Trace anomaly of the Stress-Energy Tensor in an interacting $\lambda\phi^4$ theory on curved spacetime

Workshop LQP49 – Erlangen

Beatrice Costeri

Università di Pavia

07.11.24

Motivation and state of the art in theoretical hadronic physics

Hadronic physics: D-term – the *last unknown property*

For the real Φ^4 model[1]:

- Computations at one-loop $\mathcal{O}(\lambda)$ only
- Results consistent at first perturbative order with our approach.

$\Rightarrow$ Study of the Φ^3 model for the computation of GPDs.[2]

[1] B. Maynard, *ArXiv: 2406.08857 [hep-ph]* (2024).

[2] M.V.Polyakov and P.Schweitzer, *Int. J. Mod. Phys. A* **33** 26 (2018)

State of the art in AQFT

The Stress-Energy tensor in AQFT:

- Axiomatic approach to the construction of $T_{\mu\nu}$[3]
- Local perturbative formulation of interacting quantum field theories on curved spacetimes[4] [5]
- Axiomatic QFT on curved spacetimes [6]
- Construction of $T_{\mu\nu}$ from existing freedoms[7]

[3] R.M.Wald, *Commun. Math. Phys.*, ***54*** *(1977)*

[4] R.Brunetti and K. Fredenhagen, *Commun. Math. Phys.* ***208****(3) (2000)*

[5] R.Brunetti, M.Duetsch and K.Fredenhagen, ArXiv: 0411072 (2009)

[6] S.Hollands and R.M. Wald, *Reviews in Mathematical Physics (2002-2005)*

[7] V.Moretti, *Comm. Math. Phys.* ***232*** *(2) (2003)*

The pAQFT approach

Key ideas of pAQFT:

1. $\mathcal{A}_{cl}(\mathcal{M}) := (\mathcal{F}_{\mu c}(\mathcal{M}), \cdot, *)$
2. Deformation quantization: $\star$-product
   $\rightarrow \mathcal{A}(\mathcal{M}) := (\mathcal{F}_{loc}(\mathcal{M}), \star_H, *)$
3. Wick-ordering: contractions between fields
4. Interacting theory as a deformation of the free theory
   $\rightarrow$ **perturbative** approach
5. Expectation value wrt a state as evaluation at $\phi = 0$.

Spaces of functionals

Let $\mathcal{M} \equiv (\mathcal{M}, g)$ a globally hyperbolic spacetime.

The kinematic structure of a real scalar field theory is codified in $\mathcal{E}(\mathcal{M}) \equiv C^\infty(\mathcal{M})$, which is a Fréchet space.

Space of functionals on $\mathcal{E}(\mathcal{M})$:

$$\mathcal{F}(\mathcal{M}) \equiv \mathcal{E}(\mathcal{M})'$$

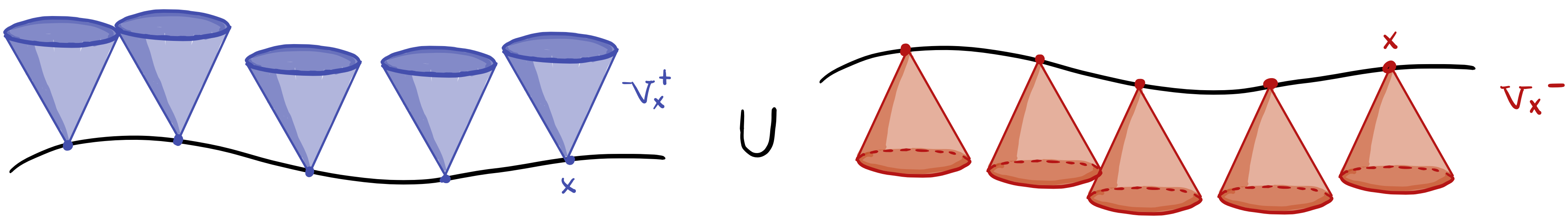
$\Rightarrow$ functional derivatives.

Microcausal functionals

We call microcausal functionals the elements of

$$\mathcal{F}_{\mu c}(\mathcal{M}) := \left\{ F \in \mathcal{F}(\mathcal{M}) \,|\, F^{(k)} \in \mathcal{E}'(\mathcal{M}^k), WF(F^{(k)}) \subset G_k(\mathcal{M}), \forall k \right\},$$

$$G_k(\mathcal{M}) := T^*(\underbrace{\mathcal{M} \times \ldots \times \mathcal{M}}_{k}) \setminus \left(\bigcup_{x \in \mathcal{M}} (V_x^+)^k \cup \bigcup_{x \in \mathcal{M}} (V_x^-)^k \right).$$



Regular and local functionals

A regular functional is an element of

$$\mathcal{F}_{reg}(\mathcal{M}) := \left\{ F \in \mathcal{F}_{\mu c}(\mathcal{M}) \mid F^{(k)} \in \mathcal{D}(\mathcal{M}^k) \hookrightarrow \mathcal{E}'(\mathcal{M}^k), \forall k \in \mathbb{N} \right\}.$$

A local functional is an element of

$$\mathcal{F}_{loc}(\mathcal{M}) := \Big\{ F \in \mathcal{F}_{\mu c}(\mathcal{M}) \,|\, F^{(1)} \in \mathcal{E}(\mathcal{M}) \hookrightarrow \mathcal{D}'(\mathcal{M}),$$
$$supp(F^{(k)}) \subset Diag_k(\mathcal{M}), \forall k \in \mathbb{N} \Big\}.$$

Henceforth, we focus on polynomial functionals.

Algebra of observables

We call classical microcausal algebra the commutative, unital, *-algebra $\mathcal{A}_{cl}(\mathcal{M}) := (\mathcal{F}_{\mu c}(\mathcal{M}), \cdot, *)$, where

- $\mathcal{F}_{\mu c}(\mathcal{M})$ is the space of microcausal functionals,
- pointwise product: $\cdot : \mathcal{F}_{\mu c}(\mathcal{M}) \times \mathcal{F}_{\mu c}(\mathcal{M}) \to \mathcal{F}_{\mu c}(\mathcal{M})$

$$(F, G) \mapsto F \cdot G := \iota^*(F \otimes G).$$

- involution: $* : \mathcal{F}_{\mu c}(\mathcal{M}) \to \mathcal{F}_{\mu c}(\mathcal{M})$

$$F^*(\varphi) := \overline{F(\varphi)}, \ \varphi \in \mathcal{E}(\mathcal{M}).$$

Algebraic setting: Hadamard parametrix

Hadamard state $\Delta_+ \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$:

$$\Delta_+(x, y) = H_\lambda(x, y) + W(x, y),$$

where W encompasses our freedom in the choice of a state.

Algebraic setting: Hadamard parametrix

Hadamard state $\Delta_+ \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$:

$$\Delta_+(x,y) = H_\lambda(x,y) + W(x,y),$$

where W encompasses our freedom in the choice of a state.

Hadamard parametrix (local form):

$$H_\lambda(x,y) = \lim_{\epsilon \to 0^+} \frac{U_d(x,y)}{\sigma_\epsilon^{\frac{d-2}{2}}(x,y)} + V_d(x,y) \frac{\ln \sigma_\epsilon(x,y)}{\lambda}.$$

H_λ codifies the universal, *local and covariant component* of any Hadamard state.

Algebraic setting: Feynman and Dyson parametrices

Singular structure of the Feynman parametrix:

$$\mathrm{WF}(H_F) = \{(x, k_x, y, -k_y) \in T^*(\mathcal{M} \times \mathcal{M}) \setminus \{0\} \mid (x, k_x) \sim_F (y, k_y)\} \cup \mathrm{WF}(\delta_{\mathrm{Diag}_2}).$$

Leading singularity in 4D $\to \sigma^{-1}(x, y)$, Powers of H_F: $\rho_{k,F} = 2k - 4$.
A renormalization procedure needs to be implemented!

Similarly for the Dyson parametrix: $H_{AF} = H_F^*$

Dynamics: Equations of motion

The **free** dynamics is ruled by the Klein-Gordon operator

$$P_0 = \Box_g - m_\phi^2 + \xi R.$$

$\Rightarrow \mathcal{E}_S(\mathcal{M})$ is the solution space.

Action of the free theory

$$S_0[\phi] = \int_{\mathcal{M}} - \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{m_\phi^2 + \xi R}{2} \phi^2 \right] f(x) d\mu_x, \quad f \in \mathcal{D}(\mathcal{M})$$

Classical Improved Stress-Energy Tensor

$$T_{\mu\nu} := \frac{-2}{\sqrt{|g|}} \frac{\delta S_0[\phi]}{\delta g_{\mu\nu}} \Rightarrow T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2} g_{\mu\nu}[\partial^\rho\phi\partial_\rho\phi + m^2\phi^2] +$$
$$+ \xi G_{\mu\nu}\phi^2 + \xi[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu]\phi^2,$$

At a functional level: $\mathbf{T}_{\mu\nu,f} \in \mathcal{F}_{loc}(\mathcal{M})$

$$\mathbf{T}_{\mu\nu,f}[\phi] = \int_{\mathcal{M}} T_{\mu\nu}(z) f^{\mu\nu}(z)\, d\mu_z, \ \forall\phi \in \mathcal{E}(\mathcal{M}),$$

where $f^{\mu\nu} \in \Gamma_0(T\mathcal{M}^{\otimes_s 2})$.

Conservation and trace of the improved Stress-Energy Tensor

For all $\phi_0 \in \mathcal{E}_S(\mathcal{M})$:

Conservation $(d = 4)$:

$$\nabla^{\mu} \mathbf{T}_{\mu\nu,f}[\phi_0] = 0.$$

Trace $(d = 4)$:

$$T_f[\phi_0] := g^{\mu\nu} \mathbf{T}_{\mu\nu,f}[\phi_0] = -m_\phi^2 \Phi_f^2[\phi_0] + \frac{1}{2}(6\xi - 1)(\Box \Phi_f^2)[\phi_0].$$

Freedom to add a term $\eta g_{\mu\nu} \phi P_0 \phi$, which **vanishes** on $\mathcal{E}_S(\mathcal{M})$ $\Rightarrow \mathbf{T}^{(\eta)}_{\mu\nu}$.

Wick ordering within the functional approach

We call (abstract) Wick-ordering of $A \in \mathrm{Pol}_{loc}(\mathcal{M})$

$$: A :_H := \alpha_H^{-1}(A) = \alpha_{-H}(A) := e^{-\frac{1}{2}\langle H, \frac{\delta}{\delta\phi(x)} \otimes \frac{\delta}{\delta\phi(y)} \rangle} A.$$

Given a Hadamard state $\Delta_+ \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$ we call a representation of $\mathrm{Pol}_{loc}(\mathcal{M})$ in $\mathrm{Pol}_{\Delta_+}(\mathcal{M})$ the map

$$\alpha_{\Delta_+} : \mathrm{Pol}_{loc}(\mathcal{M}) \to \mathrm{Pol}_{\Delta_+}(\mathcal{M})$$

where $\alpha_{\Delta_+} := e^{\frac{1}{2} D_{\hbar\Delta_+}}$, $D_{\hbar\Delta_+} := \langle \hbar\Delta_+, \frac{\delta}{\delta\phi} \otimes \frac{\delta}{\delta\phi} \rangle$.

Quantum Stress-Energy Tensor

Quantum Stress-Energy Tensor via Wick-ordering

$$
\begin{aligned}
: T^{(\eta)}_{\mu\nu} : (x) :=: \nabla_\mu \phi(x) \nabla_\nu \phi(x) : -\frac{1}{2} : g_{\mu\nu}(x) \partial^\rho \phi(x) \partial_\rho \phi(x) : + \\
- \frac{m_\phi^2}{2} : g_{\mu\nu}(x) \phi^2(x) :] + \xi : G_{\mu\nu}(x) \phi^2(x) : + \\
+ \xi : g_{\mu\nu}(x) \Box \phi^2(x) : - \xi : \nabla_\mu \nabla_\nu \phi^2(x) : \\
+ \eta : g_{\mu\nu}(x) \phi(x) P_0 \phi(x) :.
\end{aligned}
$$

The last term contributes *non-trivially* to the trace at a quantum level.

Quantum Stress-Energy Tensor[8]

Theorem 1

Let $(\mathcal{M}, g)$ with $dim\mathcal{M} = 4$. Let $:\mathbf{T}^{(\eta)}_{\mu\nu,f} :=: \mathbf{T}_{\mu\nu,f} : +\eta : g_{\mu\nu}\phi P_0 \phi :$. Then, for all $f \in \mathcal{D}(\mathcal{M})$:

1. $:\mathbf{T}^{(\eta)}_{\mu\nu,f}:$ is conserved **if and only if** $\eta = \frac{1}{3}$;
2. if $\eta = \frac{1}{3}$ and $\xi = \frac{1}{6}$, then the trace of the Stress-Energy Tensor reads v1

$$: \mathbf{T}_f := - : m_\phi^2 \Phi^2(f) : + \frac{1}{4\pi^2} \left(\int_\mathcal{M} v_1(x,x) f(x)\, d\mu_x \right) \mathbf{1}.$$

[8]V.Moretti, *Comm. Math. Phys. 232.2 (2003)*

The model

Self-interacting, real, scalar field on a globally hyperbolic spacetime, whose dynamics is ruled by

$$P_0\phi = \frac{\lambda}{(n-1)!}\phi^{(n-1)}, \quad n = 3, 4.$$

Interacting Theory in pAQFT: the strategy

Splitting of the total action[9]:

$$S = S_0 + \lambda V_h, \;\; V_h \in \mathcal{F}_{loc}(\mathcal{M}),$$

where

$$S_0[\phi] = \int_{\mathcal{M}} -\left[\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{m_\phi^2 + \xi R}{2}\phi^2\right] f(x) d\mu_x,$$

$$V_h[\phi] = \int_{\mathcal{M}} -\frac{1}{n!}\phi^n(x) h(x)\, d\mu_x,$$

for $f, h \in \mathcal{D}(\mathcal{M})$.

[9] N. Drago, T.P. Hack and N. Pinamonti, *Annales Henri Poincaré* ***18*** *(2017)*.

Interacting Theory in pAQFT: Time-ordered product

Time-ordered product $\star_F \to$ Feynman parametrix $H_F := H + i\Delta_A$, *i.e.*,

$$A \star_F B = \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \langle A^{(k)}, H_F^{\otimes k} B^{(k)} \rangle, \;\; A, B \in \mathcal{F}_{reg}(\mathcal{M}), A \succeq B$$

Renormalization of H_F^k, $k \geq 2 \Rightarrow$ connection between different renormalization schemes.

Interacting Theory in pAQFT: S-matrix and Bogoliubov map

We define the *S-matrix* as the map $S : \lambda\mathcal{F}_{loc}(\mathcal{M}) \to \mathcal{F}_{\mu c}[[\lambda]](\mathcal{M})$

$$S(\lambda V) := \sum_{n\geq 0} \frac{1}{n!}\left(\frac{i\lambda}{\hbar}\right)^n \underbrace{V \star_F \dots \star_F V}_{n}$$

with inverse

$$S^{\star-1}(\lambda V) := \sum_{n\geq 0} \frac{1}{n!}\left(-\frac{i\lambda}{\hbar}\right)^n \underbrace{V \star_{AF} \dots \star_{AF} V}_{n},$$

We call Bogoliubov map $R_{\lambda V} : \mathcal{F}_{loc}(\mathcal{M}) \to \mathcal{F}_{loc}[[\lambda]](\mathcal{M})$

$$R_{\lambda V}(F) := S^{\star-1}(\lambda V) \star [S(\lambda V) \star_F F].$$

Explicit form of interacting Stress-Energy Tensor up to $\mathcal{O}(\lambda^3)$

$$
\begin{aligned}
R^{(2)}_{\lambda V}(\mathbf{T}_{\mu\nu,f})|_{\phi=0} = \frac{\lambda^2\hbar^3}{12}\int_{\mathcal{M}^3} d\mu_x\, d\mu_y\, d\mu_z\, h(x)h(y)f^{\mu\nu}(z) \\
\{2H^3(x,y)\,[\partial_\mu H(x,z)\partial_\nu H_F(y,z) + \partial_\nu H(x,z)\partial_\mu H_F(y,z) + \\
-g_{\mu\nu}(z)\partial_\rho H(x,z)\partial^\rho H_F(y,z) \\
-(g_{\mu\nu}(z)m_\phi^2 - 2\xi G_{\mu\nu}(z))H(x,z)H_F(y,z) + \\
+4\xi g_{\mu\nu}(z)\partial_\rho H(x,z)\partial^\rho H_F(y,z) + 2\xi g_{\mu\nu}(z)(H(x,z)\Box H_F(y,z) + \\
\Box H(x,z)H_F(y,z)) - 2\xi(\partial_\mu H(x,z)\partial_\nu H_F(y,z) + \\
\partial_\nu H(x,z)\partial_\mu H_F(y,z)) - 2\xi(H(x,z)\nabla_\mu\partial_\nu H_F(y,z) \\
+\nabla_\mu\partial_\nu H(x,z)H_F(y,z)) + \eta g_{\mu\nu}(z)H(x,z)P_0 H_F(y,z) \\
+\eta g_{\mu\nu}(z)P_0 H(x,z)H_F(y,z)] + (\ldots)\}
\end{aligned}
$$

Conservation of the Interacting Stress-Energy Tensor

On Minkowski spacetime $\mathbb{M}$:

Theorem 1 Proof

Let $\mathcal{A}(\mathbb{M}) := (\mathcal{F}_{loc}(\mathbb{M}), \star, *)$ be the local and covariant algebra of functionals associated to a real scalar field. Then,

$$\nabla^{\mu} R_{\lambda V}(\mathbf{T}_{\mu\nu,f})|_{\phi=0} =_{\mathcal{O}(\lambda^3)} 0,$$

if and only if $\eta = \frac{1}{4}$.

Trace of the interacting Stress-Energy Tensor

Theorem 2 Proof

If $\eta = \frac{1}{4}$ and $\xi = \frac{1}{6}$, it holds that

$$g^{\mu\nu} R_{\lambda V}(\mathbf{T}_{\mu\nu,f})|_{\phi=0} = \frac{3}{16\pi^2}\left(\int_{\mathcal{M}} v_1(x,x) f(x)\, d\mu_x\right)\mathbf{1} - m_\phi^2 R_{\lambda V}(\Phi_f^2),$$

where $R_{\lambda V}(\mathbf{T}_{\mu\nu,f})|_{\phi=0}$ is the interacting Stress-Energy Tensor expanded up to $\mathcal{O}(\lambda^3)$.

The trace anomaly is a physical contribution even when an interaction occurs.

Conclusions and Outlooks

Conclusions[10]:

- Trace anomaly of the interacting Stress-Energy Tensor computed at $\mathcal{O}(\lambda^3)$ via pAQFT
- Explicit expression in Minkowski spacetime

[10] B.C., C. Dappiaggi and M. Goi, *In preparation.*
[11] M.B. Fröb and J. Zahn, *Journal of High Energy Physics 10 (2019).*
[12] C. Dappiaggi, T.P. Hack and N. Pinamonti, *Rev.Math.Phys. 21 (2009).*
[13] R. Larue, J, Quevillon and R. Zwicky *ArXiv: 2411.00571[hep-th].*

Conclusions and Outlooks

Conclusions[10]:

- Trace anomaly of the interacting Stress-Energy Tensor computed at $\mathcal{O}(\lambda^3)$ via pAQFT
- Explicit expression in Minkowski spacetime

Outlooks:

- Extend the computations to
  - Yukawa model[11] [12] [13]
  - Gauge theories
- Confirm the contact with the D-term

[10] B.C., C. Dappiaggi and M. Goi, *In preparation.*
[11] M.B. Fröb and J. Zahn, *Journal of High Energy Physics 10 (2019).*
[12] C. Dappiaggi, T.P. Hack and N. Pinamonti, *Rev.Math.Phys. 21 (2009).*
[13] R. Larue, J, Quevillon and R. Zwicky *ArXiv: 2411.00571[hep-th].*

Explicit expression $v_1(x,x)$

For $\xi = \frac{1}{6}$,

$$v_1(x,x) = \frac{1}{720}\left(C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + R_{\mu\nu}R^{\mu\nu} - \frac{R^2}{3} + \Box R\right) + \frac{m_\phi^2}{8}.$$

Example: Computation of the interacting Φ_f^2 up to $\mathcal{O}(\lambda^2)$

In 4−dimensions,

$$R_{\lambda V}(\Phi_f^2) =_{\mathcal{O}(\lambda^3)} \Phi_f^2 + \frac{i\lambda}{\hbar}\left(V_h \star_F \Phi_f^2 - V_h \star \Phi_f^2\right) + \\ - \frac{\lambda^2}{2\hbar^2}\Big[(V_h \star_{AF} V_h) \star \Phi_f^2 + (V_h \star_F V_h) \star_F \Phi_f^2 \\ -2V_h \star (V_h \star_F \Phi_f^2)\Big] .$$

None of the terms of the first order expansion survives when evaluated at $\phi = 0$.

Contractions:
From $(V_h \star_{AF} V_h) \star \Phi_f^2$, it is possible to obtain

$$(\Phi_h, \Phi_h, \Phi_h, \Phi_h \;\; \Phi_h, \Phi_h, \Phi_h, \Phi_h) \; \Phi_f, \Phi_f$$

At the level of integral kernels, it reads

$$-\frac{\lambda^2 \hbar^{(n-1)}}{(n-1)!} \int_{\mathcal{M}^3} H_{AF}^{(n-1)}(x,y) H(x,z) H(y,z) h(x) h(y) f(z) \, d\mu_x d\mu_y d\mu_z .$$

Analogously for $(V_h \star_F V_h) \star_F \Phi_f^2$.

For what concerns $V_h \star (V_h \star_F \Phi_f^2)$, we obtain

$$\Phi_h, \Phi_h, \Phi_h, \Phi_h \; (\Phi_h, \Phi_h, \Phi_h, \Phi_h) \; \Phi_f, \Phi_f),$$

which translates at the level of integral kernels to

$$-\frac{\lambda^2 \hbar^{(n-1)}}{(n-1)} \int_{\mathcal{M}^3} H^{(n-1)}(x,y) H(x,z) H_F(y,z) h(x) h(y) f(z) \, d\mu_x d\mu_y d\mu_z .$$

Theorem A1

Let $F_f[\phi] = \int_{\mathcal{M}} K\phi(z)K'\phi(z)f(z)d\mu_z$ be a differentiated functional, where K, K' are arbitrary finite-order differential operators. Then

$$
\begin{aligned}
V_g \star_{\mathcal{G}} F_f &= V_g \cdot F_f \\
&- \frac{\hbar^2}{(n-2)!} \int_{\mathcal{M}^2} K\mathcal{G}(x,z)K'\mathcal{G}(x,z)\phi^2(x)g(x)f(z)\, d\mu_x d\mu_z + \\
&- \frac{\hbar}{(n-1)!} \left[\int_{\mathcal{M}^2} K\mathcal{G}(x,z)K'\phi(z)\phi^3(x)g(x)f(z)\, d\mu_x d\mu_z \right. \\
&+ \left. \int_{\mathcal{M}^2} K\phi(z)K'\mathcal{G}(x,z)\phi^3(x)g(x)f(z)\, d\mu_x d\mu_z \right] .
\end{aligned}
$$

Applying this result to the term appearing in the Stress-Energy Tensor, one gets

$$
\begin{aligned}
&[2V_g \star (V_g \star_F (K\Phi K'\Phi)_f) - (V_g \star_{AF} V_g) \star (K\Phi K'\Phi)_f \\
&\quad - (V_g \star_F V_g) \star_F (K\Phi K'\Phi)_f]|_{\phi=0} = \\
&\quad = \frac{\hbar^{(n+1)}}{(n-1)!} \int_{\mathcal{M}^3} d\mu_x d\mu_y d\mu_z h(x)h(y)f(z) \\
&\quad \big[2H^3(x,y)(KH(x,z)K'H_F(y,z) + K'H(x,z)KH_F(y,z)) + \\
&\quad -H^3_{AF}(x,y)(KH(x,z)K'H(y,z) + K'H(x,z)KH(y,z)) + \\
&\quad -H^3_F(x,y)(KH_F(x,z)K'H_F(y,z) + K'H_F(x,z)KH_F(y,z))\big] \, .
\end{aligned}
$$

Using the properties of the Levi-Civita connection and the (second) Bianchi identity, one can write

$$\nabla^{\mu} R_{\lambda V}(\mathbf{T}_{\mu\nu,f})|_{\phi=0} = \frac{\lambda^2 \hbar^{(n-1)}}{2(n-1)!} \int_{\mathbb{M}^2} d\mu_y d\mu_z h(y) h(z) f^{\nu}(z)$$

$$\left\{ -2H^{(n-1)}(z,y)\partial_{\nu} H(z,y)] + 2H_F^{(n-1)}(z,y)\partial_{\nu} H_F(z,y) \right\} +$$

$$\eta \frac{\lambda^2 \hbar^{(n-1)} n}{(n-1)!} \left[\int_{\mathbb{M}^2} d\mu_y d\mu_z h(y) h(z) f^{\nu}(z) H^{(n-1)}(z,y)\partial_{\nu} H(z,y) + \right.$$

$$\left. - \int_{\mathbb{M}^2} d\mu_y d\mu_z h(y) h(z) f^{\nu}(z) H_F^{(n-1)}(z,y)\partial_{\nu} H_F(z,y) \right],$$

which vanishes for $\eta = \frac{1}{n}$, $n = 3, 4$.

Sketch of the proof of Thm. 3

If $\eta = \frac{1}{4}, \xi = \frac{1}{6}$, exploiting the relations

$$g^{\mu\nu} g_{\mu\nu} = 4, \;\; g^{\mu\nu} G_{\mu\nu} = -R,$$

one obtains

$$g^{\mu\nu} R^{(2)}_{\lambda V}(\mathbf{T}_{\mu\nu,f})|_{\phi=0} = -2m_\phi^2 \frac{\lambda^2 \hbar^3}{12} \int_{\mathcal{M}^3} d\mu_x \, d\mu_y \, d\mu_z h(x) h(y) f(z)$$
$$\big\{ 2H^3(x,y) H(x,z) H_F(y,z)$$
$$- H^3_{AF}(x,y) H(x,z) H(y,z) - H^3_F(x,y) H_F(x,z) H_F(y,z) \big\} =$$
$$= -m_\phi^2 R^{(2)}_{\lambda V}(\Phi_f^2)|_{\phi=0}.$$

Trace anomaly for $\lambda\Phi^3$

The Stress-Energy Tensor is conserved **if and only if** $\eta = \frac{1}{3}$ and its trace reads

$$g^{\mu\nu} R^{(2)}_{\lambda V}(\mathbf{T}_{\mu\nu,f})|_{\phi=0} =_{\mathcal{O}(\lambda^3)} \frac{\lambda^2\hbar^2}{6} \int_{\mathcal{M}^2} d\mu_x \, d\mu_z h(x)h(y)f(z) \\ [H^3(x,z) - H^3_F(x,z)] \\ - \left(\frac{5}{3}m^2_\phi + \frac{2}{9}R\right) R_{\lambda V}(\Phi^2_f)|_{\phi=0}.$$