informations about the unitary regime.
- It is natural to expect that a direct connection with non-unitary Wightman theories could provide a useful tool to study these models.

Non-unitary Wightman (chiral) CFTs

- Warning: by non-unitary we mean possibly non-unitary
- $\mathcal{D}$ = complex vector space
- An operator-valued distribution on $\mathcal{D}$ is a linear map

$$C^{\infty}(S^1) \ni f \mapsto \varphi(f) \in \mathrm{End}(\mathcal{D})$$

- $\mathcal{F}$ = family of operator-valued distributions on $\mathcal{D}$

- The $\mathcal{F}$-strong topology on $\mathcal{D}$ is the strongest locally convex topology such that, for any finite subset $\{\varphi_1, \ldots, \varphi_k\} \subset \mathcal{F}$ and any vector $\Phi \in \mathcal{D}$, the map

$$C^\infty(S^1)^k \ni (f_1, \ldots, f_k) \mapsto \varphi_1(f_1) \ldots \varphi_k(f_k)\Phi \in \mathcal{D}$$

 is (separately or equivalently jointly) continuous.

- If the $\mathcal{F}$-strong topology is Hausdorff we say that $\mathcal{F}$ acts regularly on $\mathcal{D}$.

- Let $\mathcal{D}^*_{\mathcal{F}}$ denote the subset of the algebraic dual $\mathcal{D}^*$ whose elements are the linear functional $\lambda \in \mathcal{D}^*$ which are $\mathcal{F}$-strongly continuous. Then, a linear functional $\lambda \in \mathcal{D}^*$ belongs to $\mathcal{D}^*_{\mathcal{F}}$ iff all the maps

$$C^\infty(S^1)^k \ni (f_1, \ldots, f_k) \mapsto \lambda\left(\varphi_1(f_1) \ldots \varphi_k(f_k)\Phi\right) \in \mathbb{C}$$

 are continuous. Moreover, $\mathcal{F}$ acts regularly on $\mathcal{D}$ iff $\mathcal{D}^*_{\mathcal{F}}$ separates points

Example

- Assume that the vector space $\mathcal{D}$ is equipped with a non-degenerate Hermitian form $(\cdot|\cdot)$ such that all the maps

$$C^\infty(S^1) \ni f \mapsto (\Psi|\varphi(f)\Phi) \in \mathbb{C}, \quad \Psi, \Phi \in \mathcal{D}, \ \varphi \in \mathcal{F}$$

are continuous.

- Assume furthermore that $\mathcal{F}$ is self-adjoint, i.e. for every $\varphi \in \mathcal{F}$ there is a $\varphi^\dagger \in \mathcal{F}$ such that

$$(\Psi|\varphi(f)\Phi) = (\varphi^\dagger(\bar{f})\Psi|\Phi), \quad \Psi, \Phi \in \mathcal{D}, \ f \in C^\infty(S^1) .$$

- Then, for any $\Psi \in \mathcal{D}$, the linear functional λ_Ψ defined by $\lambda_\Psi(\Phi) := (\Psi|\Phi)$ belongs to $\mathcal{D}^*_\mathcal{F}$. Hence, $\mathcal{D}^*_\mathcal{F}$ separates points because $(\cdot|\cdot)$ is non-degenerated so that $\mathcal{F}$ acts regularly on $\mathcal{D}$.

- The group $\text{Möb} \simeq PSU(1,1) \simeq \mathrm{PSL}(2,\mathbb{R})$ of Möbius transformations of S^1 is the group of transformations of the form

$$S^1 \ni z \mapsto \frac{az+b}{\overline{b}z+\overline{a}} \in S^1, \quad |a|^2 - |b|^2 = 1\,.$$

- By taking $a = e^{i\frac{t}{2}}$ and $b = 0$ we get the one-parameter subgroup of rotations $R(t)$ of S^1.
- Fore every $d \in \mathbb{Z}_{\geq 0}$ we have continuous action β_d of Möb on $C^\infty(S^1)$ defined by

$$(\beta_d(\gamma)f)\,(z) := \left(X_\gamma(\gamma^{-1}(z))\right)^{d-1} f(\gamma^{-1}(z)), \quad \gamma \in \text{Möb},\ f \in C^\infty(S^1)$$

where

$$X_\gamma(e^{i\vartheta}) := -i\frac{d}{d\vartheta}\log(\gamma(e^{i\vartheta}))$$

- Assume now that the vector space $\mathcal{D}$ is equipped with a representation U of Möb i.e. $U : \text{Möb} \to \mathrm{GL}(\mathcal{D})$ is a group homomorphism. An operator-valued distribution φ on $\mathcal{D}$ is Möbius covariant with dimension $d \in \mathbb{Z}_{\geq 0}$ if

$$U(\gamma)\varphi(f)U(\gamma)^{-1} = \varphi(\beta_d(\gamma)f), \quad \gamma \in \text{Möb},\ f \in C^\infty(S^1)$$

- $\Phi \in \mathcal{D}$ has conformal dimension $d \in \mathbb{Z}$ if $U(t)\Phi = e^{idt}\Phi$ for all $t \in \mathbb{R}$.

Let $\mathcal{D}$ be a vector space equipped with a family $\mathcal{F}$ of operator-valued distributions acting regularly on $\mathcal{D}$, with a representation U of Möb and with a non-zero vector Ω (the the vacuum vector). The data $(\mathcal{D}, \mathcal{F}, U, \Omega)$ forms a (not-necessarily unitary) Möbius covariant Wightman CFT on S^1 if they satisfy the following axioms

- **Möbius covariance:** For every $\varphi \in \mathcal{F}$ there is a $d \in \mathbb{Z}_{\geq 0}$ such that φ is Möbius covariant with dimension d.
- **Locality:** If f and g have disjoint support then $[\varphi_1(f), \varphi_2(g)] = 0$ for any pair $\varphi_1, \varphi_2 \in \mathcal{F}$.
- **Spectrum condition:** If $\Phi \in \mathcal{D}$ has conformal dimension $d < 0$ then $\Phi = 0$.
- **Vacuum:** Ω is U-invariant and $\mathcal{D}$ is spanned by vectors of the form $\varphi_1(f_1) \dots \varphi_k(f_k)\Omega$ with $\varphi_1, \dots \varphi_k \in \mathcal{F}$ and $f_1, \dots f_k \in C^\infty(S^1)$

Möbius vertex algebras

Let $\mathcal{V}$ be a complex vector space equipped with a non-zero vector Ω, endomorphisms L_{-1}, L_0, L_1 satisfying the $\mathfrak{sl}(2, \mathbb{C})$ commutation relations

$$[L_1, L_0] = L_1,\ [L_{-1}, L_0] = -L_{-1},\ [L_1, L_{-1}]2L_0$$

and with a linear map (the state field correspondence)

$$\mathcal{V} \ni v \mapsto Y(v, z)$$

where the fields (the vertex operators) $Y(v, z)$ are formal Laurent series (equivalently operator-valued formal distributions)

$$\sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1}, \quad v_{(n)} \in \mathrm{End}(\mathcal{V})$$

Then $\mathcal{V}$ is a ($\mathbb{Z}_{\geq 0}$-graded) Möbius vertex algebra if it satisfies following six condtions:

- 1) $\mathcal{V} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{V}(n)$, where $\mathcal{V}(n) := \ker(L_0 - n1_{\mathcal{V}})$
- 2) $Y(\Omega, z) = 1_{\mathcal{V}},\ Y(v,z)\Omega|_{z=0} = v$
- 3) $L_m \Omega = 0$, $m = -1, 0, 1$
- 4) $[L_m, Y(v,z)] = \sum_{j=0}^{m+1} \binom{m+1}{j} z^{m+1-j} Y(L_{j-1}v, z)$
- 5) $Y(L_{-1}v, z) = \frac{d}{dz} Y(v,z)$
- 6) $(z-w)^N [Y(v,z), Y(u,w)] = 0$ for N sufficiently large

- If $v \in \mathcal{V}(d)$ v is said to be homogeneous of dimension d
- If $v \in \mathcal{V}(d)$ and $L_1 v = 0$ v is said to be quasiprimary of dimension $d_v := d$
- For $v\ in \mathcal{V}$ we define the endomorphisms v_n, $n \in \mathbb{Z}$ by $Y(z^{L_0} v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n}$. Then, if $v \in \mathcal{V}(d)$ we have $Y(v,z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-d}$

From Möbius vertex algebras to Wightman CFTs

- Let $\mathcal{V}$ be a Möbius vertex algebra and let $S \subset \mathcal{V}$ be a generating set of quasiprimary vectors. This means that each $v \in S$ is quasiprimary and $\mathcal{V}$ is sapanned by vectors of the form $v^1_{m_1} \dots v^k_{m_k}\Omega$ with $v^1, \dots, v^k \in S$. The idea is now to construct a vector space $\mathcal{D}_S \supset \mathcal{V}$ so that the formal distributions $Y(v,z)$, $v \in S$, can be upgraded to Möbius covariant operator-valued distributions acting regularly on $\mathcal{D}_S$.

- We construct $\mathcal{D}_S$ as a subspace of the algebraic completion $\hat{\mathcal{V}} := \prod_{n\in\mathbb{Z}_{\geq 0}} \mathcal{V}(n)$ of $\mathcal{V} = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathcal{V}(n)$. An important role is played by the restricted dual $\mathcal{V}' := \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathcal{V}(n)^*$ of $\mathcal{V}$.

We need the following two basic technical ingredients.

- 1) **Uniformly bounded order:** Given $v^1, \dots, v^k, u \in \mathcal{V}$ and $\lambda \in \mathcal{V}'$ there exists a polynomial p such that

$$|\lambda(v^1_{m_1} \dots v^k_{m_k} u)| \leq p(m_1, \dots, m_k), \quad m_1 \dots m_k \in \mathbb{Z}.$$

The degree of p may be bounded independent of u and λ.

- 2) **Local finite dimension:** Let $v^1, \dots, v^k, u \in \mathcal{V}$. Then, for every $n \in \mathbb{Z}_{\geq 0}$, $\mathrm{span}\{v^1_{m_1} \dots v^k_{m_k}\Omega : m_1 + \cdots + m_k = -n\} \subset \mathcal{V}(n)$ is finite-dimensional.
- We consider $\hat{\mathcal{V}}$ with the weak topology induced by $\mathcal{V}' \subset \hat{\mathcal{V}}^*$
- Every $f \in C^\infty(S^1)$ can be represented by a rapidly convergent Fourier series

$$f(z) = \sum_{n \in \mathbb{Z}} \hat{f}(n) z^n$$

- Now, let $S \subset \mathcal{V}$ be a generating set of quasiprimary vectors.

- For all $v^1, \ldots, v^k \in S$ and all $f_1, \ldots, f_k \in C^\infty(S^1)$ the series

$$\sum_{m_1,\ldots,m_k} \hat{f}_1(m_1)\ldots\hat{f}_k(m_k)v^1_{m_1}\ldots v^k_{m_k}\Omega$$

converges to an element $X_{v_1,\ldots,v_k,\Omega}(f_1,\ldots,f_k) \in \hat{\mathcal{V}}$ and we define $\mathcal{D}_S$ to be the linear span of the set

$$\{X_{v^1,\ldots,v^k,\Omega}(f_1,\ldots,f_k) : k \in \mathbb{Z}_{\geq 0}, v^1,\ldots,v^k \in S, f_1,\ldots,f_k \in C^\infty(S^1)\}$$

$$\subset \hat{\mathcal{V}}$$

- For every $v \in S$, every $f \in C^\infty(S^1)$ and every $\Phi \in \mathcal{D}_S$ the series

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)v_n\Phi$$

converges to an element $Y(v,f)\Phi \in \mathcal{D}_S$ and the map $\Phi \mapsto Y(v,f)\Phi$ defines an endomorphism Y(v,f) of $\mathcal{D}_S$ (smeared vertex operator).

- $f \mapsto Y(v, f)$ is linear, i.e. an operator-valued distribution φ_v on $\mathcal{D}_S$. Moreover, $\mathcal{D}_S$ be the subspace of $\mathcal{D}$ spanned by vectors of the form $Y(v^1, f_1) \dots Y(v^k, f_k)\Omega$ with $v^1, \dots, v^k \in S$.
- We have $\mathcal{V} \subset \mathcal{D}_S \subset \hat{\mathcal{V}}$. Moreover, $iL_0, L_1 - L_{-1}, i(L_1 + L_{-1})$ gives rise to a representation of $\mathfrak{sl}(2, \mathbb{R})$ which integrates to a representation U_S of Möb on $\mathcal{D}_S$ leaving Ω invariant.
- Set, $\mathcal{F}_S := \{\varphi_v : v \in S\}$.

Theorem (A , [CRTT24])

$(\mathcal{D}_S, \mathcal{F}_S, U_S, \Omega)$ *is a Möbius covariant Wightman CFT on* S^1.

From Wightman CFTs to Möbius vertex algebras

- Let $(\mathcal{D}, \mathcal{F}, U, \Omega)$ is a Möbius covariant Wightman CFT on S^1.
- For $m \in \mathbb{Z}$ define $e_m \in \mathcal{C}^\infty(S^1)$ by $e_m(z) := z^m$ and let

$$\mathcal{V} := \mathrm{span}\{\varphi_1(e_{m_1}) \dots \varphi_k(e_{m_k})\Omega : \varphi_1, \dots, \varphi_k \in \mathcal{F}, m_1, \dots, m_k \in \mathbb{Z}\}$$

- For every $\varphi \in \mathcal{F}$ with conformal dimension d_φ we define a formal Laurent series $\hat{\varphi}(z)$ with coefficients in $\mathrm{End}(\mathcal{V})$ by

$$\hat{\varphi}(z) = \sum_{m \in \mathbb{Z}} \varphi(e_m) z^{-m-d_\varphi}$$

- The key step is the prooof that Wightman locality implies vertex algebra locality for the operator-valued formal distributions $\hat{\varphi}(z)$, $\varphi \in \mathcal{F}$. In the unitary case, and assuming each $\mathcal{V}(n)$ to be finite-dimensional $\mathcal{V}(n)$, this was done before in [CKLW18] with the additional assumption that the fields in $\mathcal{F}$ satisfy polynomial energy bounds and in [RTT22] under the weaker assumption of uniformly bounded order.

- In [CRTT24], besides unitarity, and the finite-dimensions of the $\mathcal{V}(n)$, we also drop these additional boundness assumptions. In particular this eventually shows that for a Möbius covariant Wightman CFT the uniformly bounded order condition is a consequence of the other defining conditions.

Lemma ([CRTT24])

For all $\varphi_1, \varphi_2 \in \mathcal{F}$ *we have* $(z_1 - z_2)^{d_{\varphi_1}+d_{\varphi_2}}[\hat{\varphi}_1(z_1), \hat{\varphi}_2(z_2)] = 0$.

- For every $\varphi \in \mathcal{V}$ we set $v_\varphi := \varphi(e_{-d_\varphi})\Omega$ and consider the set $S := \{v_\varphi : \varphi \in \mathcal{F}\}$.
- It follows by the existence theorem for vertex algebras that there is a unique Möbius vertex algebra structure on the vector space $\mathcal{V}$ such that, for any $\varphi \in \mathcal{F}$, v_φ are quasiprimary vectors and $Y(v_\varphi, z) = \hat{\varphi}(z)$. Moreover, the vertex algebra $\mathcal{V}$ is generated by S.

Theorem (B , [CRTT24])

$\mathcal{V}$ is a Möbius vertex algebra and $S \subset \mathcal{V}$ is a generating set of quasiprimary vector. Moreover, the Wightman CFT $(\mathcal{D}_S, \mathcal{F}_S, U_S, \Omega)$ obtained from $\mathcal{V}$ and S through Theorem A coincides with $(\mathcal{D}, \mathcal{F}, U, \Omega)$.

- Theorem A and Theorem B together imply the following.

Theorem (C , [CRTT24])

There is a natural equivalence of categories between Wightman CFTs on S^1 and Möbius vertex algebras equipped with a generating family of quasiprimary vectors.

THANK YOU!