Abstracts

Inclusions of Standard Subspaces

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(joint work with Ricardo Correa da Silva)

Standard subspaces naturally appear in the context of von Neumann algebras, where any von Neumann algebra in standard form gives rise to a standard subspace encoding its modular data, and in quantum field theory, where standard subspaces encode localization regions. From this perspective, standard subspaces appear as auxiliary objects. There is however growing evidence that standard subspaces are interesting objects in their own right – for example, they lead to an independent notion of entropy [5], can naturally be constructed on the basis of suitable Lie group representations [13], and lie at the basis of the recently introduced twisted Araki-Woods algebras [3].

In these and other applications, one is typically not interested in a single standard subspace (the set of all standard subspaces H of a complex Hilbert space $\mathcal{H}$ can easily be classified, see [9, Cor. 2.1.5]), but rather in families of standard subspaces and their intersection, inclusion and covariance properties. The topic of this talk was therefore to initiate an abstract discussion of inclusions

$$K \subset H \subset \mathcal{H}$$

of standard subspaces, without reference to von Neumann algebras or group representations. This can be seen as an analogue of the study of inclusions of von Neumann algebras, or more specifically subfactors.

We review some known results about inclusions of standard subspaces and then reported on joint work in progress with R. Correa da Silva [4].

Inclusions and irreducible inclusions. Given a standard subspace K, can we embed it properly into a larger standard subspace H, or can we properly embed a smaller standard subspace into K? This question is answered in the following lemma:

Lemma 1. [7] *Let $K \subset \mathcal{H}$ be a standard subspace. Then the following are equivalent:*

(1) *There exists a standard subspace $H \subset \mathcal{H}$ such that $K \subsetneq H$.*
(2) *There exists a standard subspace $H \subset \mathcal{H}$ such that $H \subsetneq K$.*
(3) *The modular operator Δ_K is unbounded.*

Guided by the comparison with subfactor theory, we are particularly interested in understanding *irreducible* inclusions, which by definition are inclusions $K \subset H$ with $K' \cap H = \{0\}$. Here K' denotes the symplectic complement of K. Clearly, this requires in particular $K' \cap K = \{0\}$, i.e. K must be a factorial subspace (a factor, for short). Recall that a factor has a well-defined cutting projection $P_K : K + K' \to K$, $k + k' \mapsto k$ [5].

The basic result in this regard is a reformulated version of a proposition from [7].

Proposition 2. *Let $K \subset \mathcal{H}$ be a standard subspace. Then the following are equivalent:*

(1) *There exists a standard subspace $H \subset \mathcal{H}$ such that $K \subsetneq H$ is irreducible.*
(2) *There exists a standard subspace $H \subset \mathcal{H}$ such that $H \subsetneq K$ is irreducible.*
(3) *The modular operator Δ_K is unbounded, K is a factor, and the cutting projection P_K of K is unbounded.*

This proposition states that irreducible inclusions of standard subspaces exist in abundance. A central question is then how to detect whether a given inclusion is irreducible, or how to detect whether the relative symplectic complement $K' \cap H$ is cyclic (hence standard).

Detecting irreducibility. Let K, H be a pair of standard subspaces. Then [2, Prop. 4.1]

$$K' \cap H + i(K' \cap H) = \{v \in \mathrm{dom}(S_K^* S_H) \,:\, S_K^* S_H v = v\}.$$

This characterization is however often difficult to use as it leads to intricate domain questions. The same holds true for other characterizations that we derived for $K' \cap H$ in terms of polarizers and projections [4].

Comparing with the von Neumann algebraic situation, two notions that are helpful tools in the understanding of relative commutants are the split property [6] and modular nuclearity [1]. We give standard subspace formulations for both of them and investigate their consequences in [4]. Here we focus on the nuclearity aspects.

Definition 3. An inclusion $K \subset H$ of standard subspaces is said to satisfy modular nuclearity if the real linear operator $\Delta_H^{1/4} E_K$, where $E_K : \mathcal{H} \to K$ is the real orthogonal projection onto K, is trace class.

Making use of [10, 1, 11], we then prove:

Theorem 4. [4] *Let $K \subset H$ be an inclusion of factor standard subspaces satisfying modular nuclearity. Then $\dim(K' \cap H) = \infty$.*

A class of examples. As a concrete class of examples, we consider the irreducible one-dimensional standard pair, namely the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+, \frac{dp}{p})$ and the standard subspace $H \subset \mathcal{H}$ given by the data (see [8, Sect. 4] for this and other equivalent formulations)

$$(\Delta_H^{it} \psi)(p) = \psi(e^{-2\pi t} p), \qquad (J_H \psi)(p) = \overline{\psi(p)}.$$

The one-parameter group of unitaries $(U(x)\psi)(p) = e^{ipx}\psi(p)$ acts half-sidedly by endomorphisms of H, namely $U(x)H \subset H$, $x \geq 0$. It is known that the semigroup of all unitaries $V \in \mathcal{U}(\mathcal{H})$ that commute with $U(x)$, $x \in \mathbb{R}$, and satisfy $VH \subset H$, are precisely the unitaries of the form $V = \varphi(P)$, where P is the generator of U and φ an inner function of the upper half plane satisfying the symmetry condition $\varphi(-p) = \overline{\varphi(p)}$, $p > 0$ [12, Thm. 2.3].

We are therefore presented with the family of concrete inclusions $\varphi(P)H \subset H$. In the talk it was explained that the modular nuclearity condition fails except for quite specifically chosen inner functions φ. Nonetheless it is possible to understand and sometimes explicitly compute the relative symplectic complement $\varphi(P)H'\cap H$, which can be $\{0\}$, finite-dimensional, infinite-dimensional, or cyclic depending on φ. In particular, there are interesting relations relating the number of zeros of the inner function φ and the dimension of $\varphi(P)H' \cap H$.

The structures found in this class of examples are currently being investigated alongside more general methods for analyzing relative symplectic complements of standard subspaces [4].

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