TURNPIKE THEORY AND APPLICATIONS

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Turnpike in a Nutshell

Joint work with Noboru Sakamoto and Enrique Zuazua.
INTRODUCTION
We consider the **dynamical** optimal control problem

\[
\min_u J^T(u) = \int_0^T L(y, u) dt,
\]

where:

\[
\begin{align*}
\frac{d}{dt} y &= f(y, u) \quad \text{in } (0, T) \\
y(0) &= y_0.
\end{align*}
\]

We assume the above problem is well posed as well as its **steady** analogue

\[
\min_u J_s(u) = L(y, u), \quad \text{with the constraint } f(y, u) = 0.
\]

**The turnpike property**

The control problem enjoys the **turnpike property** if the time-evolution optimal pair \((u^T, y^T)\) in long time remains exponentially close to the steady optimal pair \((\bar{u}, \bar{y})\) for most of time, except for thin initial and final boundary intervals.
THE TURNPIKE PROPERTY IN SEMILINEAR CONTROL
The turnpike property in semilinear control

The turnpike property in rotors imbalance suppression

small targets
large targets

Time-evolution optimal control problem

\[
\min_{u \in L^2((0,T) \times \omega)} J_T(u) = \frac{1}{2} \int_0^T \int_\omega |u|^2 \, dx \, dt + \frac{\beta}{2} \int_0^T \int_{\omega_0} |y - z|^2 \, dx \, dt,
\]

where:

\[
\begin{cases}
  y_t - \Delta y + f(y) = u \chi_\omega & \text{in } (0, T) \times \Omega \\
  y = 0 & \text{on } (0, T) \times \partial \Omega \\
  y(0, x) = y_0(x) & \text{in } \Omega.
\end{cases}
\]

\( \Omega \subset \mathbb{R}^n \) is a regular bounded domain, with \( n = 1, 2, 3 \). The nonlinearity \( f \) is \( C^3 \) increasing, with \( f(0) = 0 \). Hence, the behaviour is dissipative, thus avoiding blow up. \( \omega \subseteq \Omega \) is the control domain, while \( \omega_0 \subseteq \Omega \) is the observation domain. The target \( z \) is bounded and the parameter \( \beta > 0 \).

By direct methods in the Calculus of Variations, there exists an optimal control \( u^T \) minimizing \( J^T \). The corresponding optimal state is denoted by \( y^T \).
Steady optimal control problem

\[
\min_{u \in L^2(\omega)} J_s(u) = \frac{1}{2} \int_\omega |u|^2 dx + \frac{\beta}{2} \int_{\omega_0} |y - z|^2 dx,
\]

where:

\[
\begin{align*}
-\Delta y + f(y) &= u \chi_\omega \quad \text{in } \Omega \\
y &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

By direct methods in the Calculus of Variations, there exists an optimal control \( \bar{u} \) minimizing \( J_s \). The corresponding optimal state is denoted by \( \bar{y} \).
If \( \|z\|_{L^\infty} \) is small enough, the steady problem admits a unique optimal control \( \overline{u} = -\overline{q}\chi_\omega \), where \((\overline{y}, \overline{q})\) solves the Optimality System

\[
\begin{aligned}
-\Delta \overline{y} + f(\overline{y}) &= -\overline{q}\chi_\omega & \text{in } \Omega \\
\overline{y} &= 0 & \text{on } \partial\Omega \\
-\Delta \overline{q} + f'(\overline{y})\overline{q} &= \beta(\overline{y} - z)\chi_\omega & \text{in } \Omega \\
\overline{q} &= 0 & \text{on } \partial\Omega.
\end{aligned}
\]

Porretta, Alessio and Zuazua, Enrique
Remarks on long time versus steady state optimal control
*Mathematical Paradigms of Climate Science*, (2016), pp. 67 – 89
Local turnpike

Theorem (Porretta-Zuazua, 2016)

There exists $\delta > 0$ such that if the initial datum and the target fulfil the smallness condition

$$\|y_0\|_{L^\infty} \leq \delta \quad \text{and} \quad \|z\|_{L^\infty} \leq \delta,$$

there exists a solution $(y^T, q^T)$ to the Optimality System

$$\begin{cases}
    y_t^T - \Delta y^T + f(y^T) = -q^T \chi_\omega & \text{in } (0, T) \times \Omega \\
    y^T = 0 & \text{on } (0, T) \times \partial\Omega \\
    y^T(0, x) = y_0(x) & \text{in } \Omega \\
    -q_t^T - \Delta q^T + f'(y^T)q^T = \beta(y^T - z)\chi_{\omega_0} & \text{in } (0, T) \times \Omega \\
    q^T = 0 & \text{on } (0, T) \times \partial\Omega \\
    q^T(T, x) = 0 & \text{in } \Omega
\end{cases}$$

satisfying for any $t \in [0, T]$

$$\|q^T(t) - \bar{q}\|_{L^\infty(\omega)} + \|y^T(t) - \bar{y}\|_{L^\infty(\Omega)} \leq K \left[ e^{-\mu t} + e^{-\mu(T-t)} \right],$$

where $K$ and $\mu$ are $T$-independent.
Our goal is to

1. prove that in fact the turnpike property is satisfied by the optima;
2. remove the smallness condition on the initial datum.

We keep the smallness condition on the target. This leads to the smallness and uniqueness of the steady optima.
Theorem (P.-Zuazua, 2019)

Let $u^T$ be an optimal control for the time-evolution problem. There exists $\rho > 0$ such that for every $y_0 \in L^\infty(\Omega)$ and $z$ verifying $\|z\|_{L^\infty} \leq \rho$, we have for any $t \in [0, T]$

$$\|u^T(t) - \bar{u}\|_{L^\infty(\omega)} + \|y^T(t) - \bar{y}\|_{L^\infty(\Omega)} \leq K \left[ e^{-\mu t} + e^{-\mu(T-t)} \right],$$

the constants $K$ and $\mu > 0$ being independent of the time horizon $T$. 
Main ingredients of the proof

The main ingredients that our proofs require are as follows:

- prove a $L^\infty$ bound of the norm of the optimal control, uniform in the time horizon $T > 0$;
- for small data and small targets, prove that any optimal control verifies the turnpike property;
- for small targets and any data, proof of the smallness of $\|y^T(t)\|_{L^\infty(\Omega)}$ in time $t$ large. This is done by estimating the critical time needed to approach the turnpike;
- conclude concatenating the two former steps.
Numerical simulations

Graph of $t \rightarrow \| y^T(t) - \bar{y} \|_{L^\infty(\Omega)}$
Steady optimal control problem

\[
\min_{u \in L^2(\omega)} J_s(u) = \frac{1}{2} \int_{\omega} |u|^2 \, dx + \frac{\beta}{2} \int_{\omega_0} |y - z|^2 \, dx,
\]

where:

\[
\begin{align*}
-\Delta y + f(y) &= u \chi_{\omega} \quad \text{in } \Omega \\
y &= 0 \quad \text{on } \partial\Omega.
\end{align*}
\]

**Uniqueness** of the optimal control for large targets \( z \)?
Steady optimal control problem

\[
\min_{v \in \mathbb{R}} J_s(v) = \frac{1}{2} |v|^2 + \frac{\beta}{2} \int_{\omega_0} |y - z|^2 \, dx,
\]

where:

\[
\begin{cases}
-\Delta y + y^3 = vg\chi_\omega & \text{in } \Omega \\
y = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Theorem (P.-Zuazua, 2019)

Suppose \( g \in L^\infty(\omega) \setminus \{0\} \) is nonnegative. Assume \( \overline{\omega} \subset \omega_0 \). Then, there exists a target \( z \in L^\infty(\omega_0) \) such that the functional \( J_s \) admits (at least) two global minimizers.
Steady optimal control problem
Steady optimal control problem
Steady optimal control problem

The proof is based on the following local estimate

Lemma (J. Henry)

Suppose $p > 1$. Let $\omega_1$ be an open subset of $\Omega \setminus \bar{\omega}$. For any control $u \in L^2(\omega)$, let $y_u$ be the unique solution to

\[
\begin{cases}
-\Delta y + |y|^{p-1}y = u\chi_\omega & \text{in } \Omega \\
y = 0 & \text{on } \partial\Omega.
\end{cases}
\]

There exists $K$ such that, for any control $u \in L^2(\omega)$,

\[
\|y_u\|_{L^\infty(\omega_1)} \leq K. \tag{1}
\]

Henry, Jacques
Etude de la contrôlabilité de certaines équations paraboliques non linéaires
These, Paris, (1977)
We consider the corresponding time-evolution control problem

$$
\min_{v \in L^2(0, T)} J_T(v) = \frac{1}{2} \int_0^T |v|^2 dt + \frac{\beta}{2} \int_0^T \int_{\omega_0} |y - z|^2 dx dt,
$$

where:

$$
\begin{cases}
  y_t - \Delta y + y^3 = vg\chi_{\omega} & \text{in } (0, T) \times \Omega \\
  y = 0 & \text{on } (0, T) \times \partial\Omega \\
  y(0, x) = y_0(x) & \text{in } \Omega.
\end{cases}
$$
Choose initial datum $y_0 \equiv -2$. 
The corresponding optimality system reads as

\[
\begin{align*}
  y_t^T - \Delta y^T + (y^T)^3 &= -\int_\omega q^T \, dx \chi_\omega & \text{in } (0, T) \times \Omega \\
  y^T &= 0 & \text{on } (0, T) \times \partial \Omega \\
  y^T(0, x) &= y_0(x) & \text{in } \Omega \\
  -q_t^T - \Delta q^T + 3(y^T)^2 q^T &= \beta(y^T - z) \chi_{\omega_0} & \text{in } (0, T) \times \Omega \\
  q^T &= 0 & \text{on } (0, T) \times \partial \Omega \\
  q^T(T, x) &= 0 & \text{in } \Omega
\end{align*}
\]
Consider the linearization of the optimality system around one steady optimum \((\bar{y}, \bar{q})\)

\[
\begin{align*}
\eta_t^T - \Delta \eta^T + 3(\bar{y})^2 \eta^T &= -\int_{\omega} \varphi^T dx \chi_{\omega} & \text{in } (0, T) \times \Omega \\
\eta^T &= 0 & \text{on } (0, T) \times \partial \Omega \\
\eta^T(0, x) &= y_0(x) & \text{in } \Omega \\
-\varphi_t^T - \Delta \varphi^T + 3(\bar{y})^2 \varphi^T &= (\beta \chi_{\omega_0} - 6\bar{y}\bar{q}) \eta^T & \text{in } (0, T) \times \Omega \\
\varphi^T &= 0 & \text{on } (0, T) \times \partial \Omega \\
\varphi^T(T, x) &= 0 & \text{in } \Omega 
\end{align*}
\]
Time-evolution optimal control problem
Steady optimal control problem

\[
\min_{u \in L^2(\omega)} J_s(u) = \frac{1}{2} \int_\omega |u|^2 \, dx + \frac{\beta}{2} \int_{\omega_0} |y - z|^2 \, dx,
\]

where:

\[
\begin{cases}
-\Delta y + y^3 = u \chi_\omega & \text{in } \Omega \\
y = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Proposition (P.-Zuazua, 2019)

Assume \( \omega \subsetneq \omega_0 \). There exists a target \( z \in L^\infty(\omega_0) \) such that the steady functional \( J_s \) admits (at least) two stationary points. Namely, there exist two distinguished pairs \((\overline{y}, \overline{q})\) satisfying the optimality system

\[
\begin{cases}
-\Delta \overline{y} + \overline{y}^3 = -\overline{q} \chi_\omega & \text{in } \Omega \\
-\Delta \overline{q} + 3\overline{y}^2 \overline{q} = \beta(\overline{y} - z) & \text{in } \Omega \\
\overline{y} = 0, \quad \overline{q} = 0 & \text{on } \partial \Omega.
\end{cases}
\]
THE TURNPIKE PROPERTY IN ROTORS IMBALANCE SUPPRESSION
Secondment in Marposs

Figure: Marposs headquarter
Consider a rotor rotating about a fixed axis, with respect to an inertial reference frame.

Often time, rotor’s mass distribution is not homogeneous, thus producing dangerous vibrations.
A system of balancing masses is given. We determine the optimal movement of the balancing masses to minimize the imbalance of the rotor.
We introduce a **rotor-fixed reference frame** \((O; (x, y, z))\), where \(z\) coincides with the rotation axis.
The balancing masses are supposed to rotate in two planes $\pi_1$ and $\pi_2$ orthogonal to the rotation axis.

In each balancing plane $\pi_i$, the positions of the corresponding balancing masses are given by two angles $\alpha_i$ and $\gamma_i$ and their mass is $m_i$. 

(a) intermediate angle  

(b) gap angle
The **imbalance** generates a force $F$ and a momentum $N$ on the rotation axis, which can be decomposed into a force $F_1$ in plane $\pi_1$ and a force $F_2$ in $\pi_2$.

The **balancing** masses produce a force $B_1(\alpha_1, \gamma_1)$ in $\pi_1$ and a force $B_2(\alpha_2, \gamma_2)$ in $\pi_2$ to compensate the imbalance.

The global imbalance of the system made of rotor and balancing heads is given by the **imbalance indicator**

$$G(\alpha_1, \gamma_1; \alpha_2, \gamma_2) := \|B_1(\alpha_1, \gamma_1) + F_1\|^2 + \|B_2(\alpha_2, \gamma_2) + F_2\|^2.$$ 

We assume the existence of $(\bar{\alpha}_1, \bar{\gamma}_1; \bar{\alpha}_2, \bar{\gamma}_2) \in \mathbb{T}^4$, such that $G(\bar{\alpha}_1, \bar{\gamma}_1; \bar{\alpha}_2, \bar{\gamma}_2) = 0$. 
Optimal control problem

Find the trajectory \( \Phi(t) = (\alpha_1(t), \gamma_1(t); \alpha_2(t), \gamma_2(t)) \) minimizing

\[
J(\Phi) := \frac{1}{2} \int_0^\infty \left[ \|\dot{\Phi}\|^2 + \beta G(\Phi) \right] dt,
\]

over the set of admissible trajectories

\[
\mathcal{A} := \left\{ \Phi \in \bigcap_{T > 0} H^1((0, T); \mathbb{T}^4) \mid \Phi(0) = \Phi_0, \right. \\
\left. \dot{\Phi} \in L^2(0, +\infty) \text{ and } G(\Phi) \in L^1(0, +\infty) \right\}.
\]

\( \beta > 0 \) is a weighting parameter.
Proposition (Gnuffi-P.-Sakamoto, 2019)

For $i = 1, 2$, set

$$c^i := \frac{1}{2m_i r_i \omega^2} (F_{i,x}, F_{i,y})$$

Then,
1. there exists $\Phi \in \mathcal{A}$ minimizer of $J$;
2. $\Phi = (\alpha_1, \gamma_1; \alpha_2, \gamma_2)$ is $C^\infty$ smooth and, for $i = 1, 2$, the following Euler-Lagrange equations are satisfied, for $t > 0$

$$
\begin{align*}
-\ddot{\alpha}_i &= \beta \cos (\gamma_i) \left[ -c_1^i \sin (\alpha_i) + c_2^i \cos (\alpha_i) \right] \\
-\ddot{\gamma}_i &= -\beta \sin (\gamma_i) \left[ c_1^i \cos (\alpha_i) + c_2^i \sin (\alpha_i) - \cos (\gamma_i) \right] \\
\alpha_i(0) &= \alpha_{0,i}, \quad \gamma_i(0) = \gamma_{0,i}, \quad \Phi(T) \xrightarrow{T \to +\infty} 0.
\end{align*}
$$
Proposition (Gnuffi-P.-Sakamoto, 2019)

Let $\Phi$ be an optimal trajectory. Then,

1. There exists $\Phi \in \text{zero}(G)$ such that
   \[
   \Phi(t) \xrightarrow{t \to +\infty} \Phi, \quad \dot{\Phi}(t) \xrightarrow{t \to +\infty} 0. 
   \]
   and
   \[
   |G(\Phi(t))| \xrightarrow{t \to +\infty} 0. 
   \]

2. If, in addition
   \[
   m_1 r_1 > \frac{\sqrt{F_{1,x}^2 + F_{1,y}^2}}{2\omega^2} \quad \text{and} \quad m_2 r_2 > \frac{\sqrt{F_{2,x}^2 + F_{2,y}^2}}{2\omega^2},
   \]
   we have the exponential estimate for any $t \geq 0$
   \[
   \|\Phi(t) - \Phi\| + \|\dot{\Phi}(t)\| + |G(\Phi(t))| \leq C \exp(-\mu t),
   \]
   with $C, \mu > 0$ independent of $t$. 
1. the proof of (1) is based on Łojasiewicz inequality;
2. the proof of (2) relies on the Stable Manifold Theorem applied to the Pontryagin Optimality System.

Sakamoto, Noboru and Pighin, Dario and Zuazua, Enrique
The turnpike propety in nonlinear optimal control - A geometric approach
arXiv:1903.09069
Gnuffi, Matteo and Pighin, Dario and Sakamoto, Noboru
Rotors imbalance suppression by optimal control
arXiv:1907.11697

The related computational code is available in the DyCon blog at the following link:
https://deustotech.github.io/DyCon-Blog/tutorial/wp02/P0005
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