TURNPIKE THEORY AND APPLICATIONS

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Turnpike in a Nutshell

Joint work with Noboru Sakamoto and Enrique Zuazua.

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INTRODUCTION

We consider the **dynamical** optimal control problem

$$\min_u J^T(u) = \int_0^T L(y,u)dt,$$

where:

$$\begin{cases} \frac{d}{dt}y = f(y,u) & \text{in } (0,T) \\ y(0) = y_0. \end{cases}$$

We assume the above problem is well posed as well as its **steady** analogue

$$\min_u J_s(u) = L(y,u), \quad \text{with the constraint} \quad f(y,u) = 0.$$

The turnpike property

The control problem enjoys the **turnpike property** if the time-evolution optimal pair (u^T, y^T) in long time remains exponentially close to the steady optimal pair $(\overline{u}, \overline{y})$ for most of time, except for thin initial and final boundary intervals.

The turnpike property in semilinear control

Time-evolution optimal control problem

$$\min_{u \in L^2((0,T)\times\omega)} J_T(u) = \frac{1}{2}\int_0^T \int_\omega |u|^2 dxdt + \frac{\beta}{2}\int_0^T \int_{\omega_0} |y - z|^2 dxdt,$$

where:

$$\begin{cases} y_t - \Delta y + f(y) = u\chi_\omega & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x) & \text{in } \Omega. \end{cases}$$

$\Omega \subset \mathbb{R}^n$ is a regular bounded domain, with $n = 1, 2, 3$. The nonlinearity f is C^3 increasing, with $f(0) = 0$. Hence, the behaviour is **dissipative**, thus avoiding blow up. $\omega \subseteq \Omega$ is the control domain, while $\omega_0 \subseteq \Omega$ is the observation domain. The target z is bounded and the parameter $\beta > 0$.

By direct methods in the Calculus of Variations, there exists an optimal control u^T minimizing J^T. The corresponding optimal state is denoted by y^T.

Steady optimal control problem

$$\min_{u\in L^2(\omega)} J_s(u) = \frac{1}{2}\int_\omega |u|^2 dx + \frac{\beta}{2}\int_{\omega_0} |y-z|^2 dx,$$

where:

$$\begin{cases} -\Delta y + f(y) = u\chi_\omega & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega. \end{cases}$$

By direct methods in the Calculus of Variations, there exists an optimal control $\overline{u}$ minimizing J_s. The corresponding optimal state is denoted by $\overline{y}$.

Uniqueness steady optimal control

If $\|z\|_{L^\infty}$ is **small** enough, the steady problem admits a **unique** optimal control $\overline{u} = -\overline{q}\chi_\omega$, where $(\overline{y}, \overline{q})$ solves the Optimality System

$$\begin{cases} -\Delta\overline{y} + f(\overline{y}) = -\overline{q}\chi_\omega & \text{in } \Omega \\ \overline{y} = 0 & \text{on } \partial\Omega \\ -\Delta\overline{q} + f'(\overline{y})\overline{q} = \beta(\overline{y} - z)\chi_{\omega_0} & \text{in } \Omega \\ \overline{q} = 0 & \text{on } \partial\Omega. \end{cases}$$

Porretta, Alessio and Zuazua, Enrique
Remarks on long time versus steady state optimal control
Mathematical Paradigms of Climate Science, (2016), *pp.* 67 – 89

Local turnpike

Theorem (Porretta-Zuazua, 2016)

*There exists $\delta > 0$ such that if the **initial datum** and the target fulfil the **smallness condition***

$$\|y_0\|_{L^\infty} \leq \delta \quad \text{and} \quad \|z\|_{L^\infty} \leq \delta,$$

there exists a solution (y^T, q^T) to the Optimality System

$$\begin{cases} y_t^T - \Delta y^T + f(y^T) = -q^T \chi_\omega & \text{in } (0,T)\times\Omega \\ y^T = 0 & \text{on } (0,T)\times\partial\Omega \\ y^T(0,x) = y_0(x) & \text{in } \Omega \\ -q_t^T - \Delta q^T + f'(y^T)q^T = \beta(y^T - z)\chi_{\omega_0} & \text{in } (0,T)\times\Omega \\ q^T = 0 & \text{on } (0,T)\times\partial\Omega \\ q^T(T,x) = 0 & \text{in } \Omega \end{cases}$$

satisfying for any $t \in [0,T]$

$$\|q^T(t) - \overline{q}\|_{L^\infty(\omega)} + \|y^T(t) - \overline{y}\|_{L^\infty(\Omega)} \leq K\left[e^{-\mu t} + e^{-\mu(T-t)}\right],$$

where K and μ are T-independent.

Our goal is to

1. prove that in fact the turnpike property is satisfied by the optima;
2. **remove** the **smallness** condition on the **initial datum**.

We keep the smallness condition on the target. This leads to the smallness and uniqueness of the steady optima.

Global turnpike

Theorem (P.-Zuazua, 2019)

*Let u^T be an optimal control for the time-evolution problem. There exists $\rho > 0$ such that **for every** $y_0 \in L^\infty(\Omega)$ and z verifying*

$$\|z\|_{L^\infty} \leq \rho,$$

we have for any $t \in [0, T]$

$$\|u^T(t) - \overline{u}\|_{L^\infty(\omega)} + \|y^T(t) - \overline{y}\|_{L^\infty(\Omega)} \leq K \left[e^{-\mu t} + e^{-\mu(T-t)}\right],$$

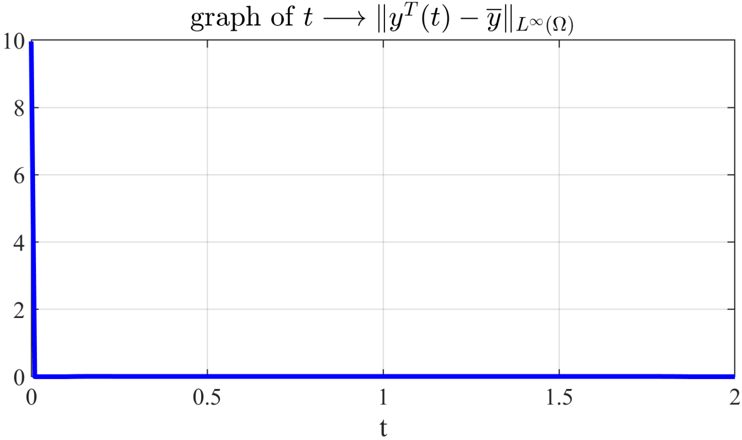
the constants K and $\mu > 0$ being independent of the time horizon T.

Main ingredients of the proof

The main ingredients that our proofs require are as follows:

- prove a L^∞ **bound** of the norm of the optimal control, **uniform in the time horizon** $T > 0$;
- for **small data** and **small targets**, prove that any optimal control verifies the turnpike property;
- for **small targets** and **any data**, proof of the smallness of $\|y^T(t)\|_{L^\infty(\Omega)}$ in time t large. This is done by estimating the critical time needed to approach the turnpike;
- conclude concatenating the two former steps.

Numerical simulations



Steady optimal control problem

$$\min_{u\in L^2(\omega)} J_s(u) = \frac{1}{2}\int_\omega |u|^2 dx + \frac{\beta}{2}\int_{\omega_0} |y-z|^2 dx,$$

where:

$$\begin{cases} -\Delta y + f(y) = u\chi_\omega & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega. \end{cases}$$

Uniqueness of the optimal control for **large** targets z?

Steady optimal control problem

$$\min_{v\in\mathbb{R}} J_s(v) = \frac{1}{2}|v|^2 + \frac{\beta}{2}\int_{\omega_0} |y - z|^2 dx,$$

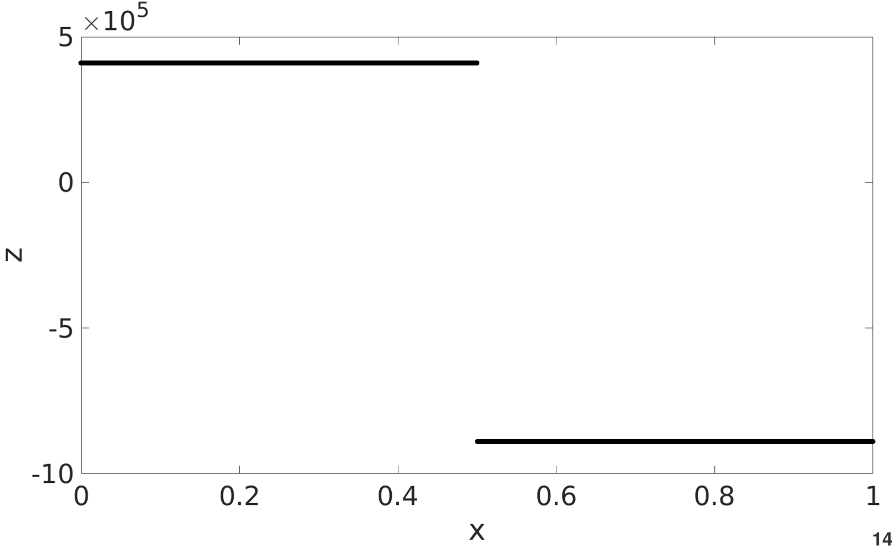
where:

$$\begin{cases} -\Delta y + y^3 = vg\chi_\omega & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega. \end{cases}$$

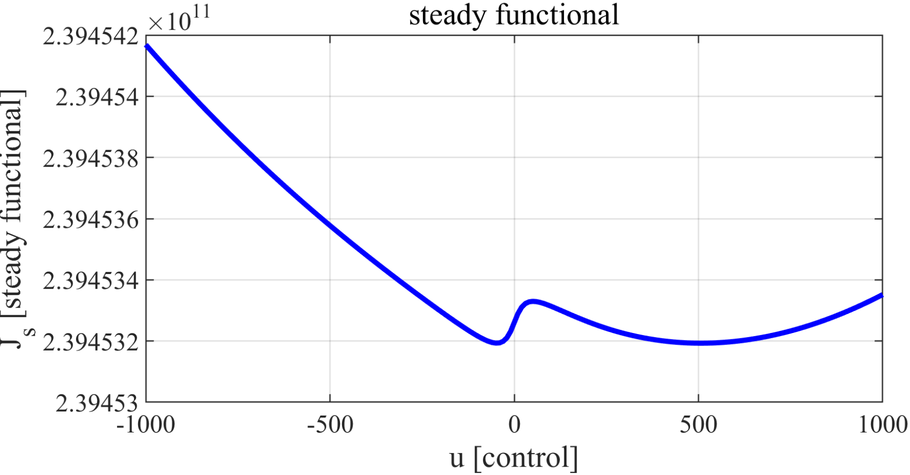
Theorem (P.-Zuazua, 2019)

Suppose $g \in L^\infty(\omega) \setminus \{0\}$ is nonnegative. Assume $\overline{\omega} \subset \omega_0$. Then, there exists a target $z \in L^\infty(\omega_0)$ such that the functional J_s admits (at least) ***two*** *global* ***minimizers****.*

Steady optimal control problem



Steady optimal control problem



Steady optimal control problem

The proof is based on the following local estimate

Lemma (J. Henry)

Suppose $p > 1$. Let ω_1 be an open subset of $\Omega \setminus \overline{\omega}$. For any control $u \in L^2(\omega)$, let y_u be the unique solution to

$$\begin{cases} -\Delta y + |y|^{p-1} y = u\chi_\omega & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega. \end{cases}$$

There exists K such that, for any control $u \in L^2(\omega)$,

$$\|y_u\|_{L^\infty(\omega_1)} \leq K. \tag{1}$$

Henry, Jacques
Etude de la contrôlabilité de certaines équations paraboliques non linéaires
These, Paris, (1977)

Time-evolution optimal control problem

We consider the corresponding time-evolution control problem

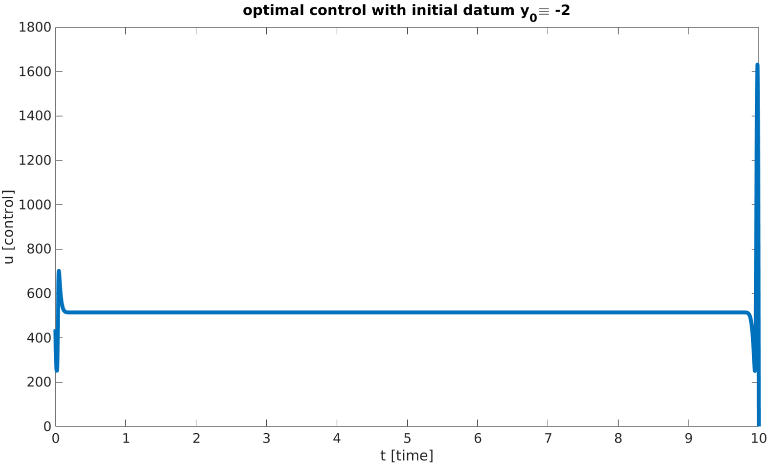
$$\min_{v\in L^2(0,T)} J_T(v) = \frac{1}{2}\int_0^T |v|^2 dt + \frac{\beta}{2}\int_0^T \int_{\omega_0} |y-z|^2 dxdt,$$

where:

$$\begin{cases} y_t - \Delta y + y^3 = vg\chi_\omega & \text{in } (0,T)\times\Omega \\ y = 0 & \text{on } (0,T)\times\partial\Omega \\ y(0,x) = y_0(x) & \text{in } \Omega. \end{cases}$$

Time-evolution optimal control problem

Choose initial datum $y_0 \equiv -2$.



Time-evolution optimal control problem

The corresponding optimality system reads as

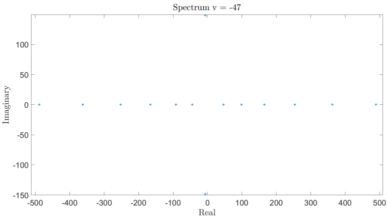
$$\begin{cases} y_t^T - \Delta y^T + (y^T)^3 = -\int_\omega q^T dx \chi_\omega & \text{in } (0,T)\times\Omega \\ y^T = 0 & \text{on } (0,T)\times\partial\Omega \\ y^T(0,x) = y_0(x) & \text{in } \Omega \\ -q_t^T - \Delta q^T + 3(y^T)^2 q^T = \beta(y^T - z)\chi_{\omega_0} & \text{in } (0,T)\times\Omega \\ q^T = 0 & \text{on } (0,T)\times\partial\Omega \\ q^T(T,x) = 0 & \text{in } \Omega \end{cases}$$

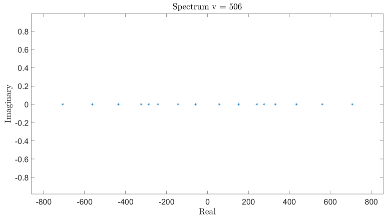
Time-evolution optimal control problem

Consider the linearization of the optimality system around one steady optimum $(\overline{y}, \overline{q})$

$$
\begin{cases}
\eta_t^T - \Delta\eta^T + 3(\overline{y})^2\eta^T = -\int_\omega \varphi^T dx \chi_\omega & \text{in } (0,T)\times\Omega \\
\eta^T = 0 & \text{on } (0,T)\times\partial\Omega \\
\eta^T(0,x) = y_0(x) & \text{in } \Omega \\
-\varphi_t^T - \Delta\varphi^T + 3(\overline{y})^2\varphi^T = (\beta\chi_{\omega_0} - 6\overline{y}\overline{q})\,\eta^T & \text{in } (0,T)\times\Omega \\
\varphi^T = 0 & \text{on } (0,T)\times\partial\Omega \\
\varphi^T(T,x) = 0 & \text{in } \Omega
\end{cases}
$$

Time-evolution optimal control problem





Steady optimal control problem

$$\min_{u\in L^2(\omega)} J_s(u) = \frac{1}{2}\int_{\omega} |u|^2 dx + \frac{\beta}{2}\int_{\omega_0} |y-z|^2 dx,$$

where:

$$\begin{cases} -\Delta y + y^3 = u\chi_\omega & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition (P.-Zuazua, 2019)

Assume $\overline{\omega} \subsetneq \omega_0$. There exists a target $z \in L^\infty(\omega_0)$ such that the steady functional J_s admits (at least) ***two stationary points****. Namely, there exist two distinguished pairs $(\overline{y}, \overline{q})$ satisfying the optimality system*

$$\begin{cases} -\Delta \overline{y} + \overline{y}^3 = -\overline{q}\chi_\omega & \text{in } \Omega \\ -\Delta \overline{q} + 3\overline{y}^2\overline{q} = \beta(\overline{y} - z) & \text{in } \Omega \\ \overline{y} = 0, \quad \overline{q} = 0 & \text{on } \partial\Omega. \end{cases}$$

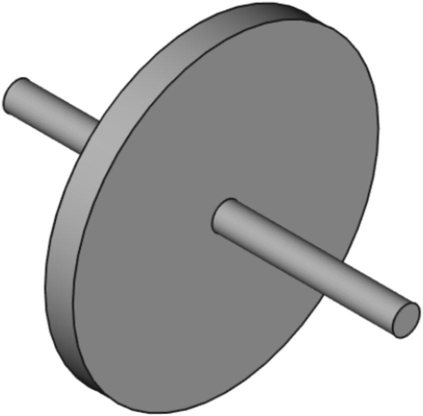
THE TURNPIKE PROPERTY IN ROTORS IMBALANCE SUPPRESSION

Secondment in Marposs

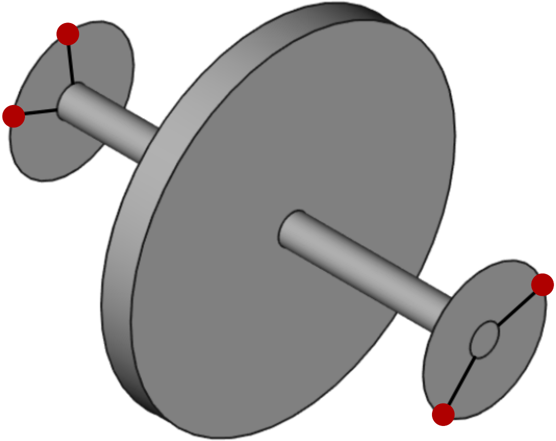
Figure: Marposs headquarter

Consider a **rotor** rotating about a **fixed axis**, with respect to an inertial reference frame.

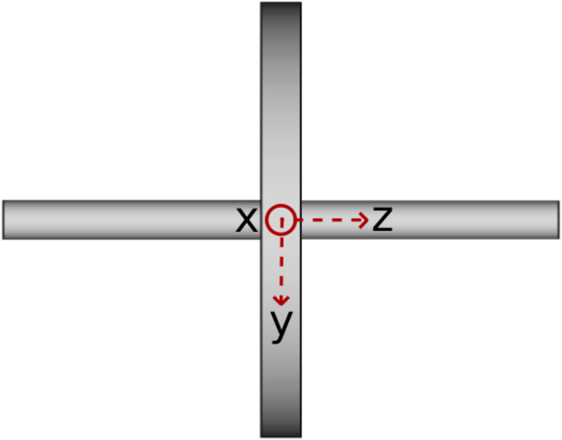
Often time, rotor's **mass** distribution is **not homogeneous**, thus producing dangerous **vibrations**.



A system of **balancing masses** is **given**. We determine the **optimal movement** of the balancing masses to **minimize the imbalance** of the rotor.

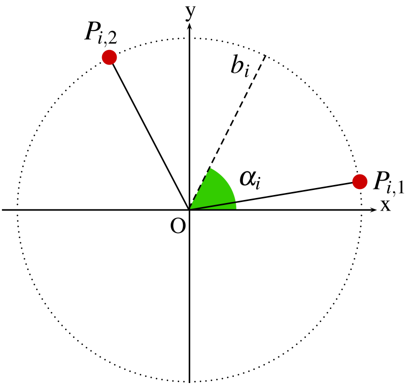


We introduce a **rotor-fixed reference frame** $(O; (x, y, z))$, where z coincides with the rotation axis.

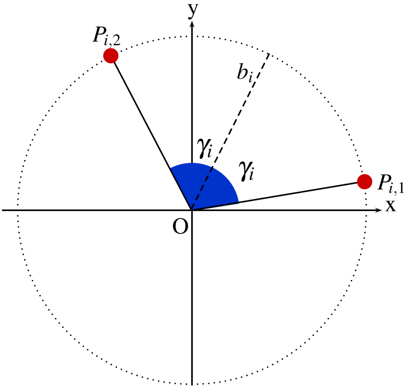


The balancing masses are supposed to rotate in two planes π_1 and π_2 orthogonal to the rotation axis.

In each balancing plane π_i, the **positions** of the corresponding balancing masses are given by two **angles** α_i and γ_i and their mass is m_i.



(a) *intermediate* angle



(b) *gap* angle

The **imbalance** generates a force F and a momentum N on the rotation axis, which can be decomposed into a force $\boldsymbol{F_1}$ in plane π_1 and a force $\boldsymbol{F_2}$ in π_2.

The **balancing** masses produce a force $\boldsymbol{B_1}(\alpha_1, \gamma_1)$ in π_1 and a force $\boldsymbol{B_2}(\alpha_2, \gamma_2)$ in π_2 to compensate the imbalance.

The global imbalance of the system made of rotor and balancing heads is given by the **imbalance indicator**

$$G(\alpha_1, \gamma_1; \alpha_2, \gamma_2) := \|B_1(\alpha_1, \gamma_1) + F_1\|^2 + \|B_2(\alpha_2, \gamma_2) + F_2\|^2 .$$

We assume the existence of $(\overline{\alpha}_1, \overline{\gamma}_1; \overline{\alpha}_2, \overline{\gamma}_2) \in \mathbb{T}^4$, such that $G(\overline{\alpha}_1, \overline{\gamma}_1; \overline{\alpha}_2, \overline{\gamma}_2) = 0$.

Optimal control problem

Find the **trajectory** $\Phi(t) = (\alpha_1(t), \gamma_1(t); \alpha_2(t), \gamma_2(t))$ **minimizing**

$$J(\Phi) := \frac{1}{2}\int_0^{\infty} \left[\|\dot{\Phi}\|^2 + \beta G(\Phi)\right] dt,$$

over the set of admissible trajectories

$$\mathscr{A} := \left\{ \Phi \in \bigcap_{T>0} H^1((0,T); \mathbb{T}^4) \;\middle|\; \Phi(0) = \Phi_0, \right.$$

$$\left. \dot{\Phi} \in L^2(0,+\infty) \quad \text{and} \quad G(\Phi) \in L^1(0,+\infty). \right\}.$$

$\beta > 0$ is a weighting parameter.

Optimal control problem

Proposition (Gnuffi-P.-Sakamoto, 2019)

For $i = 1, 2$, set

$$c^i := \frac{1}{2 m_i r_i \omega^2} \left(F_{i,x}, F_{i,y}\right)$$

Then,

1. *there **exists** $\Phi \in \mathscr{A}$ **minimizer** of J;*
2. *$\Phi = (\alpha_1, \gamma_1; \alpha_2, \gamma_2)$ is C^∞ smooth and, for $i = 1, 2$, the following Euler-Lagrange equations are satisfied, for $t > 0$*

$$\begin{cases} -\ddot{\alpha}_i = \beta \cos(\gamma_i) \left[-c_1^i \sin(\alpha_i) + c_2^i \cos(\alpha_i)\right] \\ -\ddot{\gamma}_i = -\beta \sin(\gamma_i) \left[c_1^i \cos(\alpha_i) + c_2^i \sin(\alpha_i) - \cos(\gamma_i)\right] \\ \alpha_i(0) = \alpha_{0,i}, \;\; \gamma_i(0) = \gamma_{0,i}, \;\; \dot{\Phi}(T) \underset{T \to +\infty}{\longrightarrow} 0. \end{cases}$$

Optimal control problem

Proposition (Gnuffi-P.-Sakamoto, 2019)

Let Φ be an optimal trajectory. Then,

(1) *there exists $\overline{\Phi} \in zero(G)$ such that*

$$\Phi(t) \underset{t\to+\infty}{\longrightarrow} \overline{\Phi}, \quad \dot{\Phi}(t) \underset{t\to+\infty}{\longrightarrow} 0.$$

and

$$|G(\Phi(t))| \underset{t\to+\infty}{\longrightarrow} 0.$$

(2) *If, in addition*

$$m_1 r_1 > \frac{\sqrt{F_{1,x}^2 + F_{1,y}^2}}{2\omega^2} \quad \text{and} \quad m_2 r_2 > \frac{\sqrt{F_{2,x}^2 + F_{2,y}^2}}{2\omega^2},$$

we have the ***exponential*** *estimate for any $t \geq 0$*

$$\|\Phi(t) - \overline{\Phi}\| + \|\dot{\Phi}(t)\| + |G(\Phi(t))| \leq C \exp(-\mu t),$$

with C, $\mu > 0$ independent of t.

Optimal control problem

1. the proof of (1) is based on **Łojasiewicz inequality**;
2. the proof of (2) relies on the **Stable Manifold Theorem** applied to the Pontryagin Optimality System.

Sakamoto, Noboru and Pighin, Dario and Zuazua, Enrique
The turnpike propety in nonlinear optimal control - A geometric approach
arXiv:1903.09069

Gnuffi, Matteo and Pighin, Dario and Sakamoto, Noboru
Rotors imbalance suppression by optimal control
arXiv:1907.11697

The related computational code is available in the DyCon blog at the following link:
`https://deustotech.github.io/DyCon-Blog/tutorial/wp02/P0005`

DyConBlog

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