Modular Structure and Inclusions of Twisted Araki-Woods Algebras

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joint work with Ricardo Correa da Silva

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FAU

UC Berkeley Probabilistic Operator Algebra Seminar

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2. Motivation and questions (background: mathematical physics, QFT)
3. Standardness and modular data
4. Inclusions of twisted Araki-Woods algebras ("twisted subfactors")

Construction of $\mathcal{L}_T(H)$ on twisted Fock spaces

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- **Notation:**

$$T_k := 1_{\mathcal{H}}^{\otimes(k-1)} \otimes T \otimes 1_{\mathcal{H}}^{\otimes(n-k-1)} \in \mathcal{B}(\mathcal{H}^{\otimes n}), \quad 1 \le k \le n-1$$

- **Kernels:**

$$P_{T,1} = 1, \quad P_{T,2} = 1 + T, \quad P_{T,3} = 1 + T_1 + T_2 + T_1 T_2 + T_2 T_1 + T_2 T_1 T_2,$$

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T-twisted Fock space

$$\mathcal{F}_T(\mathcal{H}) := \bigoplus_{n\geq 0} \overline{\mathcal{H}^{\otimes n} / \ker P_{T,n}}^{\langle \cdot, \cdot \rangle_{T,n}}$$

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Theorem ([Jørgensen/Schmitt/Werner; Bożejko/Speicher])

Let $T = T^* \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, $\|T\| \le 1$.

1. *If* $\|T\| \le \frac{1}{2}$, *then* T *is a strict twist.*
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3. *If*

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An example from QFT ("S-Matrix Model")

$\mathcal{H} = L^2(\mathbb{R}, d\theta)$, $s : \mathbb{R} \to S^1$, $s(-\theta) = \overline{s(\theta)}$. Then

$$(Tf)(\theta_1, \theta_2) = s(\theta_1 - \theta_2) \cdot f(\theta_2, \theta_1) \qquad \text{is a unitary twist.}$$

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- Relations ($\dim \mathcal{H} < \infty$, (e_k) ONB, $a_i := a_{L,T}(e_i)$)

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- **Left field operators:**

$$\phi_{L,T}(\xi) := a^\star_{L,T}(\xi) + a_{L,T}(\xi).$$

(Left) twisted Araki-Woods Algebra (with $H \subset \mathcal{H}$)

$$\mathcal{L}_T(H) := \{\phi_{L,T}(h) : h \in H\}'' \subset \mathcal{B}(\mathcal{F}_T(\mathcal{H}))$$

w.l.o.g.: $H \subset \mathcal{H}$ closed $\mathbb{R}$-linear subspace.

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$$S_H : H + iH \to H + iH, \qquad h_1 + ih_2 \mapsto h_1 - ih_2.$$

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- Polar decomposition: $S_H = J_H \Delta_H^{1/2}$ with J_H antiunitary and $\Delta_H > 0$.
- Tomita's Theorem for standard subspaces:

$$\Delta_H^{it} H = H, \qquad J_H H = H' = \{h' \in \mathcal{H} : \mathrm{Im}\langle h, h' \rangle = 0 \,\forall h \in H\}$$

  H' is also a standard subspace, and $(H')' = H$.

Real Hilbert spaces vs. standard subspaces

Proposition ([Shlyakhtenko '97])

There is a one to one correspondence between

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  $\mathcal{L}_T(H)$ is non-injective, of type III unless $\Delta_H = 1$

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In the following: $H \subset \mathcal{H}$ an arbitrary standard subspace (i.e. arbitrary $U(t)$ resp. modular group Δ_H^{it}), and T a twist.

Separating vacuum

Basic assumption: T and H are **compatible** in the sense $[T, \Delta_H^{it} \otimes \Delta_H^{it}] = 0$.

Lemma: If Ω is separating for $\mathcal{L}_T(H)$ and H, T are compatible, then the modular data Δ, J of $(\mathcal{L}_T(H), \Omega)$ restrict to Δ_H, J_H on $\mathcal{H}$.

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- In order to have Ω separating for $\mathcal{L}_T(H)$, need **KMS-property**. Consider n-point functions $(h_1, \ldots, h_n \in H)$

$$f_n(t) := \langle \Omega, \phi_{L,T}(h_1) \cdots \phi_{L,T}(h_{n-1}) \Delta^{it} \phi_{L,T}(h_n) \Omega \rangle_T = \langle 1\,2\,\ldots\,(n-1)\,n_t \rangle.$$

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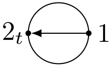
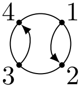
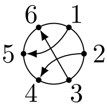
- Graphical notation (~[Bożejko/Speicher])

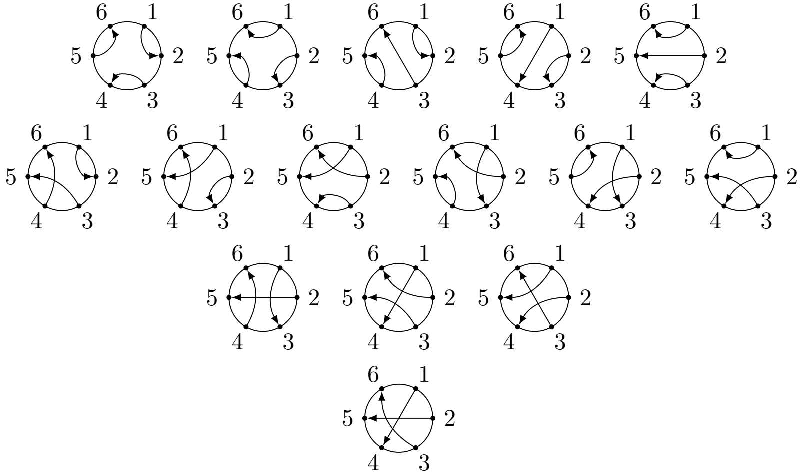






$\langle J_H h_1, \Delta_H^{it} h_2 \rangle$, $\quad \langle \bar{1}, 2 \rangle \cdot \langle \bar{3}, \Delta_H^{it} 4 \rangle$, $\quad \langle \bar{3} \otimes T(\bar{2} \otimes \bar{1}), T(4 \otimes 5) \otimes 6_t \rangle$

Six-point function $\langle 1\,2\,\ldots\,6_t \rangle$



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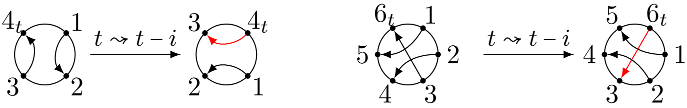

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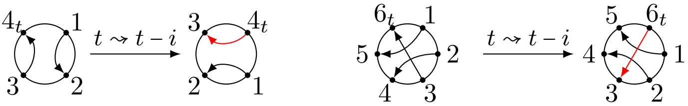

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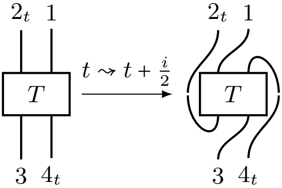

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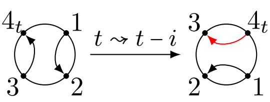
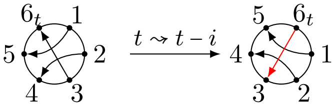
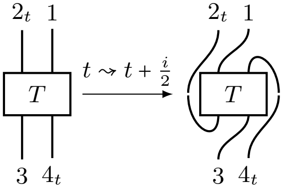




$\langle 2_t \otimes 1, T(3 \otimes 4_t) \rangle =$  $= \langle 1 \otimes 4_t, T(2_t \otimes 3) \rangle$

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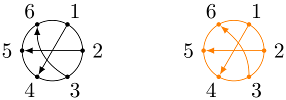
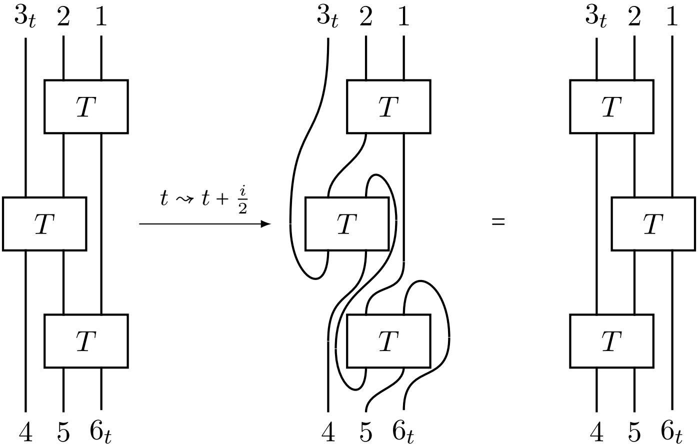
2. The two possible triple crossing terms in the 6-point function differ by a Reidemeister move of type III.




3_t 2 1
T
T
T
4 5 6_t
$t \rightsquigarrow t + \frac{i}{2}$
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T
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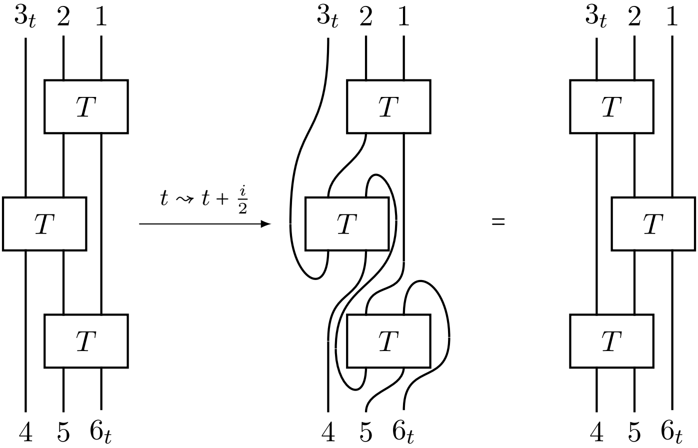

By exploiting KMS condition, one can show that one must have RHS = LHS (→ Yang-Baxter equation.)

Definition

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has an analytic continuation to the strip $\mathbb{S}_{1/2}$ (...) and

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Theorem

Let $H \subset \mathcal{H}$ be a standard subspace and T a compatible twist. The following are equivalent:

a) *Ω is separating for $\mathcal{L}_T(H)$.*

b) *T is braided and crossing symmetric w.r.t. H.*

Braided twists and left-right duality

How does the argument "YBE+crossing $\implies \Omega$ separating" work?

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For **braided** twists (YBE $T_1 T_2 T_1 = T_2 T_1 T_2$ holds), also right creation/annihilation operators exist:

$$a^\star_{R,T}(\xi)[\Psi_n] = [\Psi_n \otimes \xi],$$
$$a_{R,T}(\xi)[\Psi_n] = [a_{R,0}(\xi)(1 + T_n + \ldots + T_{n-1} \cdots T_1)\Psi_n]$$
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... generating "right" twisted Araki-Woods algebras $\mathcal{R}_T(H)$.

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Proposition

Let T be braided and crossing symmetric.

a) *The Tomita operator S of $(\mathcal{L}_T(H), \Omega)$ is given by*

$$S[\psi_1 \otimes \ldots \otimes \psi_n] = [S_H\psi_n \otimes \ldots \otimes S_H\psi_1]$$

b) *Left-right duality holds:*

$$\mathcal{L}_T(H)' = \mathcal{R}_T(H').$$

Remarks on standardness question

- From our perspective, the braided and crossing-symmetric twists are the most interesting ones (Classification unknown).
- Both the Yang-Baxter equation and crossing symmetry have their origins in physics, but can here be derived from modular theory.
- Definition of crossing is inspired by QFT crossing symmetry (scattering of particles vs. scattering of antiparticles, J_H=TCP operator)
- Result on modular data generalizes many known results [Eckmann/Osterwalder '73, Leyland/Roberts/Testard '78, Shlyakhtenko '97, Baumgärtel/Jurke/Lledo '02, Buchholz/L/Summers '11, L '12]

Inclusions

Have two maps

$$H \longmapsto \mathcal{L}_T(H), \qquad H \longmapsto \mathcal{R}_T(H)$$

from T-comp. standard subspaces $H \subset \mathcal{H}$ to v. Neumann algebras on $\mathcal{F}_T(\mathcal{H})$.

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Lemma: Proper inclusions $K \subset H$ only exist if Δ_H, Δ_K are unbounded. In particular $\dim \mathcal{H} = \infty$ is needed.

Twisted subfactors

- For $T = qF$, $-1 < q < 1$, it is known that $\mathcal{L}_{qF}(H)$ is a non-injective factor of type III if Δ_H is unbounded [Kumar, Skalski, Wasilewski '23].

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$K \subset H$. Relative commutant

$$\mathcal{C}_T(K,H) := \mathcal{L}_T(K)' \cap \mathcal{L}_T(H) = \mathcal{L}_T(K)' \cap \mathcal{R}_T(H')'.$$

In the following: Two results on $\mathcal{L}_T(K) \subset \mathcal{L}_T(H)$ in different situations,

- one "negative" (singular inclusions, $\mathcal{C}_T(K,H) = \mathbb{C}1$)
- one "positive" (large relative commutant, $\mathcal{C}_T(K,H) \neq \mathbb{C}1$)

Half-sided inclusions

Let us consider a half-sided inclusion $K \subset H$ of standard subspaces:

- have unitary one-parameter group $V(x)$ with positive generator,
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- $\mathcal{L}_T(H)$ is a III_1 factor [Wiesbrock '93].
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Singular inclusions

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Let $K \subset H$ be a half-sided inclusion of standard subspaces and T a compatible braided crossing-symmetric twist with $\|T\| < 1$. Then $\mathcal{C}_T(K, H) = \mathbb{C}1$.

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Generalization:

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Let $K \subset H$ be standard subspaces. Suppose there exist sequences of unit vectors $k_n \in K$, $h'_n \in H'$, such that

$$k_n \to 0, \quad h'_n \to 0 \quad \textit{weakly}, \qquad \langle k_n, h'_n \rangle \not\to 0.$$

- *Then $\mathcal{C}_T(K,H) = \mathbb{C}1$ (for $\|T\| < 1$).*
- *This is in particular the case when $\Delta_H^{1/4}\Delta_K^{-1/4}$ is not compact.*

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Corollary: $\mathcal{L}_T(H)$ is a factor for $\|T\| < 1$ and $\dim \mathcal{H} = \infty$.

L^2-inclusions

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Let $K \subset H$ be an inclusion such that $\|\Delta_H^{1/4}\Delta_K^{-1/4}\|_1 < 1$ (trace norm). Let T be a braided crossing symmetric compatible twist with $\|T\| < 1$. Then

a) $\mathcal{L}_T(K) \subset \mathcal{L}_T(H)$ *is split.*

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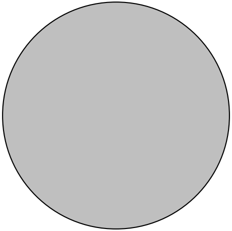
- Relation between $\Delta_H^{1/4}\Delta_K^{-1/4}$ and $\mathcal{C}_T(K,H)$ is much closer for $\|T\| < 1$ than for $\|T\| = 1$.

$\sigma(TF)$



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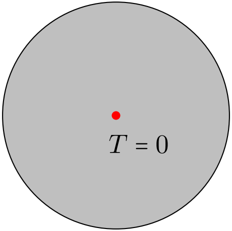

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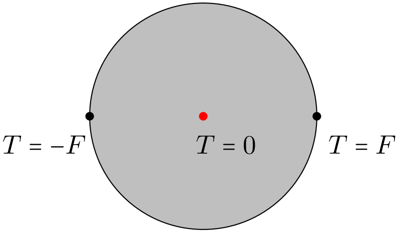

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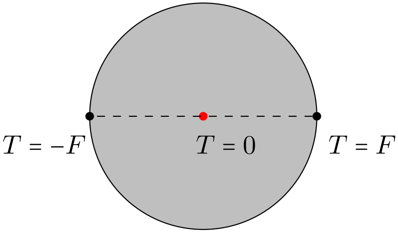

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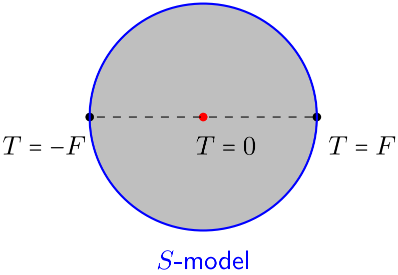

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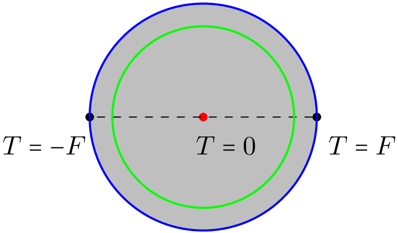

S-model

qS-model

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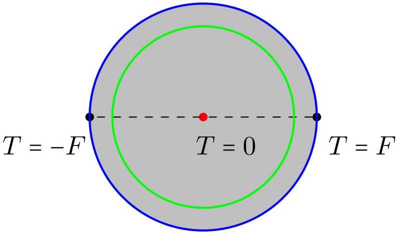

S-model

qS-model

- Do there exist inclusions $\mathcal{L}_T(K) \subset \mathcal{L}_T(H)$ that have non-trivial relative commutant but are not split?
- Interesting regime: $\Delta_H^{1/4}\Delta_K^{-1/4}$ compact, but *not* $\|\Delta_H^{1/4}\Delta_K^{-1/4}\|_1 < 1$. Can we say something about $\mathcal{C}_T(K, H)$ (avoiding split)?