

Interplay between boundary conditions and the Lorentzian Wetterich equation

Filippo Nava

University of York

Based on: C. Dappiaggi, F.N., L. Sinibaldi
(Rev. in Math. Phys., 2024)

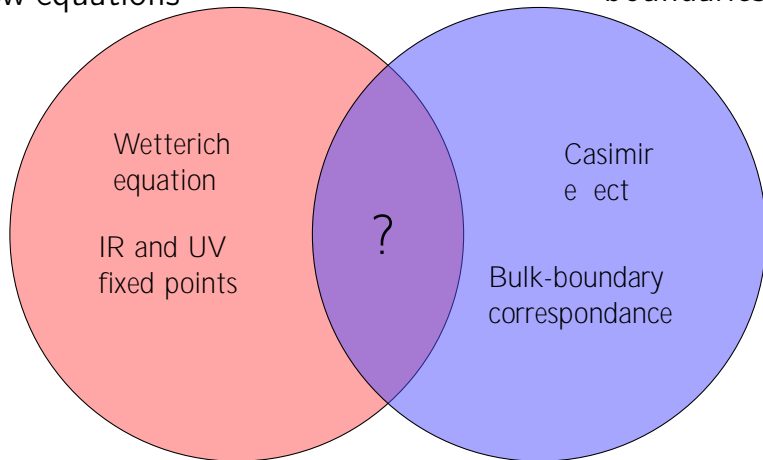
LQP 49, Erlangen 2024



Context of the work

Renormalization
flow equations

QFT with
boundaries



Renormalization flow and boundaries

Aim of the work

Studying the renormalization flow **fixed points** of the **scalar quantum field**, with **quartic self-interaction**, on a **spacetime with boundary**.

$$I(\phi) = \int_M d^d x \sqrt{g} \left[\frac{1}{2} g_{ab} \partial^a \phi \partial^b \phi + \frac{m^2}{2} \phi^2 + \frac{1}{2} R \phi^2 + \frac{\lambda}{4!} \phi^4 \right]$$

Done employing a novel renormalization flow equation (**Lorentzian Wetterich equation^a**)

^aE. D'Angelo, N. Drago, N. Pinamonti, K. Rejzner (Annales Henri Poincaré, 2024)

Renormalization flow and boundaries

Aim of the work

Studying the renormalization flow **fixed points** of the **scalar quantum field**, with **quartic self-interaction**, on a **spacetime with boundary**.

$$I(\phi) = \int_M d^d x \sqrt{g} \left[\frac{1}{2} g_{ab} \partial^a \phi \partial^b \phi + \frac{m^2}{2} \phi^2 + \frac{1}{2} R \phi^2 + \frac{\lambda}{4!} \phi^4 \right]$$

Done employing a novel renormalization flow equation (**Lorentzian Wetterich equation^a**)

^aE. D'Angelo, N. Drago, N. Pinamonti, K. Rejzner (Annales Henri Poincaré, 2024)

Renormalization flow and boundaries

Aim of the work

Studying the renormalization flow **fixed points** of the **scalar quantum field**, with **quartic self-interaction**, on a **spacetime with boundary**.

$$I(\phi) = \int_M d^d x \sqrt{g} \left[\frac{1}{2} g_{ab} \partial^a \phi \partial^b \phi + \frac{m^2}{2} \phi^2 + \frac{1}{2} R \phi^2 + \frac{\lambda}{4!} \phi^4 \right]$$

Done employing a novel renormalization flow equation (**Lorentzian Wetterich equation^a**)

^aE. D'Angelo, N. Drago, N. Pinamonti, K. Rejzner (Annales Henri Poincaré, 2024)

Spacetimes with boundary

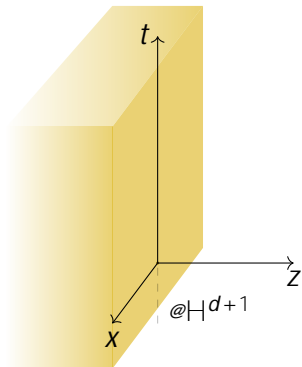
Lorentzian manifolds with a time-like boundary

M manifold with boundary, $(M; g)$ Lorentzian manifold.

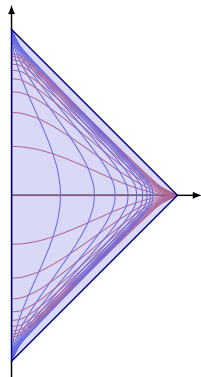
$(M; g)$ is a Lorentzian manifold with **time-like boundary** if $(@M; i^*g)$ is a Lorentzian manifold, where $i : @M \rightarrow M$.

Examples

Half Minkowski ($H^{d+1}; g$)

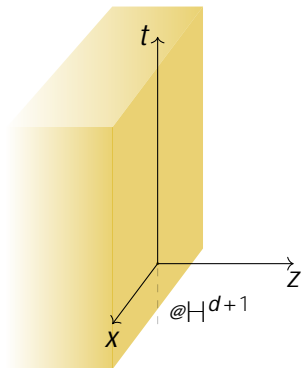


Anti de-Sitter ($PAdS_{d+1}; g$)



Half Minkowski spacetime

Half Minkowski ($H^{d+1}; \cdot$)



$$ds_{H^{d+1}}^2 = dt^2 + dz^2 + \sum_{i=1}^{d-1} dx_i^2$$

$$z \geq 0; x_i \in \mathbb{R}$$

Poincaré patch of Anti de-Sitter

$$X_0^2 - X_{d+1}^2 + \sum_{i=1}^d X_i^2 = l^2;$$

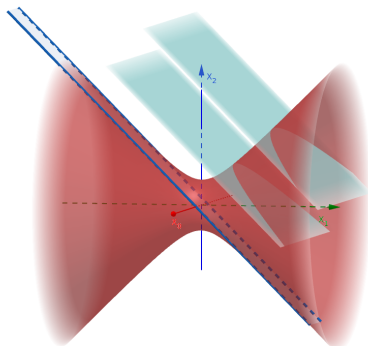
$$l^2 = \frac{d(d+1)}{2}$$

+

$$ds_{\text{PAdS}}^2 = \frac{l^2}{z^2} \left(dt^2 + dz^2 + \sum_{i=1}^d x_i^2 \right)$$

$$z > 0; x_i \in \mathbb{R}$$

Anti de-Sitter (PAdS_{d+1}; g)



Poincaré patch of Anti de-Sitter

$$X_0^2 - X_{d+1}^2 + \sum_{i=1}^d X_i^2 = l^2;$$

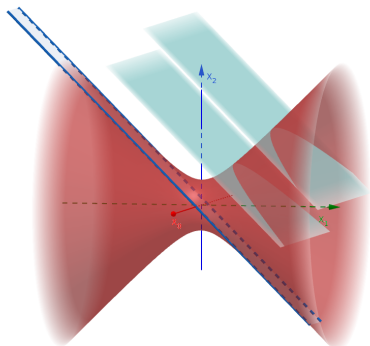
$$l^2 = \frac{d(d+1)}{2}$$

+

$$ds_{\text{PAdS}}^2 = \frac{l^2}{z^2} \left(dt^2 + dz^2 + \sum_{i=1}^d dx_i^2 \right)$$

$$z > 0; x_i \in \mathbb{R}$$

Anti de-Sitter (PAdS_{d+1}; g)



Conformal relation



$$ds_{\text{PAdS}}^2 = \frac{l^2}{z^2} \left(dt^2 + dz^2 + \sum_{i=1}^{d-1} x_i^2 \right)$$

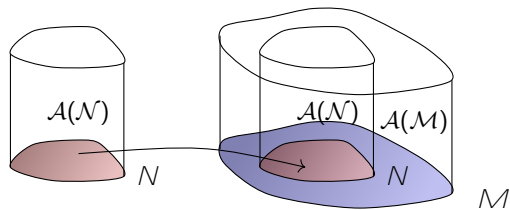
$$z > 0; x_i \in \mathbb{R}$$

$$ds_{\text{H}^{d+1}}^2 = dt^2 + dz^2 + \sum_{i=1}^{d-1} x_i^2$$

$$z \in \mathbb{R}; x_i \in \mathbb{R}$$

Algebraic QFT in the functional approach

$(M; g)$ Lorentzian manifold ! $A(M)$ algebra of observables



Functional approach

$C^1(M)$ classical field configurations

$A(M) = F(M)$ functionals on $C^1(M)$, $F : C^1(M) \rightarrow \mathbb{R}$

$F_{loc}(M)$ local functionals:

$$X_f^n(\phi) = \int_M \phi^n f; \quad f \in C_c^1(M); \quad \phi \in C^1(M)$$

Deformation quantization

Classical product: $F \cdot G$ Quantum product: $F \star G$

- 1 (Associativity) \star is an associative product;
- 2 (Classical limit) $F \star G \stackrel{\hbar \rightarrow 0}{\sim} F \cdot G$;
- 3 (Weak Dirac prescription) $\{F; G\} = \frac{1}{i\hbar}[F; G]_{\star} + O(\hbar^2)$.

Algebra structure on $F_{loc}(M)$

Klein-Gordon operator $P = \square_g + m^2 \in R$. If M is *globally hyperbolic*:

- $\mathcal{D}^0(M, M)$ weak biresolution of KG
- $\mathcal{H} \subset \mathcal{D}^0(M, M)$ with the same singular structure of \mathcal{D}^0

Unique up to the choice of a smooth function: *physical state*

Deformation quantization

Classical product: $F \cdot G$ Quantum product: $F \star G$

- 1 (Associativity) \star is an associative product;
- 2 (Classical limit) $F \star G \stackrel{\hbar \rightarrow 0}{\sim} F \cdot G$;
- 3 (Weak Dirac prescription) $\{F; G\} \stackrel{\hbar \rightarrow 0}{\sim} \frac{1}{i\hbar}[F; G]_{\star} + O(\hbar^2)$.

Algebra structure on $F_{loc}(M)$

Klein-Gordon operator $P = \Delta_g - m^2$ on R . If M is *globally hyperbolic*:

- $\mathcal{G} +$ 2-point function weak bisolution of KG
- $\mathcal{G} \in H^2(D^0(M) \times M)$ with the same singular structure of $\mathcal{G} +$

Unique up to the choice of a smooth function: *physical state*

Deformation quantization

Classical product: $F \cdot G$ Quantum product: $F \star G$

- 1 (Associativity) \star is an associative product;
- 2 (Classical limit) $F \star G \stackrel{\hbar \rightarrow 0}{\sim} F \cdot G$;
- 3 (Weak Dirac prescription) $\{F; G\} \stackrel{\hbar \rightarrow 0}{\sim} \frac{1}{i\hbar}[F; G]_{\star} + O(\hbar^2)$.

Algebra structure on $F_{loc}(M)$

Klein-Gordon operator $P = \Delta_g - m^2 \in R$. If M is *globally hyperbolic*:

- $\mathcal{G} +$ 2-point function weak bisolution of KG
- $\mathcal{G} \in H$ Hadamard parametrix with the same singular structure of $\mathcal{G} +$

Unique up to the choice of a smooth function: *physical state*

Deformation quantization

Classical product: $F \cdot G$ Quantum product: $F \star G$

- 1 (Associativity) \star is an associative product;
- 2 (Classical limit) $F \star G \stackrel{\hbar \rightarrow 0}{\sim} F \cdot G$;
- 3 (Weak Dirac prescription) $\{F; G\} = \frac{1}{i\hbar}[F; G]_{\star} + O(\hbar^2)$.

Algebra structure on $F_{loc}(M)$

Klein-Gordon operator $P = \square_g + m^2 \in R$. If M is *globally hyperbolic*:

- $\mathcal{G} +$ 2-point function weak bisolution of KG
- $\mathcal{G} + H$ Hadamard parametrix with the same singular structure of $\mathcal{G} +$

Unique up to the choice of a smooth function: **physical state**

Algebra structure on $F_{loc}(M)$ Quantum structure on $F_{loc}(M)$

Given $F, G \in F_{loc}(M)$ and \hbar a 2-point function we define

$$F \star G = F \circ G + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} F^{(n)} \circ G^{(n)}$$

Normal ordering of $X_f^2(x)$

$$: X_f^2(x) : = \lim_{x' \rightarrow y} (X_f(x) \star X_f(y)) \sim H(x; y)$$

as $\hbar \in C^1(M)$

Interacting theory

Perturbative AQFT

Interacting action functional $I_0(\phi) + V(\phi)$;

$$V(\phi) = \int_{\mathcal{M}} f(\phi); \quad \phi \in \mathcal{R}$$

Thanks to **Epstein-Glaser renormalization** we can define the **time-ordered product** T and the **local S-matrix**

$$S(V) \doteq \sum_{n=0}^{\infty} \frac{i^n}{n!} [:V: T \{ \underbrace{T \dots T}_{n \text{ times}} :V: \}]$$

AQFT on spacetimes with a boundary

The initial-value problem must be supplemented with **boundary conditions**:

$$= 0 \text{ on } @\mathcal{M} \text{ (Dirichlet)} \quad r_n = 0 \text{ on } @\mathcal{M} \text{ (Neumann)}$$

Quantum structure on $F_{loc}(\mathcal{M})$

On $(\mathbb{H}^4; \cdot)$ and $(\text{PAdS}_4; \mathbf{g})$ with Dirichlet/Neumann BCs it holds that ^a ^b:

- \mathcal{G} + 2-point function
- $\mathcal{G} H$ Hadamard function

^aO. Gannot, M. Wrochna (Journal of the Institute of Mathematics of Jussieu, 2022)

^bC. Dappiaggi, A. Marta (Mathematische Nachrichten, 2022)

Lorentzian Wetterich equation

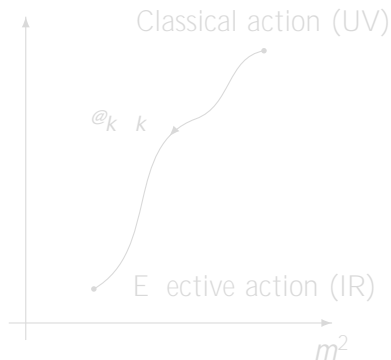
In the pAQFT formalism it holds the Lorentzian Wetterich equation¹

$$\Gamma_k(x; y) = \lim_{M \rightarrow \infty} \int_M d^d y \text{Sym}_{+;k}(x; y) H_k(x; y)$$

$$\Gamma_k^{(2)}(j) = 0 = \int_M d^d y g_{@k} m_k^2$$

$$\Gamma_k^{(4)}(j) = 0 = \int_M d^d y g_{@k} k$$

Fixed point: $\Gamma_k m_k^2 = \Gamma_k k = 0$



¹E. D'Angelo, N. Drago, N. Pinamonti, K. Rejzner (Annales Henri Poincaré, 2024)

Lorentzian Wetterich equation

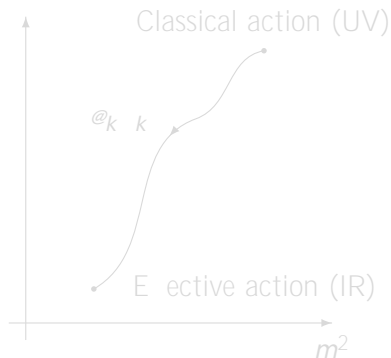
In the pAQFT formalism it holds the Lorentzian Wetterich equation¹

$$\partial_k \Gamma_k = \lim_{\epsilon \rightarrow 0} \int_M d^4y \text{Sym}_{+;k}(x;y) H_k(x;y)$$

$$\partial_k \Gamma_k^{(2)} = \int_M d^4y g_{\mu\nu} \partial_k^2 \Gamma_k$$

$$\partial_k \Gamma_k^{(4)} = \int_M d^4y g_{\mu\nu} \partial_k^4 \Gamma_k$$

Fixed point: $\partial_k \Gamma_k^{(2)} = \partial_k \Gamma_k^{(4)} = 0$



¹E. D'Angelo, N. Drago, N. Pinamonti, K. Rejzner (Annales Henri Poincaré, 2024)

Lorentzian Wetterich equation

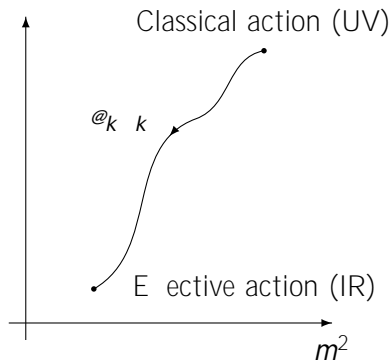
In the pAQFT formalism it holds the Lorentzian Wetterich equation¹

$$\partial_k \Gamma_k(\phi) = \lim_{\epsilon \rightarrow 0} \int_M d^4y \operatorname{Sym}_{+;k}(x;y) H_k(x;y)$$

$$\partial_k \Gamma_k^{(2)}(j=0) = \int_M d^4y \partial_k m_k^2$$

$$\partial_k \Gamma_k^{(4)}(j=0) = \int_M d^4y \partial_k k$$

Fixed point: $\partial_k m_k^2 = \partial_k k = 0$



¹E. D'Angelo, N. Drago, N. Pinamonti, K. Rejzner (Annales Henri Poincaré, 2024)

Lorentzian Wetterich equation

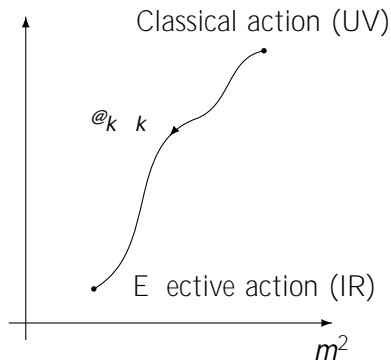
In the pAQFT formalism it holds the Lorentzian Wetterich equation¹

$$\partial_k \Gamma_k(\phi) = \lim_{\epsilon \rightarrow 0} \int_M d^4y \operatorname{Sym}_{+;k}(x;y) H_k(x;y)$$

$$\partial_k \Gamma_k^{(2)}(j=0) = \int_M d^4y g_{\partial_k} m_k^2$$

$$\partial_k \Gamma_k^{(4)}(j=0) = \int_M d^4y g_{\partial_k} k$$

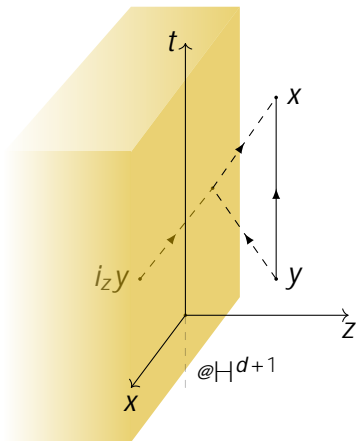
Fixed point: $\partial_k m_k^2 = \partial_k k = 0$



¹E. D'Angelo, N. Drago, N. Pinamonti, K. Rejzner (Annales Henri Poincaré, 2024)

Method of images

$$G_k(x; y) = \lim_{M \rightarrow \infty} \sum_{j=0}^M d_j g(y) \text{Sym}_{+;k}(x; y) = H_k(x; y)$$



With Dirichlet/Neumann BCs:

$$G_{\pm}(x; y) = G_{\pm}^M(x; y) \pm G_{\pm}^M(x; i_z y)$$

$$\textcircled{1} \quad H(x; y) = H^M(x; y) \pm \underbrace{G_{\pm}^M(x; i_z y)}_{\text{smooth at } x=y}$$

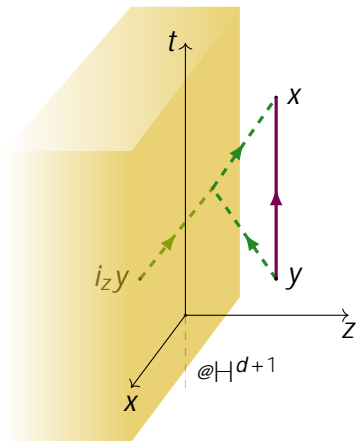
Full subtraction

$$\textcircled{2} \quad H(x; y) = H^M(x; y)$$

Minimal subtraction

Method of images

$$G_k(x; y) = \lim_{M \rightarrow \infty} \int_{\mathcal{M}} d^d g(y) \text{Sym}_{+,k}(x; y) H_k(x; y)$$



With Dirichlet/Neumann BCs:

$$G_+(x; y) = G_+^M(x; y) + G_+^M(x; i_z y)$$

$$\textcircled{1} H(x; y) = H^M(x; y) + \underbrace{H^M(x; i_z y)}_{\text{smooth at } x=y}$$

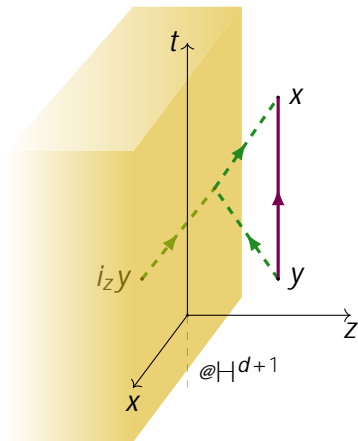
Full subtraction

$$\textcircled{2} H(x; y) = H^M(x; y)$$

Minimal subtraction

Method of images

$$H_k(x; y) = \lim_{M \rightarrow \infty} \sum_{x' \neq y}^Z k(x'; y) \text{Sym} \left[\sum_{x'' \neq y}^Z k(x''; y) H_k(x; y) \right]$$



With Dirichlet/Neumann BCs:

$$H_+(x; y) = H_+^M(x; y) + H_+^M(x; i_z y)$$

$$\textcircled{1} H(x; y) = H^M(x; y) + H^M(x; i_z y)$$

smooth at $x=y$

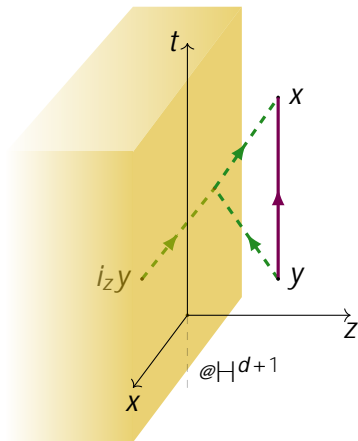
Full subtraction

$$\textcircled{2} H(x; y) = H^M(x; y) - H^M(x; i_z y)$$

Minimal subtraction

Method of images

$$H_k(x; y) = \lim_{M \rightarrow \infty} \sum_{x' \neq y}^Z k(x'; y) \text{Sym} \left[\sum_{x'}^Z g(y) \right] H_k(x; y)$$



With Dirichlet/Neumann BCs:

$$H_+(x; y) = H_+^M(x; y) + H_+^M(x; i_z y)$$

$$\textcircled{1} H(x; y) = H^M(x; y) + H^M(x; i_z y)$$

smooth at $x=y$

Full subtraction

$$\textcircled{2} H(x; y) = H^M(x; y)$$

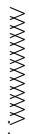
Minimal subtraction

Scaling of full Minkowski



$$k @_k m_k^2 = \frac{e_k^2}{16 \pi^2} \frac{h}{2} \log \frac{m_k^2}{\mu^2} + 1 + 1 \quad (1 \quad B(\vartheta))$$

$$(1 + m_k^2) \log \frac{m_k^2}{\mu^2} + 1 \quad B_2(\vartheta)$$



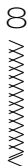
$$k @_k e_k = \frac{e_k^2}{16 \pi^2} \frac{h}{2} \frac{3}{1 + m_k^2} (1 \quad B(\vartheta))$$

$$6 \log \frac{m_k^2}{\mu^2} + 1 + 1 \quad B_2(\vartheta)$$

$$(1 + m_k^2) \log \frac{m_k^2}{\mu^2} + 1 \quad B_4(\vartheta)$$

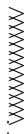
where $B(\vartheta) \doteq \frac{p_{-2}}{iMz} I_1(p_{-2} iMz)$, $m_k^2 = k^2 + m_k^2$, $e_k = k$, $\vartheta = kz$

Full subtraction - Dirichlet/Neumann BCs



$$k @_k m_k^2 = 2m_k^2 + \frac{e_k}{16} \frac{h}{2} \log m_k^2 + 1 + 1 \quad (1 \quad B(\varrho))$$

$$(1 + m_k^2) \log 1 + m_k^2 \quad B_2(\varrho)$$



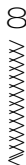
$$k @_k e_k = \frac{e_k^2}{16} \frac{h}{2} \frac{3}{1 + m_k^2} (1 \quad B(\varrho))$$

$$6 \log m_k^2 + 1 + 1 \quad B_2(\varrho)$$

$$(1 + m_k^2) \log m_k^2 + 1 \quad B_4(\varrho)$$

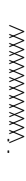
where $B(\varrho) \doteq \frac{p-2}{iMz} I_1(p-2iMz)$, $m_k^2 = k^2 + m^2$, $e_k = k$, $\varrho = kz$

Minimal subtraction - Dirichlet/Neumann BCs



$$k @_k m_k^2 = 2m_k^2 + \frac{e_k}{16} \frac{h}{2} \log m_k^2 + 1 + 1 \quad (1 \quad C(\varrho))$$

$$(1 + m_k^2) \log 1 + m_k^2 \quad C_2(\varrho)$$



$$k @_k e_k = \frac{e_k^2}{16} \frac{h}{2} \frac{3}{1 + m_k^2} (1 \quad C(\varrho))$$

$$6 \log m_k^2 + 1 + 1 \quad C_2(\varrho)$$

$$(1 + m_k^2) \log m_k^2 + 1 \quad C_4(\varrho)$$

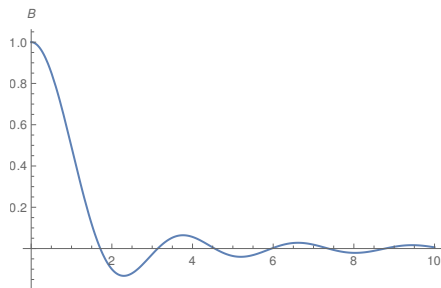
where $C(\varrho) \doteq \frac{2}{M^2} K_1(2zM)$, $m_k^2 = k^2 m^2$, $e_k = k$, $\varrho = kz$

Comparison between full and minimal subtraction

$$B(z) \in \mathbb{R}$$

$$\lim_{z \rightarrow 1} B(z) = 0$$

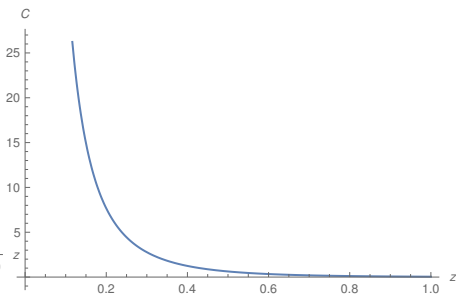
$$|B(z)| \sim 1$$



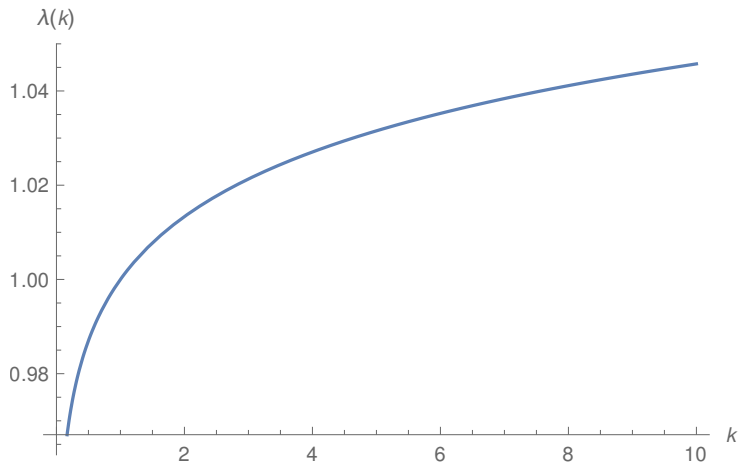
$$C(z) \in \mathbb{R}$$

$$\lim_{z \rightarrow 1} C(z) = 0$$

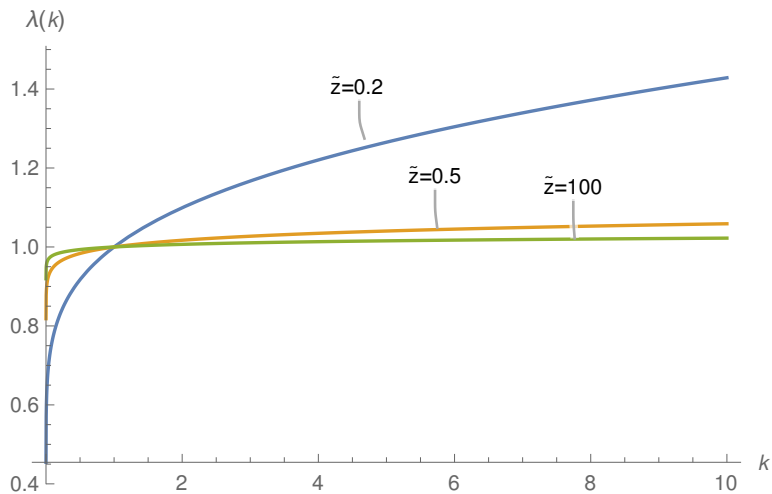
$$C(z) \sim z^{\rho+1}$$



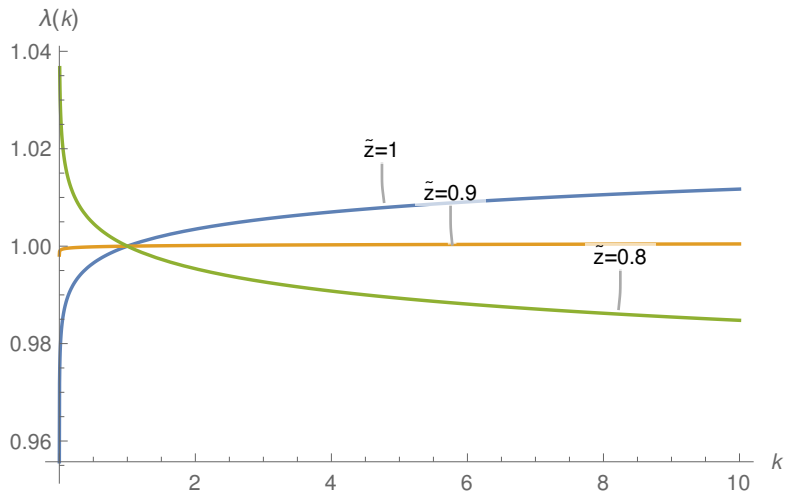
Scaling of λ on full Minkowski



Minimal subtraction - Dirichlet BCs



Minimal subtraction - Neumann BCs



Poincaré patch of Anti de-Sitter

To construct g_{PAdS} from H :

$$g_{\text{PAdS}} = H; \quad = \frac{z}{l}$$

If H is PAdS $_{d+1}$ with R satisfies $(2g - m^2 - R) = 0$

+

$$- = \frac{1-d}{2} : H^{d+1} \text{ with } R \text{ satisfies } 2 \frac{\bar{m}^2}{z^2} - = 0; \bar{m}^2 = m^2 + \left(\frac{d-1}{4d} \right) R$$

Flow on Anti de-Sitter - Dirichlet BCs

$$\begin{aligned} \sum_k k @_k m_k^2 &= \frac{e_k}{4} \quad \sim^2 \quad \frac{1}{4} \quad (\dots) \\ \sum_k k @_k e_k &= 2e_k + \frac{3e_k^2}{16} \quad \sim^2 \quad \frac{1}{4} \quad (\dots) \end{aligned} \quad \#$$

where $(\dots) / ; 0; \infty$ digamma function, $m_k^2 = l^2 m_k^2$, $e_k = k^2 l^2$
 $\sim = \frac{1}{2} \frac{1}{1 + 4k^2 l^2 + 4m_k^2}$

Large l approximation

If $l \rightarrow \infty$, as $l^2 \rightarrow 1$: $k^2 l^2 = 0$! autonomous ODE

Flow on Anti de-Sitter - Dirichlet BCs

$$\sum_k k @_k m_k^2 = \frac{e_k}{4} \quad \sim^2 \quad \frac{1}{4} \quad (\dots)$$

$$\sum_k k @_k e_k = 2e_k + \frac{3e_k^2}{16} \quad \sim^2 \quad \frac{1}{4} \quad (\dots) \quad \#$$

where $(\dots) / ; \dots$ digamma function, $m_k^2 = l^2 m_k^2$, $e_k = k^2 l^2$
 $\sim = \frac{1}{2} \frac{1}{1 + 4k^2 l^2 + 4m_k^2}$

Large l approximation

If $l \rightarrow \infty$, as $l^2 \rightarrow 1$: $k^2 l^2 = 0$! autonomous ODE

Flow on Anti de-Sitter - Dirichlet BCs

$$\sum_k k^2 \tilde{m}_k^2 = \frac{e_k}{4} \quad \sim^{-2} \quad \frac{1}{4} \quad (\dots)$$

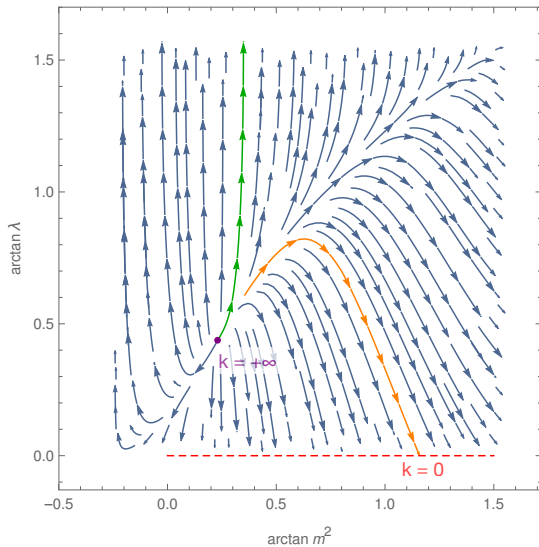
$$\sum_k k^2 e_k = 2e_k + \frac{3e_k^2}{16} \quad \sim^{-2} \quad \frac{1}{4} \quad (\dots) \quad \#$$

where $(\dots) \sim^{-2} \frac{1}{4} (\dots)$ digamma function, $\tilde{m}_k^2 = l^2 m_k^2$, $e_k = k^2 l^2$,
 $\sim = \frac{1}{2} \frac{1}{1 + 4k^2 l^2 + 4\tilde{m}_k^2}$

Large approximation

If $l \rightarrow 1$, as $l^2 \rightarrow 1$: $k^2 l^2 = 0$! autonomous ODE

Flow on Anti de-Sitter - Dirichlet BCs



Summary

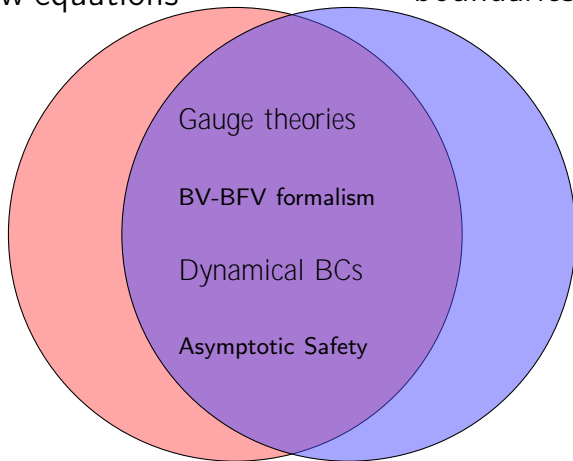
Results

- State-dependence of UV/IR behaviour
- Asymptotic freedom “near the boundary” with Neumann BCs
- Set of non-interacting, exact IR fixed points on PAdS₄
- UV fixed point under the large λ approximation

Future perspectives

Renormalization
flow equations

QFT with
boundaries

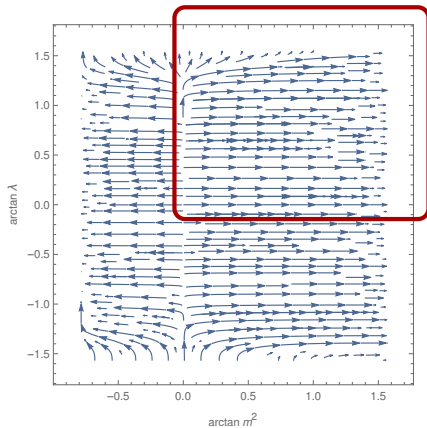
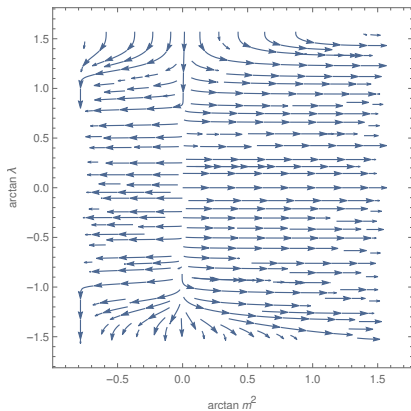


Comparison between Minkowski and the full subtraction

Half Minkowski, full subtraction:

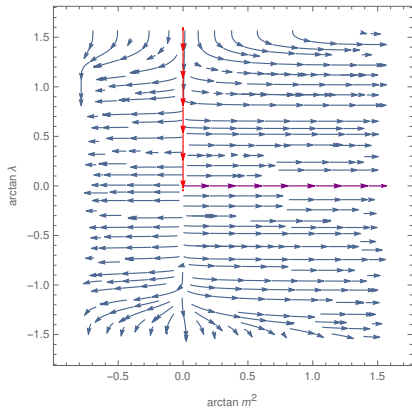
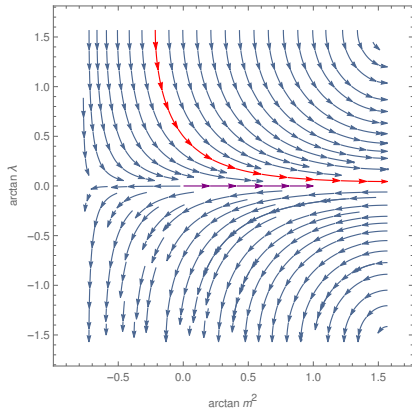
$$\epsilon = 10^{-3}$$

Minkowski



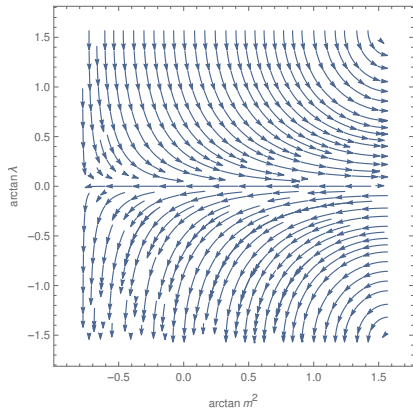
Comparison between Minkowski and the minimal subtraction (Dirichlet BCs)

Minkowski

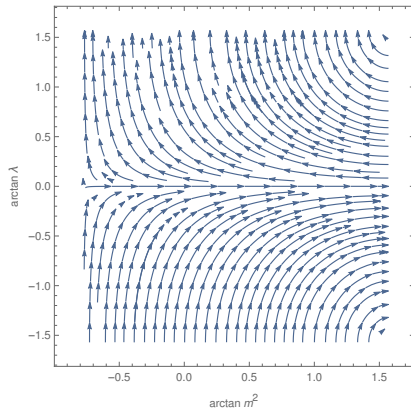
Half Minkowski: $\epsilon = 10^{-5}$ 

Comparison between Dirichlet and Neumann BCs

Dirichlet



Neumann



Hadamard parametrix

Theorem (Radzikowski)

For $x; y$ in a *convex normal neighborhood* $C \subset M$ there exist $U; V; W \in C^1(M \times M)$ such that

$$H(x; y) = \underbrace{\frac{U(x; y)}{(x; y)} + V(x; y) \log \frac{(x; y)}{2}}_{\text{fixed by the geometry}} + \underbrace{W(x; y)}_{\text{physical state}}$$

Hadamard function

Given

- convex normal neighborhood $C \subset M$;
- geodesic distance;
- Hadamard parametrix

$$H(x; y) = \frac{u(x; y)}{(2)^2 (x; y)} + v(x; y) \log \frac{(x; y)}{2} ;$$

there exists $s \in C^1(C \times C)$ such that for all $f, g \in C^1(C)$:

$$\lim_{\epsilon \rightarrow 0^+} \int_C \int_C H(x; y) P_\epsilon f(x) g(y) dx dy = \int_C \int_C s(x; y) f(x) g(y) dx dy$$