Optimal Estimates on Front Propagation for the Thin-Film Equation and Other Fourth-Order Parabolic Equations

Optimale Abschätzungen zur Ausbreitung freier Ränder für die Dünne-Filme-Gleichung und andere parabolische Gleichungen vierter Ordnung

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Abstract

This thesis is concerned with the analysis of front propagation in nonnegativity-preserving fourth-order parabolic partial differential equations. Due to the lack of a comparison principle for higher-order parabolic equations, nonnegativity-preserving equations are rare within the class of all higher-order parabolic equations. At the same time important physical phenomena are described by such equations. The thin-film equation

$$u_t = -\operatorname{div}(f(u)\nabla\Delta u) \tag{1}$$

(with $f \in C^0(\mathbb{R}_0^+)$, $f(u) \geq 0$, $f(0) = 0$) describes the evolution of a thin viscous liquid film on a solid surface driven by surface tension. Different slip conditions on the fluid-solid interface correspond to different choices of f: the case of a no-slip condition corresponds to $f(u) = u^3$, while the case of the Navier slip condition corresponds to $f(u) = u^2 + u^3$. In order to avoid unnecessary technical complications, in the present thesis we shall consider mainly the case

$$u_t = -\operatorname{div}(u^n\nabla\Delta u) \ , \tag{2}$$

where $n \in \mathbb{R}^+$. Note that in order to prevent ill-posedness of the problem, one needs to prescribe an additional boundary condition on the free boundary $\partial \operatorname{supp} u(.,t)$. Typically one prescribes the contact-angle of the solution, which in most realistic situations depends on the interfacial energies only. In the present thesis, in the analysis of the thin-film equation we shall be concerned with the case of complete wetting only, i.e. the case of zero contact-angle at the free boundary.

The second important fourth-order nonnegativity-preserving parabolic equation, the so-called Derrida-Lebowitz-Speer-Spohn equation

$$u_t = -\operatorname{div}\left(u\nabla\frac{\Delta\sqrt{u}}{\sqrt{u}}\right) \ , \tag{3}$$

is used for the description of several quantum phenomena: First, it is the equation associated with the leading-order operator of the quantum drift-diffusion equation, a drift-diffusion equation for charge transport in semiconductors augmented by a term describing quantum corrections. Secondly, it is used to characterize interface fluctuations in the Toom model, a probabilistic cellular automaton describing the evolution of a spin lattice.

The first part of the thesis is dedicated to a long-standing open problem in the theory of the thin-film equation. Solutions to degenerate parabolic equations like the thin-film equation display the finite speed of support propagation property. Moreover, if certain conditions on the initial data are satisfied – in the case of the thin-film equation if the initial droplet is "flat enough" at its boundary – a waiting time phenomenon may occur: the free boundary (i.e. the boundary of the droplet) locally does not advance for some time before the support of the solution (i.e. the droplet) starts spreading. In the case

of the thin-film equation, Dal Passo, Giacomelli and Grün [52] have given sufficient criteria for the occurrence of a waiting time phenomenon in terms of the initial data; moreover Giacomelli and Grün [26] have derived lower bounds on waiting times. However, no lower bounds on support propagation for the thin-film equation have been known; in particular, no upper bounds on waiting times have been derived. In the case of second-order parabolic equations – for instance the porous medium equation –, upper bounds on waiting times and sufficient criteria for the nonexistence of a waiting time are typically obtained using comparison arguments or Harnack inequalities. These tools being unavailable for higher-order equations, the question of optimality of the conditions by Dal Passo, Giacomelli and Grün and optimality of the estimates by Giacomelli and Grün has since remained open.

In the first part of this thesis we devise methods for proving upper bounds on waiting times for strong solutions of the thin-film equation. For $n \in (2, \frac{32}{11})$ our upper bounds coincide (up to a constant factor) with the lower bounds by Giacomelli and Grün. Therefore our bounds are optimal. In the borderline case $n = 2$ (essentially the case of Navier slip conditions) we obtain upper bounds which coincide with the lower bounds up to a logarithmic correction term. Our results are based on new monotonicity formulas for solutions to the thin-film equation of the form

$$\frac{d}{dt} \int u^{1+\alpha} |x - x_0|^\gamma \, dx \geq c \int u^{1+\alpha+n} |x - x_0|^{\gamma-4} \, dx$$

with $\alpha \in (-1, 0]$ and $\gamma < 0$. These formulas are valid as long as the support of u does not touch the point x_0; combined with a differential inequality argument due to Chipot and Sideris [17], they yield the desired upper bounds on waiting times.

In the second part of the thesis we apply the methods developed in the first part to obtain lower bounds on asymptotic support propagation rates for the thin-film equation. We prove that for $n \in (1.5, \frac{32}{11})$ at time $t > 0$ the support of any strong solution to the thin-film equation with initial data u_0 contains a ball with radius $R(t) := c(d, n) ||u_0||_{L^1}^{\frac{n}{4+d \cdot n}} t^{\frac{1}{4+d \cdot n}} - \operatorname{diam}(\operatorname{supp} u_0)$; in particular, $R(t)$ scales like the diameter of the support of the self-similar solution as t tends to infinity. Our result shows that the upper bounds on asymptotic support propagation rates due to Bernis [4], Hulshof and Shishkov [39], Bertsch, Dal Passo, Garcke and Grün [12] and Grün [35] are optimal for any initial data. This result is another important contribution to the theory of the thin-film equation: for example, in case $d = 1$ the previously known results could not even exclude the possibility that for certain (nonvanishing) initial data u_0 with $\operatorname{supp} u_0 \subset \mathbb{R}_0^+$ the solution u would satisfy $\operatorname{supp} u(., t) \subset \mathbb{R}_0^+$ for all $t > 0$.

In the third part of this thesis, we show how to prove infinite speed of support propagation of solutions to the Derrida-Lebowitz-Speer-Spohn equation using an adaption of our method. This result is also an important new contribution to the theory of the DLSS equation; at the same time it shows that our method, which we have derived to analyse front propagation in case of the thin-film equation, also applies to other higher-order nonnegativity-preserving parabolic equations.

A paper comprising the derivation of the upper bounds on waiting times for the thin-film equation has been submitted for publication to Archive for Rational Mechanics and Analysis. A second paper consisting of the derivation of the optimal lower bounds on asymptotic support propagation rates for the thin-film equation has been submitted to Journal of Differential Equations, while a third paper containing the results on infinite speed of propagation for the DLSS equation has been submitted to Nonlinear Differential Equations and Applications.

Zusammenfassung

Thema dieser Dissertation ist die Analysis der Frontausbreitung in Lösungen nichtnegativitätserhaltender parabolischer partieller Differentialgleichungen vierter Ordnung. Da für parabolische Gleichungen höherer Ordnung im Allgemeinen kein Vergleichsprinzip gilt, stellen nichtnegativitätserhaltende Gleichungen in der Klasse aller parabolischen Gleichungen höherer Ordnung eine Ausnahmeerscheinung dar. Gleichzeitig werden jedoch wichtige physikalische Phänomene durch derartige Gleichungen beschrieben. Die Dünne-Filme-Gleichung

$$u_t = -\operatorname{div}(f(u)\nabla\Delta u)$$

(mit $f \in C^0(\mathbb{R}_0^+)$, $f(u) \geq 0$, $f(0) = 0$) beschreibt die durch Oberflächenspannung getriebene zeitliche Entwicklung der Dicke eines dünnen viskosen Flüssigkeitsfilms auf einer festen Oberfläche. Unterschiedliche Mobilitätsfunktionen f entsprechen unterschiedlichen Schlupfbedingungen an der Oberfläche zwischen Fluid und Festkörper: Die Funktion $f(u) = u^3$ entspricht beispielsweise einer Haftbedingung, hingegen entspricht der Fall $f(u) = u^2 + u^3$ der Navier-Schlupfbedingung. Um den technischen Aufwand nicht unnötig zu vergrößern, betrachten wir in dieser Dissertation zumeist den Fall

$$u_t = -\operatorname{div}(u^n\nabla\Delta u) \ ,$$

wobei $n \in \mathbb{R}^+$ ein reeller Parameter ist. Um Schlechtgestelltheit des Problems zu vermeiden, muss eine weitere Randbedingung am freien Rand $\partial \operatorname{supp} u(.,t)$ gestellt werden. Zumeist wird dieses Problem gelöst, indem am freien Rand der Kontaktwinkel der Lösung (bzw. äquivalent dazu $|\nabla u|$) vorgeschrieben wird. In der vorliegenden Arbeit beschränken wir uns bei der Analysis der Dünne-Filme-Gleichung auf den Fall der vollständigen Benetzung, d.h. den Fall von Lösungen mit verschwindendem Kontaktwinkel.

Die zweite wichtige nichtnegativitätserhaltende parabolische Gleichung vierter Ordnung, die sogenannte Derrida-Lebowitz-Speer-Spohn-Gleichung

$$u_t = -\operatorname{div}\left(u\nabla\frac{\Delta\sqrt{u}}{\sqrt{u}}\right) \ ,$$

wird zur Beschreibung mehrerer Quantenphänomene herangezogen: Zum Einen erhält man die DLSS-Gleichung aus der Quanten-Drift-Diffusions-Gleichung durch Vernachlässigung aller Terme niedrigerer Ordnung; die Quanten-Drift-Diffusions-Gleichung ist dabei eine Drift-Diffusions-Gleichung für den Ladungstransport in Halbleitern, die um Quantenkorrekturen niedrigster Ordnung erweitert wurde. Zum Anderen wird die DLSS-Gleichung zur Beschreibung von Grenzflächen-Fluktuationen im Toom-Modell verwendet; das Toom-Modell ist ein probabilistischer zellulärer Automat, der die Zeitentwicklung eines Spin-Gitters beschreibt.

Der erste Teil der Arbeit ist einem lange Zeit offenen Problem aus der Theorie der Dünne-Filme-Gleichung gewidmet. Lösungen degeneriert parabolischer

Gleichungen wie der Dünne-Filme-Gleichung weisen das Phänomen der endlichen Ausbreitungsgeschwindigkeit des Trägers der Lösung auf. Wenn zudem die Anfangsdaten gewisse Bedingungen erfüllen – genauer gesagt, wenn im Fall der Dünne-Filme-Gleichung der Fluidtropfen zu Beginn am Rand „flach genug“ ist – , tritt ein Wartezeitenphänomen auf: der freie Rand (d.h. der Rand des Fluidtropfens) bewegt sich zunächst eine gewisse Zeit lang nicht vorwärts, bevor der Träger der Lösung (also der Tropfen) beginnt, sich auszubreiten. Im Fall der Dünne-Filme-Gleichung haben Dal Passo, Giacomelli und Grün [52] hinreichende Bedingungen an die Anfangsdaten für das Auftreten eines Wartezeitenphänomens hergeleitet. Zudem haben Giacomelli und Grün [26] untere Schranken für die Wartezeit bewiesen. Jedoch sind bislang keinerlei untere Schranken für die Ausbreitung des Trägers von Lösungen der Dünne-Filme-Gleichung bekannt; insbesondere gibt es bislang keine oberen Schranken für die Wartezeit. Im Fall parabolischer Gleichungen zweiter Ordnung – wie beispielsweise der Poröse-Medien-Gleichung – werden obere Schranken für Wartezeiten und hinreichende Bedingungen für das Nichtauftreten von Wartezeitenphänomenen zumeist mit Hilfe von Vergleichsprinzipien oder Harnack-Ungleichungen hergeleitet. Da diese Werkzeuge im Fall von Gleichungen höherer Ordnungen nicht verfügbar sind, ist die Frage nach der Optimalität der Bedingungen von Dal Passo, Giacomelli und Grün und der Optimalität der Abschätzungen von Giacomelli und Grün bislang offen geblieben.

Im ersten Teil dieser Dissertation entwickeln wir Methoden, um obere Schranken für die Wartezeit starker Lösungen der Dünne-Filme-Gleichung zu beweisen. Für $n \in (2, \frac{32}{11})$ stimmen unsere oberen Schranken bis auf einen konstanten Faktor mit den unteren Schranken von Giacomelli und Grün überein. Unsere Schranken sind somit optimal. Im Grenzfall $n = 2$ (im Wesentlichen der Fall von Navier-Schlupfbedingungen) erhalten wir obere Schranken, die bis auf einen logarithmischen Korrekturterm mit den unteren Schranken übereinstimmen. Unsere Ergebnisse basieren auf neuen Monotonieformeln für Lösungen der Dünne-Filme-Gleichung von der Form

$$\frac{d}{dt}\int u^{1+\alpha}|x-x_0|^{\gamma}\,dx \geq c\int u^{1+\alpha+n}|x-x_0|^{\gamma-4}\,dx$$

mit $\alpha \in (-1, 0]$ und $\gamma < 0$. Diese Formeln gelten, solange der Träger von u den Punkt x_0 nicht berührt; in Kombination mit einem Differentialungleichungsargument von Chipot und Sideris [17] liefern sie die gewünschten oberen Schranken für die Wartezeit.

Im zweiten Teil der Arbeit wenden wir die Methoden, die im ersten Teil entwickelt wurden, an, um untere Schranken für die asymptotische Ausbreitungsrate des Trägers von Lösungen der Dünne-Filme-Gleichung zu erhalten. Wir zeigen, dass für $n \in (1.5, \frac{32}{11})$ der Träger jeder starken Lösung der Dünne-Filme-Gleichung mit Anfangsdaten u_0 zum Zeitpunkt $t > 0$ eine Kugel mit Radius $R(t) := c(d,n)||u_0||_{L^1}^{\frac{n}{4+d\cdot n}} t^{\frac{1}{4+d\cdot n}} - \operatorname{diam}(\operatorname{supp} u_0)$ enthält; insbesondere hat $R(t)$ im Limes $t \to \infty$ das gleiche asymptotische Verhalten wie der Durchmesser der selbstähnlichen Lösung. Unser Ergebnis zeigt, dass die oberen Schranken für die asymptotische Ausbreitungsrate des Trägers, die von Bernis [4], Hulshof und Shishkov [39], Bertsch, Dal Passo, Garcke und Grün [12] sowie Grün [35] bewiesen wurden, optimal für alle Anfangsdaten sind. Dieses Ergebnis stellt einen weiteren wichtigen Beitrag zur Theorie der Dünne-Filme-Gleichung dar: beispielsweise konnten die zuvor bekannten Resultate im Fall $d = 1$ nicht ausschließen, dass für gewisse nichtverschwindende Anfangsdaten u_0 mit $\operatorname{supp} u_0 \subset \mathbb{R}_0^+$ die Lösung u für alle $t > 0$ die Inklusion $\operatorname{supp} u(.,t) \subset \mathbb{R}_0^+$ erfüllen würde.

Im dritten Teil dieser Arbeit beweisen wir mit Hilfe einer nichttrivialen Adaption unserer Methode unendliche Ausbreitungsgeschwindigkeit für Lösungen der Derrida-Lebowitz-Speer-Spohn-Gleichung. Dieses Resultat ist ebenso ein wichtiger neuer Beitrag zur Theorie der DLSS-Gleichung; zugleich zeigt es, dass unsere Methode, die zur Analysis der Frontausbreitung im Fall der Dünne-Filme-Gleichung entwickelt wurde, auch auf andere nichtnegativitäterhaltende parabolische Gleichungen höherer Ordnung anwendbar ist.

Ein Manuskript, das die Herleitung der oberen Schranken für die Wartezeit von Lösungen der Dünne-Filme-Gleichung enthält, ist bei der Zeitschrift „Archive for Rational Mechanics and Analysis“ eingereicht worden. Ein zweites Manuskript, das aus der Herleitung der optimalen unteren Schranken für die asymptotische Ausbreitungsgeschwindigkeit des Trägers von Lösungen der Dünne-Filme-Gleichung besteht, wurde bei der Zeitschrift „Journal of Differential Equations“ eingereicht. Ein drittes Manuskript über die unendliche Ausbreitungsgeschwindigkeit von Lösungen der DLSS-Gleichung ist bei der Zeitschrift „Nonlinear Differential Equations and Applications“ eingereicht worden.

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1 Introduction

1.1 The thin-film equation

The thin-film equation has been the subject of intensive research during the last two decades. It describes the evolution of a thin liquid film governed by the force of surface tension for various slip conditions. The thin-film equation reads

$$u_t = -\nabla \cdot (u^n \nabla \Delta u)$$

where n is a positive real parameter and $\Omega \subset \mathbb{R}^d$. More general versions have also been considered, e.g. with u^n replaced by some nonnegative mobility function $f(u)$ (see e.g. [7], [30]). The thin-film equation is the most prominent example of a higher-order nonnegativity-preserving parabolic equation.

For $n = 3$, the thin-film equation can formally be derived using a long-wave approximation of the Navier-Stokes equations with no-slip boundary conditions on the fluid-solid interface; see e.g. [48]. For Navier slip conditions (i.e. the effective boundary condition for laminar viscous flow on a rough surface, see the paper by Jäger and Mikelic [40]), the mobility function $f(u)$ is given (after rescaling) by $u^2 + u^3$ (see e.g. [33]). For $n = 1$, the thin-film equation arises as the lubrication approximation of the Hele-Shaw flow; this has been proven rigorously by Giacomelli and Otto [29].

The thin-film equation may be regarded as a higher-order analogue of the porous medium equation $u_t = \nabla \cdot (u^{m-1} \nabla u)$. Like solutions of the porous medium equation, solutions of the thin-film equation feature the finite speed of support propagation property; see the papers by Bernis [4] [5], Hulshof and Shishkov [39], Bertsch, Dal Passo, Garcke, and Grün [12], Grün [35].

As in the case of the porous medium equation, depending on the initial data solutions of the thin-film equation may also exhibit a waiting time phenomenon: if a droplet initially is "flat enough" near its boundary, the contact line of the droplet does not advance for some time before the droplet starts spreading. The first rigorous proof for the occurrence of such a waiting time phenomenon is due to Dal Passo, Giacomelli, and Grün [52]; the method used in their paper has been refined by Giacomelli and Grün [26] to yield quantitative lower bounds on waiting times. A formal analysis of the expected waiting time behaviour of solutions to the thin-film equation has been carried out by Blowey, King and Langdon [14]; it indicates that at least for $n \in (2, 3)$, the results by Dal Passo, Giacomelli and Grün should be optimal.

However, up to now only lower bounds on waiting times and upper bounds on support propagation rates for the thin-film equation have been proven. Besides being of strong independent interest, the rigorous derivation of upper bounds on waiting times and lower bounds on support propagation rates is necessary to decide whether the previously known reverse bounds are optimal in any situation.

Moreover, a technique for proving upper bounds on waiting times may help in the analysis of the competition between convection and diffusion in the thin-film equation with convection due to gravity (see e.g. the paper by Giacomelli and Shishkov [31]). It may also help in the analysis of the influence of additional second-order diffusion terms on solutions of the thin-film equation; such terms are used e.g. to model van der Waals forces (see e.g. the papers by Bertozzi and Pugh [8, 9, 10]).

A variant of the thin-film equation containing both convection and second-order diffusion terms is the stochastic thin-film equation

$$du = -\left(u^3 u_{xxx} - u^3[\Phi'(u)]_x\right)_x \, dt + \sum_k (e_k u^{\frac{3}{2}})_x dW_t^k$$

as derived by Grün, Mecke, and Rauscher [37]; here, the e_k denote a set of mutually orthogonal functions and the W^k denote a sequence of independent Brownian motions. In this equation, the second-order diffusion term models van der Waals forces; however, the noisy convection term is of different origin, modeling thermal fluctuations. Grün, Mecke, and Rauscher have shown that the stochastic thin-film equation can explain the behaviour of microscopic thin films (i.e. in a regime where thermal noise has to be taken into account) in a quantitative way. It is an interesting open question whether the presence of noisy convection in the stochastic thin-film equation has influence on support propagation behaviour (or, perhaps, influence on the propagation of the apparent support $\{u > \delta\}$). Numerical evidence by Davidovitch, Moro, and Stone [19] suggests that this might indeed be the case. Note that existence of solutions for the stochastic thin-film equation is also an open problem.

For second-order equations like the porous medium equation, assertions on qualitative behaviour of solutions may be derived (directly or indirectly) using the comparison principle; see e.g. the paper by Choi and Kim [18] for results on waiting time phenomena in the case of the Hele-Shaw and the Stefan problem or the paper by Alikakos [1] for a criterion for the nonexistence of a waiting time in the case of the porous medium equation whose proof is based upon a result by Aronson and Caffarelli [2].

For fourth-order equations, no comparison principle is available and one has to rely on integral estimates. Integral estimates have been used successfully to obtain finite speed of propagation results, lower bounds on waiting times, as well as sufficient conditions for support shrinking for various degenerate parabolic PDEs; in particular, all results on qualitative behaviour of solutions of the thin-film equation mentioned above are based on such methods. However, the only results which use integral estimates to derive upper bounds on waiting times and lower bounds on support propagation are concerned with second-order equations; see the papers by Chipot and Sideris [17] and by Djie [22]. A direct application of the approach by Chipot and Sideris to the case of the thin-film equation fails due to the structure of the fourth-order operator: terms with negative sign and terms with indefinite sign appear which cannot be controlled. Chipot and Sideris proceed by deriving a differential inequality which forces either the support to spread or some quantity which is known to stay bounded to blow up. As we shall show below, this general strategy also applies to the case of the thin-film equation; however, instead of directly testing the equation with a cutoff (as done by Chipot and Sideris), we need to work with weighted backward entropy estimates and carefully choose a singular weight function which admits a small constant in Hardy's inequality in order to obtain our differential inequality. In the multidimensional case, we are faced with the additional problem of indefinite Hessians of the weight functions.

Before stating our results and describing our methods, we shall give an overview of the mathematical literature on the thin-film equation. This list is not exhaustive; we restrict ourselves to the case of the thin-film equation without lower-order terms and omit all results on self-similar or travelling-wave solutions.

Starting with the work of Bernis and Friedman [7] who constructed nonnegative solutions in the case of one spatial dimension, in the case of complete wetting (i.e. zero contact angle of the solution at the free boundary) a quite satisfactory theory of existence of weak solutions of the thin-film equation has been established. The solutions constructed can be shown to satisfy different entropy estimates, namely the zeroth-order entropy estimates

$$\begin{aligned}&\frac{1}{\alpha(\alpha+1)}\int_\Omega u_0^{\alpha+1}\,dx \\ \geq&\frac{1}{\alpha(\alpha+1)}\int_\Omega u^{\alpha+1}(.,T)\,dx+c\int_0^T\int_\Omega\left|\nabla u^{\frac{n+\alpha+1}{4}}\right|^4+\left|D^2u^{\frac{n+\alpha+1}{2}}\right|^2\,dx\,dt\end{aligned}\tag{4}$$

for $\alpha\in(\frac{1}{2}-n,2-n)\setminus\{-1,0\}$, which for $d=1$ have been derived by Beretta, Bertsch, and Dal Passo [3] (extending the work by Bernis and Friedman who discovered these estimates in the special case $d=1$, $\alpha=1-n$), and the first-order energy estimate

$$\begin{aligned}&\frac{1}{2}\int_\Omega|\nabla u_0|^2\,dx\\ \geq&\frac{1}{2}\int_\Omega|\nabla u(.,T)|^2\,dx+\int_0^T\int_\Omega u^n|\nabla\Delta u|^2\,dx\,dt\\ \geq&\frac{1}{2}\int_\Omega|\nabla u(.,T)|^2\,dx\\ &+c\int_0^T\int_\Omega\left|\nabla\Delta u^{\frac{n+2}{2}}\right|^2+u^{n-2}\left|\nabla u\right|^2\left|D^2u\right|^2+\left|\nabla u^{\frac{n+2}{6}}\right|^6\,dx\,dt\,.\end{aligned}$$

In the one-dimensional case, for $n\in(\frac{1}{2},3)$ the second inequality in the previous formula is a consequence of the famous inequalities by Bernis [6]; in the multidimensional case, for $n\in\left(2-\sqrt{\frac{8}{8+d}},3\right)$ and $d\leq 3$ it is a consequence of a generalization by Grün [36].

Localized versions of these entropy and energy inequalities are the base of most studies of the qualitative properties of the thin-film equation; besides the theorems on finite speed of propagation and waiting times which have been mentioned above, one can show that the support of solutions is nonshrinking for $n > 1.5$. For $d = 1$ and $n > 3.5$, solutions of the thin-film equation remain strictly positive for all $t > 0$ if the initial data is strictly positive.

In multiple space dimensions techniques different from the ones used by Bernis and Friedman are required to construct solutions of the thin-film equation. Elliot and Garcke [24] and Grün [34] independently developed a method to construct solutions for degenerate parabolic fourth-order PDEs. Using these ideas Dal Passo, Garcke and Grün [51] have shown existence of solutions of the thin-film equation in multiple space dimensions for $n \in (\frac{1}{8}, 3)$ and $d \leq 3$. However, the approximation procedure by Dal Passo et al. fails to yield the dissipation relation for the first-order energy for the solutions constructed by it in case $d > 1$; since for $n \in [2, 3)$ only "backward" entropy estimates are available, in this regime it is therefore not known whether the solutions constructed by Dal Passo, Garcke and Grün satisfy the finite speed of propagation property.

Dal Passo and Garcke have constructed solutions of the thin-film equation with weak initial trace [50]. More precisely, they allow for nonnegative Radon measures with finite total mass as initial data in case $n \in (\frac{1}{8}, 3)$; for $n \in [2, 3)$, the Radon measures are additionally required to have compact support. Their result also was the first result on solutions of the thin-film equation with initial data which is not compactly supported.

Finally, Grün [36] has constructed solutions satisfying the first-order energy dissipation relation for $n \in \left(2 - \sqrt{\frac{8}{8+d}}, 3\right)$ and $d \leq 3$, thus enabling him to prove finite speed of propagation for these solutions even in the parameter range $n \in [2, 3)$ and therefore allowing for the construction of solutions for the Cauchy problem in case $n \in [2, 3)$ and $d > 1$ for the first time.

Regarding large-time behaviour of solutions, in the case of the Cauchy problem there is only a single result available: in case $n = 1$ and $d = 1$ Carrillo and Toscani [16] have shown convergence of solutions to the self-similar solution as $t \to \infty$. For bounded domains Ω, there are various results proving exponential decay of solutions to the steady-state.

A newer existence result which is worth mentioning is the existence of smooth zero contact angle solutions which are a perturbation of the steady state x_+^2; in particular, these solutions have infinite mass. See the paper by Giacomelli, Knüpfer and Otto [28]. For results derived by related methods see the papers by Giacomelli and Knüpfer [27] and by Knüpfer [47].

In the case of partial wetting (i.e. nonzero contact angle of the solution at the free boundary, which we shall not consider in the present thesis) there are fewer results; however in case $d = 1$ Otto [49] (for $n = 1$) and Bertsch, Giacomelli, and Karali [11] (for general n) have proven existence of weak solutions.

Having given an overview of the existing literature, we now turn to a description of our results.

In this thesis, for the parameter range $n \in [2, \frac{32}{11})$ we provide an answer to a long-standing open question in the theory of the thin-film equation, namely the question of optimality of the lower bounds on waiting times derived in [26]. This is accomplished by deriving corresponding upper bounds on waiting times for solutions to the thin-film equation.

In the one-dimensional case, we show that in the regime $2 < n < \frac{32}{11}$ the lower bounds on waiting times for the thin-film equation derived by Grün and Giacomelli [26] are optimal up to a constant factor. In the borderline case $n = 2$, we derive bounds which are optimal up to some additional logarithmic term: a small gap arises between the lower bounds by Giacomelli and Grün and our upper bounds. For $n \in (1, 2)$, we still obtain some estimates on waiting times. However, our estimates deteriorate quickly as n decreases and presumably are no longer optimal. See Section 3.5 for details.

In the multidimensional case, we obtain similar assertions. For $n \in (2, \frac{32}{11})$, we show that for points on $\partial \operatorname{supp} u_0$ near which $\partial \operatorname{supp} u_0$ is a C^4 manifold, the lower bounds on waiting times are sharp up to some constant factor. For $n = 2$, we obtain immediate support spreading at any point on $\partial \operatorname{supp} u_0$ near which the initial data satisfies some growth condition and near which $\partial \operatorname{supp} u_0$ is a C^4 manifold.

To the best of our knowledge, these are the first upper bounds on waiting times for nonnegativity-preserving higher-order parabolic equations.

Combining our methods with entropy production estimates, our approach also yields optimal lower bounds on asymptotic support propagation rates for the thin-film equation. In particular, for $n \in (1.5, \frac{32}{11})$ we prove that in case $\operatorname{supp} u_0 \subset B_{R_0}(x_0)$ one has $B_{R(t)}(x_0) \subset \operatorname{supp} u(.,t)$, where $R(t) := c||u_0||_{L^1}^{\frac{n}{4+d\cdot n}} t^{\frac{1}{4+d\cdot n}} - R_0$. This is also a significant advance as compared to the previously known results: for example, in case $d = 1$ the previous results could not even exclude the possibility that for some nonvanishing initial data u_0 with $\operatorname{supp} u_0 \subset \mathbb{R}_0^+$ the solution u to the Cauchy problem would satisfy $\operatorname{supp} u(.,t) \subset \mathbb{R}_0^+$ for all $t > 0$.

We now provide a short description of our method. Our technique relies on weighted backward entropy estimates. In these estimates, positive and negative terms on the right-hand side are in competition. Choosing the singular weight carefully such that the constant in Hardy's inequality becomes small enough, we ensure that the positive terms dominate. This yields a differential inequality for the weighted entropy, forcing the weighted entropy to blow up after a certain time in case the support does not spread; thus we obtain a contradiction and therefore an upper bound on the waiting time of the solution. In multiple space dimensions, a naive attempt to generalize the ideas from the one-dimensional case fails (at least when trying to prove optimal upper bounds on waiting times). Instead we need to design the weight carefully in such a way that the derivatives of the weight in directions tangent to $\partial \operatorname{supp} u_0$ are much smaller than the derivatives of the weight in direction perpendicular to $\partial \operatorname{supp} u_0$.

In the proof of our optimal lower bounds on asymptotic support propagation rates for the thin film equation, we also first derive a weighted backward entropy estimate; however, to obtain optimal lower bounds on asymptotic support propagation rates a different choice of the (singular) weight function becomes necessary, at least for $d > 1$. An application of a variant of Hardy's inequality yields a differential inequality; using the known upper bounds on support propagation rates, we deduce that the support of the solution must reach the singularity of the weight after some time, as otherwise the differential inequality would force the weighted entropy to blow up. Finally, to ensure optimality of the lower bounds on asymptotic support propagation rates we need to combine our technique with some entropy production estimates.

1.2 The Derrida-Lebowitz-Speer-Spohn equation

The Derrida-Lebowitz-Speer-Spohn equation

$$u_t = -\nabla \cdot \left(u \nabla \frac{\Delta \sqrt{u}}{\sqrt{u}} \right) \tag{5}$$

is another example of a nonnegativity-preserving fourth-order parabolic equation. The DLSS equation is of mathematical interest due to three reasons: first, as shown by Derrida, Lebowitz, Speer, and Spohn [21] it arises in the study of interfaces in the Toom model, a stochastic cellular automaton describing the evolution of a spin lattice. Secondly, it is the equation associated with the leading order operator of quantum drift-diffusion models, i.e. drift-diffusion models for charge transport in semiconductor devices which include lowest-order quantum corrections; for a derivation of the quantum drift-diffusion equation, see the article by Degond, Gallego, Mehats, Ringhofer [20] and the references therein. Thirdly, it is the equation associated with the gradient flow of the Fisher information $\int |\nabla \sqrt{u}|^2 \, dx$ with respect to the Wasserstein distance as shown by Gianazza, Savare, and Toscani [32].

Regarding existence of weak nonnegative solutions, for $d \leq 3$ a satisfactory theory of existence has been developed independently by Jüngel and Matthes [43] (for periodic boundary conditions) and by Gianazza, Savare, and Toscani [32] (for variational boundary conditions); see also [13, 44, 46] for other existence results. Recently, uniqueness of weak solutions in some class of sufficiently regular solutions has been shown by the author [25]. Note that the entropies found by Jüngel and Matthes [41] imply that the solutions of the DLSS equation constructed by them [43] belong to this class of uniqueness.

To the best of our knowledge, almost all results on qualitative behaviour of weak solutions to the DLSS equation are concerned with large time behaviour or perturbations of a steady state (see e.g. Jüngel and Toscani [45], Caceres, Carrillo, and Toscani [15], Gualdani, Jüngel, and Toscani [38], Dolbeault, Gentil, and Jüngel [23]). Regarding positivity of solutions of the DLSS equation, only for $d = 1$ it is known that $\int |\log u_0| \, dx < \infty$ implies

$\int |\log u(.,t)|\,dx < \infty$ for all $t > 0$. However, to the best of our knowledge there are in particular no results on the short-time behaviour of the support in case $\operatorname{supp} u_0 \neq \Omega$.

In the present work, we prove that weak solutions to the DLSS equation with periodic boundary conditions display infinite speed of support propagation; more precisely, we prove that for any nontrivial solution $\operatorname{supp} u$ necessarily equals $\Omega \times [0, \infty)$.

Deriving lower bounds on the support of nonnegativity-preserving higher-order parabolic equations is a difficult task due to the lack of a comparison principle; to the best of the author's knowledge, no lower bounds on support propagation for higher-order nonnegativity-preserving parabolic equations are available, apart from the results developed in this thesis.

To prove infinite speed of propagation for solutions of the DLSS equation, we use a nontrivial adaption of our method developed for proving upper bounds on waiting times for the thin-film equation. In the case of the one-dimensional DLSS equation with periodic boundary conditions, we roughly insert $|x|^\gamma$ (where $\gamma < 0$) as a test function into the weak formulation of the equation and apply Hardy's inequality to deduce a differential inequality which shows that $\int u(.,t)|x|^\gamma\,dx$ grows exponentially with time. The growth constant of the exponential function is seen to behave like $|\gamma|^3$; thus passing to the limit $\gamma \to -\infty$ we obtain a contradiction. In the multidimensional case, we need the reformulation of the DLSS equation in terms of $\sqrt{u}$ which the author has proven to hold for weak solutions within the class of uniqueness in [25]. We then use $|x|^\gamma$ multiplied by a cutoff as a test function and apply Hardy's inequality to obtain a differential inequality; letting again $\gamma \to -\infty$, we conclude.

1.3 Notation

In this section, we introduce some notation which will be used throughout the thesis.

General Notation

$\mathbb{N}$	set of positive integers
$\mathbb{N}_0$	set of nonnegative integers
$\mathbb{Z}$	set of integers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^+$	set of positive real numbers
$\mathbb{R}^+_0$	set of nonnegative real numbers
$(.)_+$	$\max(.,0)$
$(.)_-$	$\min(.,0)$

I	interval $[0, \infty)$ (to be interpreted as a time interval)
d	number of spatial dimensions
Ω	domain in $\mathbb{R}^d$
$\partial\Omega$	boundary of the domain Ω
A^c	complement of the set A
$\overline{A}$	closure of the set A
$\mathrm{diam}(A)$	diameter of the set A
$U \subset\subset W$	$\overline{U}$ is a compact set with $\overline{U} \subset W$ and we have $\mathrm{dist}(U, \partial W) > 0$
$B_r(x_0)$	open ball with radius r and center x_0
X'	dual of Banach space X
$\langle a, b \rangle$	duality pairing of $b \in X$ with $a \in X'$
ρ_δ	standard smoothing kernel with respect to space, generally assumed to be symmetric
$\operatorname{supp} u$	support of the function or distribution u; denotes the essential support if u is an L^p function
$f\big\vert_a^b$	$f(b) - f(a)$
$f * g$	convolution of functions f and g; also used to denote the convolution of a distribution g with a smooth function f

Notation for Vectors and Matrices

Id	identity matrix
A^T	transpose of the matrix A
$\vec{e}_i$	i-th vector of the standard basis of $\mathbb{R}^d$
$\lvert v \rvert$	length of the vector v
$\lvert A \rvert$	norm $\sqrt{\sum_{i,j} \lvert A_{ij} \rvert^2}$ of the matrix A
ab	multiplication of scalar and scalar
$a \cdot b$	multiplication of scalar and scalar
av	multiplication of scalar and vector
$a \cdot v$	multiplication of scalar and vector
$v \cdot w$	scalar product of vectors v and w
$A \cdot v$	matrix-vector multiplication Av

$v \cdot A$	vector-matrix multiplication $v^T A$
$A : B$	trace of the product matrix AB^T
$v \otimes w$	tensor product of vectors v and w

Notation for Derivatives

All notations for derivatives may refer to both the classical derivative and the weak derivative.

∇u	(spatial) gradient of u
$D^2 u$	second (spatial) derivative of u (Hessian)
Δu	(spatial) Laplacian of u
$\Delta^2 u$	(spatial) Laplacian of Δu
$\partial_i u$	derivative of u with respect to spatial standard basis vector $\vec{e}_i$
$\partial_{\vec{v}} u$	derivative of u in direction $\vec{v}$
u_t	derivative of u with respect to time
u_x	spatial derivative of u (in case of one spatial dimension)
$\mathrm{div}(v)$	divergence of the vector field v

Notation for Function Spaces

$L^p(A)$	space of all Lebesgue measurable functions f on A with $\int_A \lvert f\rvert^p \, dx < \infty$
$W^{k,p}(\Omega)$	Sobolev space of all L^p functions on Ω whose weak derivatives up to order k belong to $L^p(\Omega)$
$H^k(\Omega)$	$W^{k,2}(\Omega)$
$L^p_{loc}(\Omega)$	space of all Lebesgue measurable functions f on Ω with $\int_U \lvert f\rvert^p \, dx < \infty$ for all $U \subset\subset \Omega$
$W^{k,p}_{loc}(\Omega)$	space of all functions f whose restrictions to U belong to $W^{k,p}(U)$ for all $U \subset\subset \Omega$
$H^k_{loc}(\Omega)$	$W^{k,2}_{loc}(\Omega)$
$W^{k,p}_{per}((0,1)^d)$	periodic Sobolev space $W^{k,p}$, i.e. closure in $W^{k,p}((0,1)^d)$ of the set of all smooth 1-periodic functions on $\mathbb{R}^d$
$H^k_{per}((0,1)^d)$	$W^{k,2}_{per}((0,1)^d)$

$L^p(I;X)$	space of all strongly measurable mappings $f : [0,\infty) \to X$ with $\int_{[0,\infty)} \|\|f(t)\|\|_X^p \; dt < \infty$, where X is a Banach space
$L^p_{loc}(I;X)$	space of all strongly measurable mappings $f : [0,\infty) \to X$ with $\int_{[0,T)} \|\|f(t)\|\|_X^p \; dt < \infty$ for all $T > 0$
$L^p(I;L^q_{loc}(\Omega))$	space of all strongly measurable mappings $f : [0,\infty) \to L^q_{loc}(\Omega)$ with $\int_{[0,\infty)} \|\|f(t)\|\|_{L^q(U)}^p \; dt < \infty$ for all $U \subset\subset \Omega$
$L^p_{loc}(I;L^q_{loc}(\Omega))$	space of all strongly measurable mappings $f : [0,\infty) \to L^q_{loc}(\Omega)$ with $\int_{[0,T)} \|\|f(t)\|\|_{L^q(U)}^p \; dt < \infty$ for all $T > 0$ and all $U \subset\subset \Omega$
$L^p(I;W^{k,q}_{loc}(\Omega))$	space of all strongly measurable mappings $f : [0,\infty) \to W^{k,q}_{loc}(\Omega)$ with $\int_{[0,\infty)} \|\|f(t)\|\|_{W^{k,q}(U)}^p \; dt < \infty$ for all $U \subset\subset \Omega$
$L^p_{loc}(I;W^{k,q}_{loc}(\Omega))$	space of all strongly measurable mappings $f : [0,\infty) \to W^{k,q}_{loc}(\Omega)$ with $\int_{[0,T)} \|\|f(t)\|\|_{W^{k,q}(U)}^p \; dt < \infty$ for all $T > 0$ and all $U \subset\subset \Omega$
$C^k(A)$	space of all k times continuously differentiable functions on A which are bounded and whose derivatives up to order k are bounded
$C^k_{loc}(A)$	space of all k times continuously differentiable functions on A
$C^k_c(A)$	space of all compactly supported k times continuously differentiable functions on A
$C^\infty_c(A)$	space of all smooth compactly supported functions on A
$C^0(I;X)$	space of all bounded continuous mappings $f : [0,\infty) \to X$
$C^0_{loc}(I;X)$	space of all continuous mappings $f : [0,\infty) \to X$
$W^{1,p}(I;X)$	space of all $f \in L^p(I;X)$ whose weak derivative (with respect to time) exists and belongs to $L^p(I;X)$
$H^1(I;X)$	$W^{1,2}(I;X)$
$W^{1,p}_{loc}(I;X)$	space of all $f \in L^p_{loc}(I;X)$ whose weak derivative (with respect to time) exists and belongs to $L^p_{loc}(I;X)$
$H^1_{loc}(I;X)$	$W^{1,2}_{loc}(I;X)$

Note that the space $C^\infty_c(\Omega \times I)$ also contains functions ϕ which do *not* vanish for $t = 0$, since $I = [0,\infty)$. This is in contrast to the space $C^\infty_c(\Omega \times (0,\infty))$: all functions in the latter space have zero trace on $t = 0$.

If no confusion may arise, we use $\int_0^T \int_\Omega f$ as a shorthand notation for the integral $\int_0^T \int_\Omega f \; dx \; dt$.

Throughout the thesis, if an integral of the form $\int_\Omega fg\ dx$ involves a function g which is only defined a.e. on $\{f \neq 0\}$, we use $\int_\Omega fg\ dx$ as a shorthand notation for $\int_{\Omega\cap\{f\neq 0\}} fg\ dx$. In particular, expressions like $\int_\Omega u^{-\beta}|\nabla u|^4\ dx$ (note that $\nabla u = 0$ a.e. on $\{u = 0\}$) or $\int_\Omega u^{2\beta}|D^2u|^2\ dx$ (if we have only $u^{1+\beta} \in H^2$ and thus D^2u is only defined on $\{u \neq 0\}$) are to be understood in this way. More generally, by convention we define $f \cdot g = 0$ on the set $\{f = 0\} \cap \{g \text{ undefined}\}$.

We also use the convention from measure theory $0 \cdot \infty = 0$. Moreover, we use the conventions $0^0 = 1$ and $\frac{1}{0} = \infty$ as well as $\frac{0}{0} = 0$.

2 Main results

2.1 Upper bounds on waiting times for the thin-film equation

We now recall the definition of strong solutions of the thin-film equation which we shall work with and state our main results concerning the waiting-time behaviour of solutions to the thin-film equation.

The strongest concept of solution for which global existence for fairly general initial data is known is characterized by the dissipation of both the first-order energy and the zeroth-order entropies. Existence of such strong energy solutions (the author decided to use this name in order to distinguish this notion of solution from the weaker notion of strong solution in [51]) of the thin-film equation has been shown by Bernis for the case of the one-dimensional Cauchy problem [5]. In case $d = 2$ or $d = 3$, proving existence of these solutions is much more demanding. In this case the proof has been carried out by Grün [36].

Assume $d \leq 3$. Let $\Omega \subset \mathbb{R}^d$ be a bounded convex domain, a bounded domain with boundary of class $C^{1,1}$, or let $\Omega = \mathbb{R}^d$. Let $u_0 \in H^1(\Omega)$ be nonnegative and have bounded support.

Definition 1. *Set $I := [0, \infty)$. We call $u \in L^\infty(I; H^1(\Omega) \cap L^1(\Omega))$, $u \geq 0$, a strong energy solution to the thin-film equation if the following conditions are satisfied:*

a) we have $\nabla u^{\frac{n+2}{6}} \in L^6(\Omega \times I)$, $\chi_{\{u>0\}} u^{\frac{n-2}{2}} \nabla u \otimes D^2 u \in L^2(\Omega \times I)$, $u^{\frac{n}{2}} \nabla \Delta u \in L^2(\Omega \times I)$

b) for some $\alpha \in (\max\{-1, \frac{1}{2} - n\}, 2 - n) \setminus \{0\}$, we have $D^2 u^{\frac{1+n+\alpha}{2}} \in L^2_{loc}(I; L^2(\Omega))$ and $\nabla u^{\frac{1+n+\alpha}{4}} \in L^4_{loc}(I; L^4(\Omega))$

c) it holds that $u \in H^1_{loc}\left(I; (W^{1,p}(\Omega))'\right)$ for all $p > \frac{4d}{2d+n(2-d)}$

d) for any $\xi \in L^2(I; W^{1,\infty}(\Omega))$ and any $T > 0$ it holds that

$$\int_0^T \langle u_t, \xi \rangle \, dt = \int_0^T \int_{\Omega \cap \{u>0\}} u^n \nabla \Delta u \cdot \nabla \xi \, dx \, dt$$

e) u attains its initial data u_0 in the sense that $\lim_{t \to \infty} u(.,t) = u_0(.)$ in $L^1(\Omega)$

Note that the solutions constructed by Grün [36] satisfy the α entropy estimate (4) for any $\alpha \in (\max\{-1, \frac{1}{2} - n\}, 2 - n) \setminus \{0\}$, not just for a particular α. Besides $d \leq 3$ and $n \in \left(2 - \sqrt{\frac{8}{8+d}}, 3\right)$, the existence result of [36] required either that Ω be bounded and convex with smooth boundary or that $\Omega = \mathbb{R}^d$.

Thus, for such Ω and any nonnegative initial data $u_0 \in H^1(\Omega)$ with bounded support there exists a strong energy solution u of the thin-film equation which satisfies all α entropy estimates for $\alpha \in (\max\{-1, \frac{1}{2} - n\}, 2 - n)$, $\alpha \neq 0$.

The notion of "waiting time" refers to the phenomenon that depending on the initial data, it may happen that the free boundary of a solution of a degenerate parabolic equation does not advance in the neighbourhood of some point $x_0 \in \partial \operatorname{supp} u_0$ for some time T^*. More precisely, we define:

Definition 2. *Let $u \in L^\infty(I; H^1(\Omega)) \cap H^1_{loc}(I; (W^{1,p}(\Omega))')$ be a solution of the thin-film equation and let $x_0 \in \partial \operatorname{supp} u_0$ be some point. We then define the waiting time T^* of u at x_0 as*

$$T^* := \lim_{\epsilon \to 0} \inf\{t > 0 : \operatorname{supp} u(.,t) \cap B_\epsilon(x_0) \not\subset \operatorname{supp} u_0 \cap B_\epsilon(x_0)\} .$$

Note that the regularity $u \in L^\infty(I; H^1(\Omega)) \cap H^1_{loc}(I; [W^{1,p}(\Omega)]')$ implies $u \in C^0_{loc}(I; L^2(V))$ for any bounded open set $V \subset \Omega$ with smooth boundary; see e.g. Corollary 4 in [53]. Thus the essential support $\operatorname{supp} u(.,t)$ is well-defined for all $t \geq 0$.

In the one-dimensional case, our main result reads as follows:

Theorem 3. *Let $d = 1$ and $x_0 \in \mathbb{R}$. Let u be a strong energy solution of the Cauchy problem for the thin-film equation with compactly supported nonnegative initial data $u_0 \in H^1(\mathbb{R}^d)$.*

a) Suppose $n \in \left(2, \frac{32}{11}\right)$. Assume $\operatorname{supp} u_0 \cap (-\infty, x_0) = \emptyset$. Then there exist constants $\alpha \in (-1, -\frac{1}{2})$ with $\alpha + n < 2$ and $C > 0$ which depend only on n such that the following holds: If u satisfies the α entropy estimate, then $T := \inf\{t > 0 : (-\infty, x_0) \cap \operatorname{supp} u(.,t) \neq \emptyset\}$ is bounded by

$$T \leq C(n) \inf_{\epsilon > 0} \epsilon^{4 - \frac{n}{1+\alpha}} \left[\int_{\mathbb{R}} u_0^{1+\alpha} |x - x_0 + \epsilon|^{-2} \, dx \right]^{-\frac{n}{1+\alpha}} .$$

b) Suppose $n \in \left(2, \frac{32}{11}\right)$. Assume $\operatorname{supp} u_0 \cap (x_0 - \delta, x_0) = \emptyset$ for some $\delta > 0$ and $x_0 \in \partial \operatorname{supp} u_0$. Then there exist constants $\alpha \in (-1, -\frac{1}{2})$ and $C > 0$ which depend only on n and satisfy $\alpha + n < 2$ such that the following holds: If u satisfies the α entropy estimate, then the waiting time T^* of u at x_0 is bounded by

$$T^* \leq C(n) \left[\limsup_{\epsilon \to 0} \fint_{(x_0, x_0+\epsilon)} \left[\frac{1}{\epsilon^{\frac{4}{n}}} u_0 \right]^{1+\alpha} dx \right]^{-\frac{n}{1+\alpha}} .$$

c) Assume that $n = 2$. Suppose furthermore that u satisfies the α entropy estimate for $\alpha = -\frac{1}{2}$. Assume that $x_0 \in \partial \operatorname{supp} u_0$ and that $(x_0 - \delta, x_0) \cap$

$\operatorname{supp} u_0 = \emptyset$ *holds for some* $\delta > 0$*. Then the waiting time* T^* *of* u *at* x_0 *is bounded by*

$$T^* \le C \left[\limsup_{\epsilon \to 0} \fint_{(x_0, x_0+\epsilon)} \left[\frac{1}{\epsilon^2 |\log \epsilon|^{\frac{5}{2}}} u_0 \right]^{\frac{1}{2}} dx \right]^{-4} .$$

Assertion (b) gives an upper bound on the waiting time at some point $x_0 \in \partial \operatorname{supp} u_0$ in terms of the growth of u_0 near x_0; in contrast, assertion (a) also applies to points x_0 away from $\operatorname{supp} u_0$, giving an upper bound on the time at which $\operatorname{supp} u$ spreads beyond x_0.

As a corollary, one obtains easily:

Corollary 4. *Suppose* $d = 1$*. Let* u *be a strong energy solution of the Cauchy problem for the thin-film equation with compactly supported nonnegative initial data* $u_0 \in H^1(\mathbb{R})$*. Let a point* $x_0 \in \partial \operatorname{supp} u_0$ *be given such that* $\operatorname{supp} u_0 \cap (x_0 - \delta, x_0) = \emptyset$ *for some* $\delta > 0$*.*

a) Let $n \in \left(2, \frac{32}{11}\right)$*. Then there exists* $\alpha \in (-1, -\frac{1}{2})$ *depending only on* n *with* $n + \alpha < 2$ *such that the following assertion holds if* u *satisfies the* α *entropy estimate: If*

$$u_0(x) \ge S \cdot (x - x_0)_+^{\frac{4}{n}}$$

is satisfied for all $x \in B_\epsilon(x_0)$ *for some* $\epsilon > 0$ *and some* $S > 0$*, then the waiting time* T^* *of* u *at* x_0 *is bounded from above by*

$$T^* \le C(n) S^{-n} .$$

b) Let $n = 2$*. Assume that* u *satisfies the* α *entropy estimate for* $\alpha = -\frac{1}{2}$*. Then the following assertion holds: If*

$$u_0(x) \ge S \cdot |\log |x - x_0||^{\frac{5}{2}} (x - x_0)_+^2$$

is satisfied for all $x \in B_\epsilon(x_0)$ *for some* $\epsilon > 0$ *and some* $S > 0$*, then the waiting time* T^* *of* u *at* x_0 *is bounded from above by*

$$T^* \le C S^{-2} .$$

Of course, analogous assertions hold in the mirrored case $x_0 \in \partial \operatorname{supp} u_0$, $\operatorname{supp} u_0 \cap (x_0, x_0 + \delta) = \emptyset$.

In the case of several spatial dimensions, we obtain the following result:

Theorem 5. *Let* u *be a strong energy solution of the thin-film equation on some domain* $\Omega \subset \mathbb{R}^d$*,* $d \le 3$*, with nonnegative initial data* $u_0 \in H^1(\Omega)$*. Assume that* $\operatorname{supp} u_0$ *is bounded. Let* M *denote the closure of some domain* *with boundary of class* C^4. *Suppose that* $\operatorname{supp} u_0 \subset M$. *Let* $x_0 \in \partial M \cap \partial \operatorname{supp} u_0 \cap \Omega$ *be some point.*

Define H *to be the tangent plane of* ∂M *at* x_0 *and let* P *denote the orthogonal projection onto* H. *Abbreviate* $\operatorname{dist}_C(x, x_0) := \max(|Px - x_0|, \operatorname{dist}(x, H))$.

a) *Let* $n \in \left(2, \frac{32}{11}\right)$. *Then there exists a constant* $\alpha \in (-1, -\frac{1}{2})$ *depending only on* n *with* $n + \alpha < 2$ *such that the following holds: Provided that* u *satisfies the* α *entropy estimate and provided that we have*

$$W := \limsup_{r\to 0} \limsup_{h\to 0} ⨍_{\{x:\operatorname{dist}_C(x,x_0)<r,\operatorname{dist}(x,\partial M)<h\}} \left[\frac{1}{h^{\frac{4}{n}}} u_0\right]^{1+\alpha} dx > 0 \ ,$$

the waiting time T^* *of* u *at* x_0 *is bounded from above by*

$$T^* \leq C(d, n) W^{-\frac{n}{1+\alpha}} \ .$$

b) *Let* $n = 2$. *Suppose that* u *satisfies the* α *entropy estimate for* $\alpha = -\frac{1}{2}$. *If*

$$\limsup_{h\to 0} ⨍_{\{x:\operatorname{dist}_C(x,x_0)<\frac{1}{|\log h|},\operatorname{dist}(x,\partial M)<h\}} \left[\frac{1}{|\log h|^{14+2d} h^2} u_0\right]^{\frac{1}{2}} dx > 0 \ ,$$

then u *has no waiting time at* x_0.

The following corollary follows easily:

Corollary 6. *Let* u *be a strong energy solution of the thin-film equation on a domain* $\Omega \subset \mathbb{R}^d$, $d \leq 3$, *with nonnegative initial data* $u_0 \in H^1(\Omega)$. *Assume that* $\operatorname{supp} u_0$ *is bounded. Let* $x_0 \in \partial \operatorname{supp} u_0 \cap \Omega$ *be some point with the property that in some neighbourhood of* x_0 *the set* $\partial \operatorname{supp} u_0$ *is a manifold of class* C^4.

a) *Suppose* $n \in \left(2, \frac{32}{11}\right)$. *Then there exists a constant* $\alpha \in (-1, -\frac{1}{2})$ *depending only on* n *with* $n + \alpha < 2$ *such that the following holds: Provided that* u *satisfies the* α *entropy estimate and provided that there exist constants* $R > 0$, $S > 0$ *such that for any* $x \in B_R(x_0) \cap \operatorname{supp} u_0$ *we have*

$$u_0(x) \geq S \cdot \operatorname{dist}(x, \partial \operatorname{supp} u_0)^{\frac{4}{n}} \ ,$$

the waiting time T^* *of* u *at* x_0 *is bounded from above by*

$$T^* \leq C(d, n) S^{-n} \ .$$

b) Let $n=2$. Suppose that u satisfies the α entropy estimate for $\alpha=-\frac{1}{2}$. Provided that there exist constants $R>0$, $S>0$ such that for any $x\in B_R(x_0)\cap\operatorname{supp}u_0$ we have

$$u_0(x)\geq S\cdot|\log\operatorname{dist}(x,\partial\operatorname{supp}u_0)|^{14+2d}\cdot\operatorname{dist}(x,\partial\operatorname{supp}u_0)^2\ ,$$

u has no waiting time at x_0.

Remark 7. *Note that for $n=2$, the growth condition on u_0 known to be sufficient for the nonexistence of a waiting time is a bit stricter in the multidimensional case than in the one-dimensional case. This is likely due to a limitation of our technique. The author is not sure whether the one-dimensional result represents the optimal growth condition either. However, the condition is of course optimal up to some logarithmic factor.*

The proof for the multidimensional case also applies to the one-dimensional situation, thereby providing upper bounds on waiting times for solutions of the thin-film equation on domains $\Omega\neq\mathbb{R}$. However, as the proof is much more technical, we prefer to give the proof in the case of the one-dimensional Cauchy problem separately.

In the regime of strong slippage $n\in(1,2)$, we obtain:

Theorem 8. *Let $d=1$, $n\in(1,2)$ and let $u_0\in L^1(\mathbb{R})$ be nonnegative and compactly supported. Let u be a solution of the Cauchy problem for the thin-film equation with weak initial trace u_0 constructed as in [50] as the limit of a certain approximating sequence u_δ. Suppose additionally that this approximating sequence satisfies $u_\delta(.,0)\rightarrow u_0$ strongly in $L^1(\mathbb{R})$. Let $x_0\in\partial\operatorname{supp}u_0$ be the point with $(-\infty,x_0)\cap\operatorname{supp}u_0=\emptyset$.*

Then there exist constants $\alpha\in(-\frac{1}{2},0]$ and $\beta\in(0,2)$ depending only on n such that the following assertion holds: if there exists $\tau>0$ with

$$\limsup_{\epsilon\rightarrow 0}\fint_{(x_0,x_0+\epsilon)}\left[u_0(x)\epsilon^{-\beta+\tau}\right]^{1+\alpha}\,dx>0\ ,$$

then u has no waiting time at x_0, i.e. $\inf\{t>0:\operatorname{supp}u(.,t)\cap(-\infty,x_0)\neq\emptyset\}=0$.

As n approaches 2, the constant α tends to $-\frac{1}{2}$ and the constant β tends to 2. For n approaching 1, both α and β tend to 0.

See Chapter 6 for a discussion of our results.

2.2 Optimal lower bounds on asymptotic support propagation rates for the thin-film equation

In this section, we shall state our results on lower bounds on asymptotic support propagation rates for the thin-film equation. Existence of strong energy solutions as constructed in [36] (see Definition 1) is only known for $n \in \left(2-\sqrt{\frac{8}{8+d}}, 3\right)$; however, our results on asymptotic support propagation apply to $n \in (1, \frac{32}{11})$. We therefore also need to work with the notion of strong solutions to the thin-film equation as introduced by Bertsch, Dal Passo, Garcke and Grün [12], since existence of such strong solutions is guaranteed for $n \in (\frac{1}{8}, 2)$.

The definition of strong solutions to the thin-film equation from [12] reads as follows:

Definition 9. *Let $u_0 \in H^1(\mathbb{R}^d)$, $1 \leq d \leq 3$, be nonnegative and compactly supported and let $n \in (\frac{1}{8}, 2)$. A nonnegative function $u \in L^\infty(I; H^1(\mathbb{R}^d))$ is called a strong solution of the Cauchy problem for the thin-film equation if the following conditions are satisfied:*

a) $u \in H^1_{loc}\left(I; \left[W^{1,p}(\mathbb{R}^d)\right]'\right)$ *for all* $p > \frac{4d}{2d+n(2-d)}$

b) for some $\alpha \in (\max\{-1, \frac{1}{2}-n\}, 2-n) \setminus \{0\}$, *we have* $D^2 u^{\frac{1+n+\alpha}{2}} \in L^2_{loc}(I; L^2(\mathbb{R}^d))$ *and* $\nabla u^{\frac{1+n+\alpha}{4}} \in L^4_{loc}(I; L^4(\mathbb{R}^d))$

c) for any $\xi \in C^\infty_c(\mathbb{R}^d \times I)$ *we have*

$$\begin{aligned}
&\int_0^T \langle u_t, \xi \rangle \; dt \\
&= \int_0^T \int_{\{u>0\}} u^n \nabla u \cdot \nabla \Delta \xi \; dx \; dt + n \int_0^T \int_{\{u>0\}} u^{n-1} \nabla u \cdot D^2 \xi \cdot \nabla u \; dx \; dt \\
&\quad + \frac{n}{2} \int_0^T \int_{\{u>0\}} u^{n-1} |\nabla u|^2 \Delta \xi \; dx \; dt \\
&\quad + \frac{n(n-1)}{2} \int_0^T \int_{\{u>0\}} u^{n-2} |\nabla u|^2 \nabla u \cdot \nabla \xi \; dx \; dt
\end{aligned}$$

for all $T > 0$.

d) u attains the initial data in the sense that $u(.,t) \to u_0$ in $L^1(\mathbb{R}^d)$ as $t \to 0$

Note that the solutions constructed by Bertsch, Dal Passo, Garcke and Grün [12] satisfy the α entropy estimate (4) for any $\alpha \in (\max\{-1, \frac{1}{2}-n\}, 2-n) \setminus \{0\}$, not just for a particular α; thus for any nonnegative initial data $u_0 \in H^1(\mathbb{R}^d)$ with compact support, in case $n \in (\frac{1}{8}, 2)$ there exists a strong solution satisfying all α entropy estimates for $\alpha \in (\max\{-1, \frac{1}{2}-n\}, 2-n)$, $\alpha \neq 0$.

The following upper bounds on asymptotic support propagation have been obtained by Bernis [4] for $n \in (0, 2)$ and $d = 1$, by Hulshof and Shishkov [39] for $n \in [2, 3)$ and $d = 1$, by Bertsch, Dal Passo, Garcke and Grün [12] for $d \in \{2, 3\}$ and $n \in (\frac{1}{8}, 2)$, and by Grün [35] for $n \in \left(2-\sqrt{\frac{8}{8+d}}, 3\right)$ and $d \in \{2, 3\}$:

Theorem 10. *Let $u_0 \in H^1(\mathbb{R}^d)$, $1 \le d \le 3$, be nonnegative and compactly supported. Let u be a strong solution of the Cauchy problem for the thin-film equation obtained by the procedure in [12] and $n \in (\frac{1}{8}, 2)$ or let u be a strong energy solution of the Cauchy problem for the thin-film equation and $n \in \left(2 - \sqrt{\frac{8}{8+d}}, 3\right)$. Assume that $\operatorname{supp} u_0 \subset B_{R_0}(x_0)$ for some $R_0 > 0$ and some $x_0 \in \mathbb{R}^d$. Then for any $t > 0$ we have the estimate $\operatorname{supp} u(.,t) \subset B_{R(t)}(x_0)$ with*

$$R(t) := R_0 + C(n,d)||u_0||_{L^1(\mathbb{R}^d)}^{\frac{n}{4+d\cdot n}} t^{\frac{1}{4+d\cdot n}} .$$

Our main result provides more or less the reverse estimate. It reads:

Theorem 11. *Let $u_0 \in H^1(\mathbb{R}^d)$ be nonnegative and compactly supported. Assume $d \le 3$ and $n \in (1, \frac{32}{11})$. Let*

- *$n \in \left(2 - \sqrt{\frac{8}{8+d}}, \frac{32}{11}\right)$ and let u be a strong energy solution of the Cauchy problem for the thin-film equation satisfying all α entropy estimates for $\alpha > -1$, or*
- *let $n \in (1,2)$ and let u be a strong solution of the Cauchy problem constructed as in [12].*

Let $x_0 \in \mathbb{R}^d$ be a point. Set

$$T^* := \inf \left\{ T \ge 0 : \inf_{0 \le t \le T} \operatorname{dist}(x_0, \operatorname{supp} u(.,t)) = 0 \right\}.$$

Then there exists a constant $C(d,n)$ depending only on d and n such that the following estimate holds:

$$T^* \le C(d,n) \left[\operatorname{dist}(x_0, \operatorname{supp} u_0) + \operatorname{diam}(\operatorname{supp} u_0)\right]^{4+d\cdot n} ||u_0||_{L^1(\mathbb{R}^d)}^{-n}$$

Recall that the regularity $u \in L^\infty(I; H^1(\mathbb{R}^d)) \cap H^1_{loc}(I; [W^{1,p}(\mathbb{R}^d)]')$ implies $u \in C^0_{loc}(I; L^2(V))$ for any bounded open set $V \subset \mathbb{R}^d$ with smooth boundary (see e.g. Corollary 4 in [53]). Thus the essential support $\operatorname{supp} u(.,t)$ is well-defined for all $t \ge 0$.

Note that in the special case $n = 1$ and $d = 1$, a similar estimate could be inferred from the much stronger assertion by Carrillo and Toscani [16] who prove asymptotic decay of the solution to the self-similar solution in this case. However, to the best of our knowledge up to now no generalization of their result to $d > 1$ or $n \neq 1$ is available; moreover, for $d > 1$ such a convergence result would not imply our theorem.

For $n > 1.5$ the support of solutions to the thin-film equation is nondecreasing with respect to time, i.e. we have $\operatorname{supp} u(.,t_1) \subset \operatorname{supp} u(.,t_2)$ for all $t_1, t_2 \in I$ with $t_1 \le t_2$. This has been proven for strong solutions constructed by the usual approximation procedure in case $d = 1$ in [3]; for strong solutions constructed as in [12] and $d \leq 3$ it follows by the considerations in [12] (though only the weaker assertion $\operatorname{supp} u_0 \subset \operatorname{supp} u(.,t)$ for $t > 0$ is explicitly stated in this paper). For strong energy solutions constructed as in [36] and $d \leq 3$, the estimate $\operatorname{supp} u(.,t_1) \subset \operatorname{supp} u(.,t_2)$ for $0 \leq t_1 \leq t_2$ is a consequence of the entropy estimates in [36] (though it is not stated explicitly in this paper). Given a solution with nondecreasing support, our previous theorem implies:

Corollary 12. *Let $u_0 \in H^1(\mathbb{R}^d)$ be nonnegative and compactly supported. Assume $1 \leq d \leq 3$ and $1.5 < n < \frac{32}{11}$. Let*

- *$n \in \left(1.5, \frac{32}{11}\right)$ and let u be a strong energy solution of the Cauchy problem for the thin-film equation satisfying all α entropy estimates for $\alpha > -1$, or*
- *let $n \in (1.5, 2)$ and let u be a strong solution of the Cauchy problem constructed as in [12].*

Suppose that $\operatorname{supp} u(.,t_1) \subset \operatorname{supp} u(.,t_2)$ holds for all $0 \leq t_1 \leq t_2$.

Let $x_s \in \operatorname{supp} u_0$ be some point. Then there exists a constant $c(d,n)$ depending only on n and d such that for any $t > 0$ with $R(t) > 0$ we have

$$B_{R(t)}(x_s) \subset \operatorname{supp} u(.,t) \ ,$$

where

$$R(t) := c(d,n) ||u_0||_{L^1(\mathbb{R}^d)}^{\frac{n}{4+d\cdot n}} t^{\frac{1}{4+d\cdot n}} - \operatorname{diam}(\operatorname{supp} u_0) \ .$$

2.3 Infinite speed of propagation of solutions to the Derrida-Lebowitz-Speer-Spohn equation

We now recall the definition of weak solutions to the DLSS equation and state our results concerning infinite speed of propagation for the DLSS equation.

Jüngel and Matthes [43] have introduced the following definition of weak solutions of the DLSS equation with periodic boundary conditions:

Definition 13. *Let $1 \leq d \leq 3$ and $\Omega = (0,1)^d$. Let $u_0 \in L^1(\Omega)$ be given with $u_0 \geq 0$. We call a nonnegative function $u \in L^\infty(I; L^1(\Omega))$ with $u \in W^{1,1}_{loc}(I; [H^2_{per}(\Omega)]')$ and $\sqrt{u} \in L^2_{loc}(I; H^2_{per}(\Omega))$ a weak solution to the DLSS equation with initial data u_0 and periodic boundary conditions if for all $\psi \in L^\infty(I; H^2_{per}(\Omega))$ and all $T > 0$ we have*

$$\int_0^T \langle \partial_t u, \psi \rangle \, dt + \int_0^T \int_\Omega (\sqrt{u} D^2 \sqrt{u} - \nabla \sqrt{u} \otimes \nabla \sqrt{u}) : D^2 \psi \, dx \, dt = 0 \quad (6)$$

and if in addition $u(.,0) = u_0(.)$ in $[H^2_{per}(\Omega)]'$.

The weak formulation of the DLSS equation (6) is formally derived from (5) using the the usual rules for differentiation and integration by parts.

Existence of weak solutions for nonnegative measurable initial data u_0 with $u_0 \log u_0 \in L^1(\Omega)$ has been shown in [43].

Our first result deals with the one-dimensional case. It reads as follows:

Theorem 14. *Let u be a weak solution of the DLSS equation on $\Omega = (0,1)$ with periodic boundary conditions. Suppose that $||u_0||_{L^1(\Omega)} > 0$. We then have* $\operatorname{supp} u = \Omega \times [0,\infty)$.

In the case of several spatial dimensions, we need to impose an additional regularity assumption on the solution, namely $u^{\frac{1}{4}} \in L^2_{loc}(I; H^2_{per}(\Omega))$. However, this regularity is implied by the entropy estimates for the DLSS equation and thus the assumption is satisfied for the solutions constructed by Jüngel and Matthes [43]. Moreover, note that this additional regularity is precisely the additional regularity required of a solution in order to belong to the class of uniqueness; see [25].

Our result in the multidimensional case reads:

Theorem 15. *Let $d = 2$ or $d = 3$ and $\Omega = (0,1)^d$. Let u be a weak solution to the DLSS equation with periodic boundary conditions; suppose that u has the additional regularity $u^{\frac{1}{4}} \in L^2_{loc}(I; H^2_{per}(\Omega))$. Assume that $||u_0||_{L^1(\Omega)} > 0$. Then we have* $\operatorname{supp} u = \Omega \times [0,\infty)$.

3 Proof of the upper bounds on waiting times for the thin-film equation

3.1 Derivation of entropy estimates with explicit constants

Our technique strongly relies on "backward" entropy estimates, i.e. entropy estimates for $-1 < \alpha < 0$. However, for our approach to work we need explicit constants in the entropy estimates which cannot be found in the literature. Therefore we now derive these estimates, starting with the weak formulation of the thin-film equation and using the additional regularity inferred from entropy estimates.

This section is devoted to the proof of the following two lemmas:

Lemma 16. *Let $\Omega \subset \mathbb{R}^d$ be a domain and $d \leq 3$. Let u be a strong energy solution of the thin-film equation which satisfies the α entropy estimate. Suppose that $\operatorname{supp} u_0$ is bounded. Assume $-1 < \alpha < 0$ and $1 < n < 3$. Defining $b := n + \alpha$, the formula*

$$\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\
&= \left(b - \frac{1}{2}n\right) \int_{t_1}^{t_2} \int_\Omega u^{b-1}|\nabla u|^2 \Delta\psi + b \int_{t_1}^{t_2} \int_\Omega u^{b-1} \nabla u \cdot D^2\psi \cdot \nabla u \\
&\quad - \frac{1}{b+1} \int_{t_1}^{t_2} \int_\Omega u^{b+1} \Delta^2 \psi + (n-b) \int_{t_1}^{t_2} \int_\Omega u^{b-1} |D^2 u|^2 \psi \\
&\quad + \left(b - \frac{n}{2}\right)(b-1)(2-b) \int_{t_1}^{t_2} \int_\Omega u^{b-3} |\nabla u|^4 \psi \\
&\quad + (2n - 3b)(b-1) \int_{t_1}^{t_2} \int_\Omega u^{b-2} \nabla u \cdot D^2 u \cdot \nabla u\ \psi \\
&\quad + \left(\frac{n}{2} - b\right)(b-1) \int_{t_1}^{t_2} \int_\Omega u^{b-2} |\nabla u|^2 \Delta u\ \psi
\end{aligned} \tag{7}$$

holds for any $\psi \in C_c^\infty(\Omega)$ and a.e. $t_2 > t_1 > 0$ as well as a.e. $t_2 > 0$ in case $t_1 = 0$.

Lemma 17. *Let $\Omega \subset \mathbb{R}^d$ be a domain and $d \leq 3$. Let $1 \leq b \leq 2$. For any nonnegative u with $u^{\frac{b+1}{2}} \in H^2(\Omega)$ and $u^{\frac{b+1}{4}} \in W^{1,4}(\Omega)$, we have*

$$\begin{aligned}
&\int_\Omega u^{b-1}|D^2 u|^2 \psi + (b-1) \int_\Omega u^{b-2} \nabla u \cdot D^2 u \cdot \nabla u\ \psi + \int_\Omega u^{b-1} \nabla u \cdot D^2\psi \cdot \nabla u \\
&= \int_\Omega u^{b-1} |\Delta u|^2 \psi + (b-1) \int_\Omega u^{b-2} |\nabla u|^2 \Delta u\ \psi + \int_\Omega u^{b-1} |\nabla u|^2 \Delta\psi
\end{aligned}$$

for any $\psi \in C_c^\infty(\Omega)$.

The latter lemma allows replacing $\int_\Omega u^{b-1}|D^2u|^2\psi$ by $\int_\Omega u^{b-1}|\Delta u|^2\psi$, which will become important in the multidimensional case.

For the reader's convenience, we first give a formal derivation of the formula in Lemma 16. Formally inserting $\phi := u^\alpha\psi$ with $\psi \in C_c^\infty(\Omega)$ as a test function in the thin-film equation, we obtain by repeated formal integrations by parts

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}\psi \, dx\bigg|_{t_1}^{t_2} \\
=&\int_{t_1}^{t_2}\int_\Omega u^b \nabla\Delta u \cdot \nabla\psi + \alpha \int_{t_1}^{t_2}\int_\Omega u^{b-1}\nabla\Delta u\cdot\nabla u\ \psi \\
=&-b\int_{t_1}^{t_2}\int_\Omega u^{b-1}\nabla u\cdot D^2u\cdot\nabla\psi \\
&-\int_{t_1}^{t_2}\int_\Omega u^b D^2u : D^2\psi \\
&+(n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|D^2u|^2\psi \\
&+(n-b)(b-1)\int_{t_1}^{t_2}\int_\Omega u^{b-2}\nabla u\cdot D^2u\cdot\nabla u\ \psi \\
&+(n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}\nabla u\cdot D^2u\cdot\nabla\psi \\
=&\frac{1}{2}b(b-1)\int_{t_1}^{t_2}\int_\Omega u^{b-2}|\nabla u|^2\nabla u\cdot\nabla\psi \\
&+\frac{1}{2}b\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\nabla u|^2\Delta\psi \\
&+b\int_{t_1}^{t_2}\int_\Omega u^{b-1}\nabla u\cdot D^2\psi\cdot\nabla u \\
&+\int_{t_1}^{t_2}\int_\Omega u^b\nabla u\cdot\nabla\Delta\psi \\
&+(n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|D^2u|^2\psi \\
&+(n-b)(b-1)\int_{t_1}^{t_2}\int_\Omega u^{b-2}\nabla u\cdot D^2u\cdot\nabla u\ \psi \\
&-\frac{1}{2}(n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\nabla u|^2\Delta\psi \\
&-\frac{1}{2}(n-b)(b-1)\int_{t_1}^{t_2}\int_\Omega u^{b-2}|\nabla u|^2\nabla u\cdot\nabla\psi
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}\psi \; dx\bigg|_{t_1}^{t_2} \\
&= \left(\frac{1}{2}(n-b) - \frac{1}{2}b\right)(b-1)(b-2)\int_{t_1}^{t_2}\int_\Omega u^{b-3}|\nabla u|^4\psi \\
&\quad + \left(-\frac{1}{2}b + \frac{1}{2}(n-b)\right)(b-1)\bigg(2\int_{t_1}^{t_2}\int_\Omega u^{b-2}\nabla u\cdot D^2u\cdot\nabla u\;\psi \\
&\qquad\qquad + \int_{t_1}^{t_2}\int_\Omega u^{b-2}|\nabla u|^2\Delta u\;\psi\bigg) \\
&\quad + (n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|D^2u|^2\psi \\
&\quad + \left(b-\frac{1}{2}n\right)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\nabla u|^2\Delta\psi \\
&\quad + b\int_{t_1}^{t_2}\int_\Omega u^{b-1}\nabla u\cdot D^2\psi\cdot\nabla u \\
&\quad - \frac{1}{b+1}\int_{t_1}^{t_2}\int_\Omega u^{b+1}\Delta^2\psi \\
&\quad + (n-b)(b-1)\int_{t_1}^{t_2}\int_\Omega u^{b-2}\nabla u\cdot D^2u\cdot\nabla u\;\psi \\
&= \left(b-\frac{1}{2}n\right)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\nabla u|^2\Delta\psi \\
&\quad + b\int_{t_1}^{t_2}\int_\Omega u^{b-1}\nabla u\cdot D^2\psi\cdot\nabla u \\
&\quad - \frac{1}{b+1}\int_{t_1}^{t_2}\int_\Omega u^{b+1}\Delta^2\psi \\
&\quad + (n-b)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|D^2u|^2\psi \\
&\quad + \left(b-\frac{n}{2}\right)(b-1)(2-b)\int_{t_1}^{t_2}\int_\Omega u^{b-3}|\nabla u|^4\psi \\
&\quad + (2n-3b)(b-1)\int_{t_1}^{t_2}\int_\Omega u^{b-2}\nabla u\cdot D^2u\cdot\nabla u\;\psi \\
&\quad + \left(\frac{n}{2}-b\right)(b-1)\int_{t_1}^{t_2}\int_\Omega u^{b-2}|\nabla u|^2\Delta u\;\psi\;.
\end{aligned}
$$

We now make this calculation rigorous for strong energy solutions. The remainder of the proof of Lemma 16 is purely technical and may be skipped on first reading.

Proof of Lemma 16. By conservation of mass and the finite speed of support propagation property (see e.g. [35]), the α entropy estimate implies that $u^{\frac{1+n+\alpha}{4}} \in L^4_{loc}(I;W^{1,4}(\Omega))$ and $u^{\frac{1+n+\alpha}{2}} \in L^2_{loc}(I;H^2(\Omega))$.

Define $\Omega_\delta := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$ (in case $\Omega = \mathbb{R}^d$ this choice implies $\Omega_\delta = \Omega$). Denoting a standard mollifier with respect to space by ρ_δ, we notice that $(\rho_\delta * u) \in H^1_{loc}(I; C^2(\Omega_\delta))$: for any $\xi \in C_c^\infty(\Omega_\delta \times (0,\infty))$ we have

$$\begin{aligned}
&\int_0^\infty \int_{\Omega_\delta} (\rho_\delta * u(.,t))(x) \frac{d}{dt}\xi(x,t)\ dx\ dt \\
&= \int_0^\infty \int_{\Omega_\delta} \int_\Omega \rho_\delta(x-y) u(y,t) \frac{d}{dt}\xi(x,t)\ dy\ dx\ dt \\
&= \int_0^\infty \int_\Omega u(x,t) \left(\rho_\delta * \frac{d}{dt}\xi(.,t)\right)(x)\ dx\ dt \\
&= \int_0^\infty \int_\Omega u(x,t) \frac{d}{dt}(\rho_\delta * \xi)(x,t)\ dx\ dt \\
&= -\int_0^\infty \left\langle \frac{d}{dt}u(x,t), \rho_\delta * \xi \right\rangle\ dt \\
&= -\int_0^\infty \left\langle \rho_\delta * \frac{d}{dt}u(x,t), \xi \right\rangle\ dt\ ,
\end{aligned}$$

where we have used the symmetry of ρ_δ twice and where we have extended ξ to $\mathbb{R}^d \times (0,\infty)$ by zero. This shows that the weak derivative of $\rho_\delta * u$ with respect to time exists and belongs to $L^2_{loc}(I; C^2(\Omega_\delta))$ (since we have $u \in H^1_{loc}(I; (W^{1,p}(\Omega))')$ and since the mollification of a distribution is smooth); moreover, for all $t_2 > t_1 \geq 0$ we have the representation $\int_{t_1}^{t_2} \langle (\rho_\delta * u)_t, \xi\rangle\ dt = \int_{t_1}^{t_2} \langle u_t, \rho_\delta * \xi\rangle\ dt$ which holds for any $\xi \in C_c^\infty(\Omega_\delta \times (0,\infty))$ and for any $\xi \in L^2_{loc}(I; L^2(\Omega))$ with $\operatorname{dist}(\bigcup_{t\geq 0} \operatorname{supp} \xi(.,t), \partial\Omega) > \delta$ by approximation.

Thus for $\delta > 0$ small enough (depending on ψ; recall that $\operatorname{supp}\psi \subset\subset \Omega$) and $\epsilon > 0$, the function $\rho_\delta * [(\rho_\delta * u + \epsilon)^\alpha \psi]$ is an admissible test function in the weak formulation of the thin-film equation (see Definition 1). Taking $\psi \in C_c^\infty(\Omega)$, we can therefore compute for $\delta > 0$ small enough

$$\begin{aligned}
&\frac{1}{1+\alpha} \int_\Omega (\rho_\delta * u + \epsilon)^{1+\alpha} \psi\ dx \Big|_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2} \langle (\rho_\delta * u)_t, (\rho_\delta * u + \epsilon)^\alpha \psi \rangle\ dt \\
&= \int_{t_1}^{t_2} \langle u_t, \rho_\delta * [(\rho_\delta * u + \epsilon)^\alpha \psi] \rangle\ dt \\
&= \int_{t_1}^{t_2} \int_{\Omega \cap \{u>0\}} u^n \nabla\Delta u \cdot \nabla \left(\rho_\delta * [(\rho_\delta * u + \epsilon)^\alpha \psi]\right)\ dx\ dt\ . \qquad (8)
\end{aligned}$$

We now pass to the limit $\delta \to 0$. Convergence of the left-hand side for a.e. t_1, t_2 and a.e. t_2 in case $t_1 = 0$ is immediate.

Recall that by our definition of strong energy solutions (Definition 1) we have $\nabla u^{\frac{n+2}{6}} \in L^6(I; L^6(\Omega))$ and $u^{\frac{n}{2}} \nabla\Delta u \in L^2(I; L^2(\Omega))$. By the Sobolev embedding and conservation of mass, we have $u^{\frac{n+2}{6}} \in L^6_{loc}(I; L^6(K))$ for any compact set $K \subset\subset \Omega$. Note that therefore $\nabla u = \frac{6}{n+2} u^{\frac{4-n}{6}} \nabla u^{\frac{n+2}{6}} \in L^{n+2}_{loc}(I; L^{n+2}(K))$ since $\frac{n+2}{6} + (4-n)\cdot\frac{1}{6} = 1$. Moreover, $u^{\frac{n}{2}} = \left(u^{\frac{n+2}{6}}\right)^{\frac{3n}{n+2}} \in L^{\frac{2(n+2)}{n}}_{loc}(I; L^{\frac{2(n+2)}{n}}(K))$.

We calculate $\nabla(\rho_\delta * u + \epsilon)^\alpha = \alpha(\rho_\delta * u + \epsilon)^{\alpha-1}\cdot(\rho_\delta * \nabla u)$ and notice that $(\rho_\delta * u + \epsilon)^{-1+\alpha} \leq \epsilon^{-1+\alpha}$ since $\alpha \leq 0$. Putting these results together and rewriting the term on the right-hand side of (8) as

$$\int_{t_1}^{t_2}\int_{\Omega\cap\{u>0\}} u^{\frac{n}{2}}\cdot u^{\frac{n}{2}}\nabla\Delta u \cdot \left(\rho_\delta * \left[\alpha(\rho_\delta * u+\epsilon)^{-1+\alpha}\cdot\psi\cdot(\rho_\delta * \nabla u)\right]\right)\,dx\,dt$$
$$+\int_{t_1}^{t_2}\int_{\Omega\cap\{u>0\}} u^{\frac{n}{2}}\cdot u^{\frac{n}{2}}\nabla\Delta u \cdot \left(\rho_\delta * \left[(\rho_\delta * u+\epsilon)^{\alpha}\cdot\nabla\psi\right]\right)\,dx\,dt\ ,$$

we obtain convergence of the right-hand side of (8) since $\frac{n}{2(n+2)} + \frac{1}{2} + \frac{1}{n+2} = 1$, since the convergence

$$\begin{aligned}
&\left|\left|(\rho_\delta * [(\rho_\delta * u+\epsilon)^{-1+\alpha}\cdot(\rho_\delta*\nabla u)]) - (u+\epsilon)^{-1+\alpha}\nabla u\right|\right|_{L^{n+2}([0,T];L^{n+2}(K))}\\
\leq &\left|\left|\rho_\delta * \left[(\rho_\delta * u+\epsilon)^{-1+\alpha}\cdot(\rho_\delta*\nabla u) - (u+\epsilon)^{-1+\alpha}\nabla u\right]\right|\right|_{L^{n+2}([0,T];L^{n+2}(K))}\\
&+\left|\left|\rho_\delta * \left[(u+\epsilon)^{-1+\alpha}\nabla u\right] - (u+\epsilon)^{-1+\alpha}\nabla u\right|\right|_{L^{n+2}([0,T];L^{n+2}(K))}\\
\leq &\left|\left|(\rho_\delta * u+\epsilon)^{-1+\alpha}\cdot(\rho_\delta*\nabla u) - (u+\epsilon)^{-1+\alpha}\nabla u\right|\right|_{L^{n+2}([0,T];L^{n+2}(K'))}\\
&+\left|\left|\rho_\delta * \left[(u+\epsilon)^{-1+\alpha}\nabla u\right] - (u+\epsilon)^{-1+\alpha}\nabla u\right|\right|_{L^{n+2}([0,T];L^{n+2}(K))}\\
\underset{\delta\to 0}{\to}&\, 0
\end{aligned}$$

holds for every $T>0$ and every $K \subset\subset \Omega$ (here in the second step we have used the fact that mollification does not increase the L^p norms; $K' \subset\subset \Omega$ denotes a domain with $K \subset\subset K'$; both inequalities are only valid for small enough $\delta > 0$), and since the convergence

$$\begin{aligned}
&||\rho_\delta * [(\rho_\delta * u+\epsilon)^{\alpha}\cdot\nabla\psi] - (u+\epsilon)^{\alpha}\cdot\nabla\psi||_{L^{n+2}([0,T];L^{n+2}(K))}\\
\leq &||\rho_\delta * [(\rho_\delta * u+\epsilon)^{\alpha}\cdot\nabla\psi - (u+\epsilon)^{\alpha}\cdot\nabla\psi]||_{L^{n+2}([0,T];L^{n+2}(K))}\\
&+||\rho_\delta * [(u+\epsilon)^{\alpha}\cdot\nabla\psi] - (u+\epsilon)^{\alpha}\cdot\nabla\psi||_{L^{n+2}([0,T];L^{n+2}(K))}\\
\leq &||(\rho_\delta * u+\epsilon)^{\alpha}\cdot\nabla\psi - (u+\epsilon)^{\alpha}\cdot\nabla\psi||_{L^{n+2}([0,T];L^{n+2}(K'))}\\
&+||\rho_\delta * [(u+\epsilon)^{\alpha}\cdot\nabla\psi] - (u+\epsilon)^{\alpha}\cdot\nabla\psi||_{L^{n+2}([0,T];L^{n+2}(K))}\\
\underset{\delta\to 0}{\to}&\, 0
\end{aligned}$$

holds for every $T>0$ and every $K \subset\subset \Omega$ (again using the fact that mollification does not increase L^p norms; $K' \subset\subset \Omega$ again denotes a domain with $K \subset\subset K'$, and again both inequalities only hold for small enough $\delta > 0$).

Therefore the formula

$$\frac{1}{1+\alpha}\int_\Omega (u+\epsilon)^{1+\alpha}\,dx\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\Omega\cap\{u>0\}} u^n\nabla\Delta u\cdot\nabla\left[(u+\epsilon)^\alpha\psi\right]\,dx\,dt \quad (9)$$

is valid for a.e. $t_1, t_2 \in I$ with $t_1 < t_2$ and a.e. $t_2 \in I$ in case $t_1 = 0$.

By expressions like $[(u+\epsilon)^\alpha]'$ we denote the derivative with respect to u of the function in brackets evaluated at u, i.e. in the this case $\alpha(u+\epsilon)^{\alpha-1}$. Given an arbitrary smooth strictly positive function u and a smooth compactly supported function $\psi \in C_c^\infty(\Omega)$, we compute

$$
\begin{aligned}
&\int_\Omega u^n \nabla\Delta u \cdot \nabla\left((u+\epsilon)^\alpha \psi\right)\, dx \\
=&\int_\Omega (u+\epsilon)^\alpha u^n \nabla\Delta u \cdot \nabla\psi\, dx + \alpha \int_\Omega (u+\epsilon)^{\alpha-1} u^n \nabla\Delta u \cdot \nabla u\, \psi\, dx \\
=&-\int_\Omega [(u+\epsilon)^\alpha u^n]' \nabla u \cdot D^2 u \cdot \nabla\psi\, dx - \int_\Omega (u+\epsilon)^\alpha u^n D^2 u : D^2\psi\, dx \\
&- \alpha \int_\Omega (u+\epsilon)^{\alpha-1} u^n |D^2 u|^2 \psi\, dx - \alpha \int_\Omega [(u+\epsilon)^{\alpha-1} u^n]' \nabla u \cdot D^2 u \cdot \nabla u\, \psi\, dx \\
&- \alpha \int_\Omega (u+\epsilon)^{\alpha-1} u^n \nabla u \cdot D^2 u \cdot \nabla\psi\, dx \\
=&\frac{1}{2}\int_\Omega [(u+\epsilon)^\alpha u^n]'' |\nabla u|^2 \nabla u \cdot \nabla\psi\, dx + \frac{1}{2}\int_\Omega [(u+\epsilon)^\alpha u^n]' |\nabla u|^2 \Delta\psi\, dx \\
&+ \int_\Omega [(u+\epsilon)^\alpha u^n]' \nabla u \cdot D^2\psi \cdot \nabla u\, dx + \int_\Omega (u+\epsilon)^\alpha u^n \nabla u \cdot \nabla\Delta\psi\, dx \\
&- \alpha \int_\Omega (u+\epsilon)^{\alpha-1} u^n |D^2 u|^2 \psi\, dx - \alpha \int_\Omega [(u+\epsilon)^{\alpha-1} u^n]' \nabla u \cdot D^2 u \cdot \nabla u\, \psi\, dx \\
&+ \frac{\alpha}{2}\int_\Omega (u+\epsilon)^{\alpha-1} u^n |\nabla u|^2 \Delta\psi\, dx + \frac{\alpha}{2}\int_\Omega [(u+\epsilon)^{\alpha-1} u^n]' |\nabla u|^2 \nabla u \cdot \nabla\psi\, dx \\
=&-\frac{1}{2}\int_\Omega \left[[(u+\epsilon)^\alpha u^n]''' + \alpha[(u+\epsilon)^{\alpha-1} u^n]''\right] |\nabla u|^4 \psi\, dx \\
&- \frac{1}{2}\int_\Omega \left[[(u+\epsilon)^\alpha u^n]'' + \alpha[(u+\epsilon)^{\alpha-1} u^n]'\right] |\nabla u|^2 \Delta u\, \psi\, dx \\
&- \int_\Omega \left[[(u+\epsilon)^\alpha u^n]'' + 2\alpha[(u+\epsilon)^{\alpha-1} u^n]'\right] \nabla u \cdot D^2 u \cdot \nabla u\, \psi\, dx \\
&- \alpha \int_\Omega (u+\epsilon)^{\alpha-1} u^n |D^2 u|^2 \psi\, dx \\
&+ \frac{1}{2}\int_\Omega \left[[(u+\epsilon)^\alpha u^n]' + \alpha(u+\epsilon)^{\alpha-1} u^n\right] |\nabla u|^2 \Delta\psi\, dx \\
&+ \int_\Omega [(u+\epsilon)^\alpha u^n]' \nabla u \cdot D^2\psi \cdot \nabla u\, dx - \int_\Omega \int_0^u (v+\epsilon)^\alpha v^n\, dv\, \Delta^2\psi\, dx\ .
\end{aligned}
$$

Considering $\rho_\delta * u$ and passing to the limit $\delta \to 0$, one can prove that for any

$u \in H^3_{loc}(\Omega)$ with $\inf_\Omega u > 0$ and any $\psi \in C_c^\infty(\Omega)$ we have

$$
\begin{aligned}
&\int_\Omega u^n \nabla\Delta u \cdot \nabla\left((u+\epsilon)^\alpha \psi\right)\, dx \\
=&-\frac{1}{2}\int_\Omega \left[[(u+\epsilon)^\alpha u^n]''' + \alpha[(u+\epsilon)^{\alpha-1}u^n]''\right] |\nabla u|^4 \psi\, dx \\
&-\frac{1}{2}\int_\Omega \left[[(u+\epsilon)^\alpha u^n]'' + \alpha[(u+\epsilon)^{\alpha-1}u^n]'\right] |\nabla u|^2 \Delta u\ \psi\, dx \\
&-\int_\Omega \left[[(u+\epsilon)^\alpha u^n]'' + 2\alpha[(u+\epsilon)^{\alpha-1}u^n]'\right] \nabla u \cdot D^2 u \cdot \nabla u\ \psi\, dx \qquad (10) \\
&-\alpha\int_\Omega (u+\epsilon)^{\alpha-1}u^n |D^2 u|^2 \psi\, dx \\
&+\frac{1}{2}\int_\Omega \left[[(u+\epsilon)^\alpha u^n]' + \alpha(u+\epsilon)^{\alpha-1}u^n\right] |\nabla u|^2 \Delta\psi\, dx \\
&+\int_\Omega [(u+\epsilon)^\alpha u^n]' \nabla u \cdot D^2\psi \cdot \nabla u\, dx - \int_\Omega \int_0^u (v+\epsilon)^\alpha v^n\, dv\ \Delta^2\psi\, dx \\
=:&\ I + II + III + IV + V + VI + VII\ .
\end{aligned}
$$

Suppose now that $u \in L^1(\Omega)$ satisfies $u \geq 0$ and $\nabla u^{\frac{n+2}{6}} \in L^6(\Omega)$ and $u^{\frac{n}{2}}\nabla\Delta u \in L^2(\Omega)$ as well as $u^{\frac{n-2}{2}}\nabla u \otimes D^2 u \in L^2(\Omega)$; moreover, assume that $\nabla u^{\frac{1+n+\alpha}{4}} \in L^4(\Omega)$, $D^2 u^{\frac{1+n+\alpha}{2}} \in L^2(\Omega)$.

In this case, due to $d \leq 3$ and the Sobolev embedding we see that $u^{\frac{n+2}{6}}$ (and therefore also u) is continuous, so the sets $A_\delta := \{x \in \Omega : u(x) > \frac{1}{2}\delta\}$ are open and we have $\nabla\Delta u \in L^2(A_\delta \cap K)$, $\nabla u \otimes D^2 u \in L^2(A_\delta \cap K)$, $\nabla u \in L^6(A_\delta \cap K)$ for any $\delta > 0$ and any domain $K \subset\subset \Omega$. Take a smooth monotonous function ν with $0 \leq \nu \leq 1$, $\nu \equiv 0$ for $x < 0$ and $\nu \equiv 1$ for $x > 1$. Let

$$
f_\delta(v) := \int_0^v \nu\left(\frac{s-\delta}{\delta}\right)\, ds + \delta\ .
$$

Using the fact that $f_\delta(u(.)) \equiv \delta$ in some neighbourhood of $\Omega \setminus A_\delta$, we infer that $\nabla\Delta f_\delta(u) = 0$ in some neighbourhood of $\Omega \setminus A_\delta$. Using the regularity $\nabla\Delta u \in L^2(A_\delta \cap K)$ for any $K \subset\subset \Omega$, by elliptic regularity we obtain $u \in H^3_{loc}(A_\delta)$ and therefore

$$
\begin{aligned}
&\nabla\Delta f_\delta(u) \\
=&f_\delta'(u)\nabla\Delta u + f_\delta''(u)(\nabla u \Delta u + 2D^2 u \cdot \nabla u) + f_\delta'''(u)|\nabla u|^2\nabla u
\end{aligned}
$$

in A_δ. Thus, recalling that all derivatives of f_δ are bounded and taking into account that $\nabla u \otimes D^2 u \in L^2(A_\delta \cap K)$ and $\nabla u \in L^6(A_\delta \cap K)$, we see that $\nabla\Delta f_\delta(u) \in L^2(K)$ for any domain $K \subset\subset \Omega$ and therefore $f_\delta(u) \in H^3_{loc}(\Omega)$. As $f_\delta(u) \in H^3_{loc}(\Omega)$ and $f_\delta(u) \geq \delta$, formula (10) applies to $f_\delta(u)$. We now pass to the limit $\delta \to 0$.

It is easy to check that $(f_\delta(u)-u)\to 0$ in $L^\infty(\Omega)$: we have $|f_\delta(v)-v|\le \delta+\int_0^v \left|\nu\left(\frac{s-\delta}{\delta}\right)-1\right|\,ds \le 3\delta$. Moreover, we obtain by dominated convergence

$$\nabla f_\delta(u)=f_\delta'(u)\nabla u=\frac{6}{n+2}f_\delta'(u)u^{\frac{4-n}{6}}\nabla u^{\frac{n+2}{6}}\to \frac{6}{n+2}u^{\frac{4-n}{6}}\nabla u^{\frac{n+2}{6}}=\nabla u$$

strongly in $L^{n+2}(K)$ for any $K\subset\subset\Omega$ as $f_\delta'(u)$ is bounded uniformly and converges pointwise to 1 a.e. on $\{\nabla u\neq 0\}$. Since $n\ge 1$, this establishes convergence of the terms V, VI, and VII.

Convergence of term I is shown similarly: denoting expressions like the derivative of $(v+\epsilon)^{\alpha-1}v^n$ evaluated at $f_\delta(u)$ by $[(f_\delta(u)+\epsilon)^{\alpha-1}f_\delta(u)^n]'$, we see that

$$I=-\frac{1}{2}\int_\Omega \frac{[[(f_\delta(u)+\epsilon)^\alpha f_\delta(u)^n]'''+\alpha[(f_\delta(u)+\epsilon)^{\alpha-1}f_\delta(u)^n]'']}{f_\delta(u)^{n+\alpha-3}} \cdot f_\delta(u)^{n+\alpha-3}|\nabla f_\delta(u)|^4\psi\,dx\ . \tag{11}$$

Note that an estimate of the form

$$\frac{|[(v+\epsilon)^\alpha v^n]'''+\alpha[(v+\epsilon)^{\alpha-1}v^n]''|}{v^{\alpha+n-3}}\le C(\alpha,n) \tag{12}$$

can be shown to hold: recalling that $\alpha\le 0$, by the Leibniz formula we have

$$\begin{aligned}&\left|[(v+\epsilon)^\alpha v^n]'''+\alpha[(v+\epsilon)^{\alpha-1}v^n]''\right|\\ =&\left|\sum_{i=0}^{3}c_i(\alpha,n)(v+\epsilon)^{\alpha-i}v^{n+i-3}\right|\le C(\alpha,n)v^{\alpha+n-3}\ .\end{aligned}$$

By dominated convergence we get

$$\begin{aligned}&[f_\delta(u)]^{\frac{n+\alpha-3}{4}}\nabla f_\delta(u)=[f_\delta(u)]^{\frac{n+\alpha-3}{4}}f_\delta'(u)\frac{4}{n+1+\alpha}u^{\frac{3-n-\alpha}{4}}\nabla u^{\frac{n+1+\alpha}{4}}\\ &\to \frac{4}{n+1+\alpha}\nabla u^{\frac{n+1+\alpha}{4}}=u^{\frac{n+\alpha-3}{4}}\nabla u\end{aligned} \tag{13}$$

strongly in $L^4(\Omega)$ (note that $\nabla u^{\frac{n+1+\alpha}{4}}=0$ a.e. on $\{u=0\}$); to obtain the dominating function we have made use of the estimate $\frac{u}{f_\delta(u)}\le 2$ which holds since $f_\delta(v)\ge \delta+(v-2\delta)_+$ and of the fact that $\alpha\le 0$, $n<3$. Combining this convergence property with formula (11) and estimate (12) as well as pointwise convergence a.e. of $f_\delta(u)$, we deduce convergence of term I (note that $\nabla f_\delta(u)=0=\nabla u$ a.e. on $\{u=0\}$).

We now turn our attention to the terms II, III and IV which involve second derivatives. We calculate

$$[f_\delta(u)]^{\frac{\alpha+n-1}{2}}D^2 f_\delta(u)=[f_\delta(u)]^{\frac{\alpha+n-1}{2}}f_\delta'(u)D^2u+[f_\delta(u)]^{\frac{\alpha+n-1}{2}}f_\delta''(u)\nabla u\otimes\nabla u\ . \tag{14}$$

The second term on the right-hand side equals

$$\frac{16}{(n+1+\alpha)^2} f_\delta''(u)\cdot f_\delta(u)^{\frac{\alpha+n-1}{2}}\cdot u^{\frac{6-2n-2\alpha}{4}}\cdot \nabla u^{\frac{n+1+\alpha}{4}}\otimes \nabla u^{\frac{n+1+\alpha}{4}}$$

By the definition of $f_\delta(v)$, it holds that $f_\delta''(v)=0$ for any $v<\delta$ and any $v>2\delta$; moreover $|f_\delta''(v)|\leq C\delta^{-1}$ and $\frac{1}{2}v\leq f_\delta(v)$. We thus see that $f_\delta''(u)\cdot f_\delta(u)\leq C\delta^{-1}\cdot 2\delta\leq C$ and that $f_\delta(u)^{\frac{\alpha+n-3}{2}}\cdot u^{\frac{6-2n-2\alpha}{4}}\leq C$ (since $\alpha\leq 0$ and $n<3$). As the second term on the right-hand side in (14) converges to zero pointwise a.e., by dominated convergence we therefore see that it converges to zero strongly in $L^2(\Omega)$ as $\delta\to 0$.

Convergence of the first term on the right-hand side of (14) to the function $\chi_{\{u\neq 0\}}u^{\frac{n+\alpha-1}{2}}D^2u$ is immediate by dominated convergence: we have $\chi_{\{u\neq 0\}}u^{\frac{n+\alpha-1}{2}}D^2u\in L^2(\Omega)$ and

$$f_\delta'(u)\frac{[f_\delta(u)]^{\frac{n+\alpha-1}{2}}}{u^{\frac{n+\alpha-1}{2}}}\leq C(n,\alpha,d)$$

since $f_\delta'(v)=0$ for $v<\delta$, $|f_\delta'(v)|\leq 1$ for any v, and $\frac{1}{2}v\leq f_\delta(v)\leq \delta+v$ for any v.

In case $n+\alpha>1$, we have (by our convention of setting $f\cdot g=0$ if $f=0$ and g is undefined) $\chi_{\{u\neq 0\}}u^{\frac{n+\alpha-1}{2}}D^2u=u^{\frac{n+\alpha-1}{2}}D^2u$. In case $n+\alpha=1$ we also see that $\chi_{\{u\neq 0\}}u^{\frac{n+\alpha-1}{2}}D^2u=\chi_{\{u\neq 0\}}D^2u=D^2u=u^{\frac{n+\alpha-1}{2}}D^2u$ a.e. as otherwise we would obtain the inequality $\lim_{\delta\to 0}||D^2f_\delta(u)||_{L^2(\Omega)}^2=\int_\Omega \chi_{\{u\neq 0\}}|D^2u|^2\,dx<\int_\Omega |D^2u|^2\,dx=||D^2u||_{L^2(\Omega)}$ which clearly contradicts the lower semicontinuity of the L^2 norm with respect to convergence in the sense of distributions. Thus we have proven

$$[f_\delta(u)]^{\frac{\alpha+n-1}{2}}D^2f_\delta(u)\to u^{\frac{\alpha+n-1}{2}}D^2u \tag{15}$$

strongly in $L^2(\Omega)$ as $\delta\to 0$.

Using the strong convergence (15) in connection with the convergence result regarding the first derivative (13), the convergence of $(f_\delta(u)-u)$ in $L^\infty(K)$ for any $K\subset\subset\Omega$, and the estimates

$$\frac{|[(v+\epsilon)^\alpha v^n]''+\alpha[(v+\epsilon)^{\alpha-1}v^n]'|}{v^{\alpha+n-2}}\leq C(n,\alpha) \tag{16}$$

and

$$\frac{|[(v+\epsilon)^\alpha v^n]''+2\alpha[(v+\epsilon)^{\alpha-1}v^n]'|}{v^{\alpha+n-2}}\leq C(n,\alpha) \tag{17}$$

as well as

$$\frac{|(v+\epsilon)^{\alpha-1}v^n|}{v^{n+\alpha-1}}\leq C(n,\alpha)\ , \tag{18}$$

we establish convergence of the terms II, III, IV by rewriting these terms analogous to the rearrangement (11) of term I.

It remains to prove convergence of the left-hand side in (10). It is easily seen that $(f_\delta(u)+\epsilon)^\alpha \rightarrow (u+\epsilon)^\alpha$ in $W^{1,6}(K)$ for any $K \subset\subset \Omega$. We calculate

$$\begin{aligned}
&[f_\delta(u)]^{\frac{n}{2}}\nabla\Delta f_\delta(u) = [f_\delta(u)]^{\frac{n}{2}}\Delta[f_\delta'(u)\nabla u]\\
=&[f_\delta(u)]^{\frac{n}{2}}\nabla\cdot[f_\delta''(u)\nabla u\otimes\nabla u + f_\delta'(u)D^2u] \qquad (19)\\
=&[f_\delta(u)]^{\frac{n}{2}}[f_\delta'''(u)|\nabla u|^2\nabla u + 2f_\delta''(u)D^2u\cdot\nabla u + f_\delta''(u)\Delta u\nabla u + f_\delta'(u)\nabla\Delta u]\ .
\end{aligned}$$

Using the regularity $\nabla u^{\frac{n+2}{6}} \in L^6(\Omega)$, $u^{\frac{n-2}{2}}\nabla u\otimes D^2u \in L^2(\Omega)$, $u^{\frac{n}{2}}\nabla\Delta u \in L^2(\Omega)$, the fact that $f_\delta''(v) = 0$ and $f_\delta'''(v) = 0$ for $v \notin (\delta, 2\delta)$, and the estimates $|f_\delta''| \leq C(d,n)\delta^{-1}$, $|f_\delta'''| \leq C(d,n)\delta^{-2}$, $|f_\delta' - 1| \leq 1$, we see that the convergence $[f_\delta(u)]^{\frac{n}{2}}\nabla\Delta f_\delta(u) \rightarrow u^{\frac{n}{2}}\nabla\Delta u$ strongly in $L^2(\Omega)$ holds by the dominated convergence theorem: estimating the first term on the right-hand side of (19), we get

$$\begin{aligned}
&[f_\delta(u)]^{\frac{n}{2}}\cdot|f_\delta'''(u)|\cdot|\nabla u|^3 \leq [f_\delta(u)]^{\frac{n}{2}}u^{\frac{4-n}{2}}\cdot|f_\delta'''(u)|\cdot|\nabla u^{\frac{n+2}{6}}|^3\\
\leq& C(d,n)\chi_{\{u\in(\delta,2\delta)\}}\delta^{\frac{n}{2}}\delta^{\frac{4-n}{2}}\delta^{-2}|\nabla u^{\frac{n+2}{6}}|^3 = C(d,n)\chi_{\{u\in(\delta,2\delta)\}}|\nabla u^{\frac{n+2}{6}}|^3
\end{aligned}$$

which implies pointwise convergence to 0 a.e. and yields the dominating function $C(d,n)|\nabla u^{\frac{n+2}{6}}|^3$. The second term and the third term on the right-hand side of (19) can be treated similarly. Regarding the fourth term, we immediately obtain convergence a.e. to the desired limit; moreover, we notice that the fourth term is bounded from above by $C(d,n)u^{\frac{n}{2}}|\nabla\Delta u|$ since $f_\delta(v) \leq 2v$ for any $v > \delta$ and since $f_\delta'(v) = 0$ for $v \leq \delta$.

This finishes the proof of (10) under the weakened regularity assumptions. Note that by our convention of setting $f\cdot g = 0$ if $f = 0$ and g is undefined, the domain of integration in the integrals of (10) is now effectively $\Omega\cap\{u > 0\}$.

Now assume that u is a strong energy solution of the thin-film equation satisfying the α entropy estimate. We may then rewrite (9) using (10): for a.e. $t \in I$ we have $\nabla u^{\frac{n+2}{6}} \in L^6(\Omega)$, $u^{\frac{n}{2}}\nabla\Delta u \in L^2(\Omega)$, $u^{\frac{n-2}{2}}\nabla u\otimes D^2u \in L^2(\Omega)$, $\nabla u^{\frac{1+n+\alpha}{4}} \in L^4(\Omega)$, $D^2u^{\frac{1+n+\alpha}{2}} \in L^2(\Omega)$; thus, for a.e. $t \in I$ formula (10) can be applied. We obtain

$$
\begin{aligned}
&\frac{1}{1+\alpha}\int_\Omega (u+\epsilon)^{1+\alpha}\psi\ dx\Big|_{t_1}^{t_2}\\
=&-\frac{1}{2}\int_{t_1}^{t_2}\int_\Omega \Big[[(u+\epsilon)^\alpha u^n]''' + \alpha[(u+\epsilon)^{\alpha-1}u^n]''\Big]\,|\nabla u|^4\psi\ dx\ dt\\
&-\frac{1}{2}\int_{t_1}^{t_2}\int_\Omega \Big[[(u+\epsilon)^\alpha u^n]'' + \alpha[(u+\epsilon)^{\alpha-1}u^n]'\Big]\,|\nabla u|^2\Delta u\ \psi\ dx\ dt\\
&-\int_{t_1}^{t_2}\int_\Omega \Big[[(u+\epsilon)^\alpha u^n]'' + 2\alpha[(u+\epsilon)^{\alpha-1}u^n]'\Big]\,\nabla u\cdot D^2u\cdot\nabla u\ \psi\ dx\ dt\\
&-\alpha\int_{t_1}^{t_2}\int_\Omega (u+\epsilon)^{\alpha-1}u^n|D^2u|^2\psi\ dx\ dt\\
&+\frac{1}{2}\int_{t_1}^{t_2}\int_\Omega \Big[[(u+\epsilon)^\alpha u^n]' + \alpha(u+\epsilon)^{\alpha-1}u^n\Big]\,|\nabla u|^2\Delta\psi\ dx\ dt\\
&+\int_{t_1}^{t_2}\int_\Omega [(u+\epsilon)^\alpha u^n]'\nabla u\cdot D^2\psi\cdot\nabla u\ dx\ dt\\
&-\int_{t_1}^{t_2}\int_\Omega\int_0^u (v+\epsilon)^\alpha v^n\ dv\ \Delta^2\psi\ dx\ dt\ .
\end{aligned}
$$

Passing to the limit $\epsilon \to 0$, by dominated convergence the desired result is shown: we just need to use the inequalities (12), (16), (17), (18) in connection with pointwise convergence a.e. (recall that $\{\nabla u = 0\}$ a.e. on $\{u = 0\}$) and the regularity $\nabla u^{\frac{1+n+\alpha}{4}} \in L^4_{loc}(I; L^4(\Omega))$, $D^2 u^{\frac{1+n+\alpha}{2}} \in L^2_{loc}(I; L^2(\Omega))$ to deal with the first four terms on the right-hand side. Using additionally the inequalities

$$\frac{|[(v+\epsilon)^\alpha v^n]' + \alpha(v+\epsilon)^{\alpha-1}v^n|}{v^{\alpha+n-1}} \le C(\alpha,n)$$

and

$$\frac{|[(v+\epsilon)^\alpha v^n]'|}{v^{\alpha+n-1}} \le C(\alpha,n)\ ,$$

we can prove convergence of the fifth and the sixth term. The last term is immediately seen to converge to the desired limit.

□

Proof of Lemma 17. Using the fact that $\operatorname{supp}\psi \subset\subset \Omega$, for smooth strictly positive u we calculate

$$
\begin{aligned}
&\int_\Omega u^{b-1}|D^2u|^2\psi + (b-1)\int_\Omega u^{b-2}\nabla u\cdot D^2u\cdot\nabla u\ \psi + \int_\Omega u^{b-1}\nabla u\cdot D^2\psi\cdot\nabla u\\
&= -\int_\Omega u^{b-1}\nabla\Delta u\cdot\nabla u\ \psi - \int_\Omega u^{b-1}\nabla u\cdot D^2u\cdot\nabla\psi + \int_\Omega u^{b-1}\nabla u\cdot D^2\psi\cdot\nabla u\\
&= \int_\Omega u^{b-1}|\Delta u|^2\psi + (b-1)\int_\Omega u^{b-2}|\nabla u|^2\Delta u\ \psi + \int_\Omega u^{b-1}\Delta u\nabla u\cdot\nabla\psi\\
&\quad - \int_\Omega u^{b-1}\nabla u\cdot D^2u\cdot\nabla\psi + \int_\Omega u^{b-1}\nabla u\cdot D^2\psi\cdot\nabla u\\
&= \int_\Omega u^{b-1}|\Delta u|^2\psi + (b-1)\int_\Omega u^{b-2}|\nabla u|^2\Delta u\ \psi\\
&\quad - 2\int_\Omega u^{b-1}\nabla u\cdot D^2u\cdot\nabla\psi - (b-1)\int_\Omega u^{b-2}|\nabla u|^2\nabla\psi\cdot\nabla u\\
&= \int_\Omega u^{b-1}|\Delta u|^2\psi + (b-1)\int_\Omega u^{b-2}|\nabla u|^2\Delta u\ \psi + \int_\Omega u^{b-1}|\nabla u|^2\Delta\psi\ .
\end{aligned}
$$

For strictly positive u with $u\in H^2(\Omega)$, the formula is seen to hold by approximation (note that by the Sobolev embedding $u\in H^2(\Omega)$ implies $u\in L^\infty(K)$ and $\nabla u\in L^6(K)$ for any $K\subset\subset\Omega$ as we have $d\le 3$).

The formula carries over to the case of nonnegative u with $u^{\frac{b+1}{2}}\in H^2(\Omega)$ and $u^{\frac{b+1}{4}}\in W^{1,4}(\Omega)$ by considering $f_\delta(u)$ (for the definition see the proof of the previous lemma) and passing to the limit $\delta\to 0$: $u^{\frac{b+1}{4}}$ (and therefore u) is continuous, thus the set A_δ is open (A_δ being defined as in the proof of the previous lemma). We have $f_\delta(u)\equiv\delta$ on some neighbourhood of $\Omega\setminus A_\delta$ and we have $u\in H^2(A_\delta\cap K)\cap W^{1,4}(A_\delta\cap K)$ for any $K\subset\subset\Omega$ which implies $f_\delta(u)\in H^2(A_\delta\cap K)$ for any $K\subset\subset\Omega$. Thus, we have $f_\delta(u)\in H^2(K)$ for any $K\subset\subset\Omega$; moreover, $f_\delta(u)\ge\delta$. Therefore the formula holds with $f_\delta(u)$ in place of u.

We then pass to the limit $\delta\to 0$; the limit is calculated using the convergence properties (13) and (15), whose proof only required the regularity $u^{\frac{b+1}{2}}\in H^2(\Omega)$, $u^{\frac{b+1}{4}}\in W^{1,4}(\Omega)$, and the convergence property $||u-f_\delta(u)||_{L^\infty(\Omega)}\to 0$. □

3.2 Derivation of a simplified entropy estimate

The idea of the proof of our main results is to use Lemma 16 and Lemma 17 to derive a differential inequality for the quantity $\int_\Omega u^{1+\alpha}\psi\ dx$. As a first step, we would like to show nonnegativity of (roughly) the sum of the last four terms in formula (7) using Young's inequality. To do so, we make use of Lemma 17 to partly replace $\int_\Omega u^{b-1}|D^2u|^2\psi$ by $\int_\Omega u^{b-1}|\Delta u|^2\psi$.

We set $b:=n+\alpha$ and introduce the following conditions:

(H1) Assume that $1\le b\le 2$.

(H2) Suppose that $\frac{n}{2} \leq b \leq n$.

(H3) Assume that $n - 1 < b$.

(H4) Suppose that the inequality

$$(n-b)\left(b-\frac{n}{2}\right)(b-1)(2-b) \geq \frac{1}{4}\left[\left(\frac{5n}{2}-4b\right)(b-1)\right]^2$$

is satified.

The set of $(b,n) \in \mathbb{R} \times \mathbb{R}$ for which (H1) to (H4) are satisfied is depicted in Section 3.5.

Lemma 18. *Let $n \in [1,3)$, $\alpha \in (-1,0)$, and let u be a strong energy solution of the thin-film equation on a domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$, with nonnegative initial data $u_0 \in H^1(\Omega)$. Assume that $\operatorname{supp} u_0$ is bounded. Suppose that u satisfies the α entropy estimate. Set $b := n + \alpha$ and assume that (H1) to (H4) are satisfied.*

Let $\psi \in C_c^4(\Omega)$; assume that $\psi \geq 0$. Then for a.e. $t_1, t_2 \in [0,\infty)$ with $t_2 \geq t_1$ and for a.e. $t_2 \in [0,\infty)$ in case $t_1 = 0$ we have

$$\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\
&\geq \left(\frac{2}{3}b - \frac{1}{6}n\right)\int_{t_1}^{t_2}\int_\Omega u^{b-1}|\nabla u|^2 \Delta\psi + \left(\frac{4}{3}b - \frac{1}{3}n\right)\int_{t_1}^{t_2}\int_\Omega u^{b-1}\nabla u \cdot D^2\psi \cdot \nabla u \\
&\quad - \frac{1}{b+1}\int_{t_1}^{t_2}\int_\Omega u^{b+1}\Delta^2\psi\ .
\end{aligned}$$

Recall that the regularity of solutions $u \in L^\infty(I; H^1(\Omega)) \cap H^1_{loc}(I; [W^{1,p}(\Omega)]')$ implies $u \in C^0_{loc}(I; L^2(V))$ for any bounded open set $V \subset \Omega$ with smooth boundary (see e.g. Corollary 4 in [53]). Thus by approximation, the formula in our lemma holds for all $t_2 \geq t_1 \geq 0$, not just a.e..

Proof. Assume for the moment that $\psi \in C_c^\infty(\Omega)$, $\psi \geq 0$. Recall that by Lemma 17 we have

$$\begin{aligned}
&\int_\Omega u^{b-1}|D^2u|^2\psi + (b-1)\int_\Omega u^{b-2}\nabla u \cdot D^2u \cdot \nabla u\ \psi + \int_\Omega u^{b-1}\nabla u \cdot D^2\psi \cdot \nabla u \\
&= \int_\Omega u^{b-1}|\Delta u|^2\psi + (b-1)\int_\Omega u^{b-2}|\nabla u|^2\Delta u\ \psi + \int_\Omega u^{b-1}|\nabla u|^2\Delta\psi
\end{aligned}$$

for a.e. $t > 0$. Since by (H2) and (H3) it holds that $-1 < \alpha \leq 0$, formula (7) states that

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\
&= \left(b - \frac{1}{2}n\right) \int_{t_1}^{t_2} \int_\Omega u^{b-1}|\nabla u|^2 \Delta\psi + b \int_{t_1}^{t_2} \int_\Omega u^{b-1} \nabla u \cdot D^2\psi \cdot \nabla u \\
&\quad - \frac{1}{b+1} \int_{t_1}^{t_2} \int_\Omega u^{b+1} \Delta^2\psi + (n-b) \int_{t_1}^{t_2} \int_\Omega u^{b-1}|D^2 u|^2 \psi \\
&\quad + \left(b - \frac{n}{2}\right)(b-1)(2-b) \int_{t_1}^{t_2} \int_\Omega u^{b-3} |\nabla u|^4 \psi \\
&\quad + (2n-3b)(b-1) \int_{t_1}^{t_2} \int_\Omega u^{b-2} \nabla u \cdot D^2 u \cdot \nabla u\ \psi \\
&\quad + \left(\frac{n}{2} - b\right)(b-1) \int_{t_1}^{t_2} \int_\Omega u^{b-2} |\nabla u|^2 \Delta u\ \psi\ .
\end{aligned}
$$

We now multiply the formula from Lemma 17 by $\frac{1}{3}(n-b)$ and add it to this equation, resulting in

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\
&= \left(b - \frac{1}{2}n + \frac{n-b}{3}\right) \int_{t_1}^{t_2} \int_\Omega u^{b-1}|\nabla u|^2 \Delta\psi \\
&\quad + \left(b - \frac{n-b}{3}\right) \int_{t_1}^{t_2} \int_\Omega u^{b-1} \nabla u \cdot D^2\psi \cdot \nabla u - \frac{1}{b+1} \int_{t_1}^{t_2} \int_\Omega u^{b+1} \Delta^2\psi \\
&\quad + \frac{2}{3}(n-b) \int_{t_1}^{t_2} \int_\Omega u^{b-1}|D^2 u|^2 \psi + \frac{1}{3}(n-b) \int_{t_1}^{t_2} \int_\Omega u^{b-1} |\Delta u|^2 \psi \\
&\quad + \left(\frac{2}{3} + \frac{1}{3}\right)\left(b - \frac{n}{2}\right)(b-1)(2-b) \int_{t_1}^{t_2} \int_\Omega u^{b-3} |\nabla u|^4 \psi \\
&\quad + \left(\frac{5}{3}n - \frac{8}{3}b\right)(b-1) \int_{t_1}^{t_2} \int_\Omega u^{b-2} \nabla u \cdot D^2 u \cdot \nabla u\ \psi \\
&\quad + \left(\frac{5}{6}n - \frac{4}{3}b\right)(b-1) \int_{t_1}^{t_2} \int_\Omega u^{b-2} |\nabla u|^2 \Delta u\ \psi\ .
\end{aligned}
$$

We now see that the expressions

$$
\begin{aligned}
&\frac{2}{3}(n-b) \int_{t_1}^{t_2} \int_\Omega u^{b-1}|D^2 u|^2 \psi + \frac{2}{3}\left(b - \frac{n}{2}\right)(b-1)(2-b) \int_{t_1}^{t_2} \int_\Omega u^{b-3} |\nabla u|^4 \\
&+ \left(\frac{5}{3}n - \frac{8}{3}b\right)(b-1) \int_{t_1}^{t_2} \int_\Omega u^{b-2} \nabla u \cdot D^2 u \cdot \nabla u\ \psi
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{3}(n-b) \int_{t_1}^{t_2} \int_\Omega u^{b-1}|\Delta u|^2 \psi + \frac{1}{3}\left(b - \frac{n}{2}\right)(b-1)(2-b) \int_{t_1}^{t_2} \int_\Omega u^{b-3} |\nabla u|^4 \\
&+ \left(\frac{5}{6}n - \frac{4}{3}b\right)(b-1) \int_{t_1}^{t_2} \int_\Omega u^{b-2} |\nabla u|^2 \Delta u\ \psi
\end{aligned}
$$

are nonnegative: Young's inequality implies nonnegativity of these terms if $n-b \geq 0$, $1 \leq b \leq 2$, $b \geq \frac{n}{2}$ and

$$(n-b)\left(b-\frac{n}{2}\right)(b-1)(2-b) \geq \frac{1}{4}\left(\frac{5}{2}n-4b\right)^2(b-1)^2$$

are satisfied. These conditions however were precisely part of our assumptions. This proves the lemma for $\psi \in C_c^\infty(\Omega)$ with $\psi \geq 0$.

For $\psi \in C_c^4(\Omega)$ with $\psi \geq 0$, we consider the mollifications $\rho_\delta * \psi$ which belong to $C_c^\infty(\Omega)$ (at least if $\delta > 0$ is small enough); passing to the limit $\delta \to 0$, we obtain $\rho_\delta * \psi \to \psi$ in $C^4(\Omega)$. Using the regularity of u and dominated convergence, this is sufficient for passing to the limit in all expressions of our inequality. □

3.3 The case of one spatial dimension

An application of Hardy's inequality to the right-hand side of the integral estimate obtained in the previous lemma and a careful choice of the weight function ψ will enable us to derive upper bounds on waiting times for the thin-film equation for $n \in [2, \frac{32}{11})$.

Lemma 19 (Hardy's inequality)**.** *For $v \in H^1(\mathbb{R})$ with $\operatorname{supp} v \subset\subset \mathbb{R}\setminus\{0\}$ and any $\psi \in C^2_{loc}(\mathbb{R}\setminus\{0\})$ with $\psi_{xx} > 0$ on $\mathbb{R}\setminus\{0\}$ the inequality*

$$\int_{\mathbb{R}} v^2 \psi_{xx}\, dx \leq 4\int_{\mathbb{R}} |v_x|^2 \frac{|\psi_x|^2}{\psi_{xx}}\, dx$$

holds.

Proof. We calculate

$$\int_{\mathbb{R}} v^2\psi_{xx}\, dx = -2\int_{\mathbb{R}} v v_x \psi_x\, dx \leq 2\left(\int_{\mathbb{R}} |v_x|^2\frac{|\psi_x|^2}{\psi_{xx}}\, dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} v^2\psi_{xx}\, dx\right)^{\frac{1}{2}}.$$

The desired inequality follows. □

Lemma 20. *Let $d=1$, $n \in (1,3)$, $\alpha \in (-1,0)$ and let u be a strong energy solution of the Cauchy problem (i.e. $\Omega = \mathbb{R}$) for the thin-film equation with compactly supported nonnegative initial data $u_0 \in H^1(\mathbb{R})$. Set $b := n+\alpha$. Assume that conditions (H1), (H2), (H3), (H4) (preceding Lemma 18) are satisfied. Suppose that u satisfies the α entropy estimate. Given $\gamma \leq -1$, suppose furthermore that*

(H5) The condition

$$\left(2b-\frac{1}{2}n\right)\frac{\gamma-3}{(b+1)^2(\gamma-2)} - \frac{1}{b+1} \geq \tau$$

is satisfied for some $\tau > 0$.

Let $x_1 \notin \operatorname{supp} u_0$*,* $\epsilon > 0$*,* $T > 0$*. Define* $K := \overline{\bigcup_{t\in[0,T]} \operatorname{supp} u(.,t)}$*. Then in case*

$$T \geq \frac{1}{n\tau}\left[\int_{K\setminus(x_1-\epsilon,x_1+\epsilon)} |x-x_1|^{\gamma+4\frac{1+\alpha}{n}}\ dx\right]^{\frac{n}{1+\alpha}} \cdot \left[\int_{\mathbb{R}} u_0^{1+\alpha}|x-x_1|^{\gamma}\ dx\right]^{-\frac{n}{1+\alpha}}$$

we have $\operatorname{dist}(x_1, \overline{\bigcup_{t\in[0,T]} \operatorname{supp} u(.,t)}) < \epsilon$*.*

Proof. By our assumptions, Lemma 18 is applicable.

We argue by contradiction. Suppose that $B_\epsilon(x_1)\cap \operatorname{supp} u(.,t) = \emptyset$ holds for all $0 \leq t \leq T$. Hardy's inequality (Lemma 19) applied with $\psi = \frac{d^2}{dx^2}|x-x_1|^\gamma$ reads

$$\int_{\mathbb{R}} v^2 |x-x_1|^\gamma{}_{xxxx}\ dx \leq \frac{4(\gamma-2)}{\gamma-3}\int_{\mathbb{R}} |v_x|^2 |x-x_1|^\gamma{}_{xx}\ dx\ .$$

We now use $\psi := |x-x_1|^\gamma$ as a test function in Lemma 18; this is possible since ψ is C^∞ on some neighborhood of the compact set $\overline{\bigcup_{t\in[0,T]} \operatorname{supp} u(.,t)}$. Rewriting $\int_{\mathbb{R}} u^{b-1}|u_x|^2\ \psi_{xx}\ dx = \frac{4}{(b+1)^2}\int_{\mathbb{R}} |(u^{\frac{b+1}{2}})_x|^2 \psi_{xx}\ dx$ and using the previous inequality we therefore obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_1|^\gamma\ dx - \int_{\mathbb{R}} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_1|^\gamma\ dx\\
\geq&\left(\left(2b-\frac{1}{2}n\right)\frac{\gamma-3}{(b+1)^2(\gamma-2)} - \frac{1}{b+1}\right)\int_{t_1}^{t_2}\int_{\mathbb{R}} u^{b+1}|x-x_1|^\gamma{}_{xxxx}\ dx\ dt\\
=&\left(\left(2b-\frac{1}{2}n\right)\frac{\gamma-3}{(b+1)^2(\gamma-2)} - \frac{1}{b+1}\right)\\
&\quad\cdot\gamma(\gamma-1)(\gamma-2)(\gamma-3)\int_{t_1}^{t_2}\int_{\mathbb{R}} u^{b+1}|x-x_1|^{\gamma-4}\ dx\ dt\\
\geq&\tau\int_{t_1}^{t_2}\int_{\mathbb{R}} u^{b+1}|x-x_1|^{\gamma-4}\ dx\ dt
\end{aligned}$$

where in the last step we have used condition (H5) and $\gamma \leq -1$. Now notice that due to Hölder's inequality one can estimate

$$\begin{aligned}
&\int_{\mathbb{R}} u^{1+\alpha}|x-x_1|^\gamma\ dx\\
\leq&\left(\int_{\mathbb{R}} u^{b+1}|x-x_1|^{\gamma-4}\ dx\right)^{\frac{1+\alpha}{b+1}}\left(\int_{\operatorname{supp} u}|x-x_1|^{\gamma+4\frac{1+\alpha}{n}}\ dx\right)^{\frac{n}{b+1}}
\end{aligned}$$

where we have used the definition $b = \alpha+n$. Putting these estimates together, using (H3) which implies $\alpha > -1$ we arrive at the differential inequality

$$\begin{aligned}
&\int_{\mathbb{R}} u^{1+\alpha}(.,t_2)|x-x_1|^\gamma \, dx - \int_{\mathbb{R}} u^{1+\alpha}(.,t_1)|x-x_1|^\gamma \, dx \\
\geq &(1+\alpha)\tau \left[\int_{K\cap(\mathbb{R}\setminus B_\epsilon(x_1))} |x-x_1|^{\gamma+4\frac{1+\alpha}{n}} \, dx\right]^{-\frac{n}{1+\alpha}} \\
&\cdot \int_{t_1}^{t_2} \left(\int_{\mathbb{R}} u^{1+\alpha}|x-x_1|^\gamma \, dx\right)^{\frac{1+b}{1+\alpha}} \, dt
\end{aligned}$$

where we have used the assumption $\operatorname{supp} B_\epsilon(x_1) \cap \operatorname{supp} u(.,t) = \emptyset$ for $t < T$. The solution of the differential equation $\frac{d}{dt}z(t) = q \cdot [z(t)]^m$ is given by $z(t) = [z(0)^{1-m} - (m-1)\cdot q \cdot t]^{\frac{1}{1-m}}$. Using the comparison principle, we therefore obtain blow-up of the quantity $\int_{\mathbb{R}} u^{1+\alpha}(.,t)|x-x_1|^\gamma \, dx$ by no later than

$$\begin{aligned}
\hat{T} =&\frac{1}{n\tau} \left[\int_{K\cap(\mathbb{R}\setminus B_\epsilon(x_1))} |x-x_1|^{\gamma+4\frac{1+\alpha}{n}} \, dx\right]^{\frac{n}{1+\alpha}} \\
&\cdot \left[\int_{\mathbb{R}} u_0^{1+\alpha}|x-x_1|^\gamma \, dx\right]^{-\frac{n}{1+\alpha}} .
\end{aligned}$$

As $\int_{\mathbb{R}} u(.,t) \, dx = \int_{\mathbb{R}} u_0 \, dx < \infty$ and $\alpha \in (-1,0)$ as well as $\gamma \leq -1$, by Hölder's inequality we see that $\int_{\mathbb{R}\setminus B_\epsilon(x_1)} u^{1+\alpha}|x-x_1|^\gamma \, dx$ must remain bounded. Therefore we have obtained the desired contradiction. □

We are now in position to prove the main theorem in the one-dimensional case.

Proof of Theorem 3. Assertion a) is an easy consequence of the previous lemma: We choose $b := \frac{9}{20}n + \frac{12}{20}$, i.e. $\alpha = -\frac{11}{20}n + \frac{12}{20}$, and $\gamma := -2$. Condition (H3) is then equivalent to $\frac{11}{20}n < \frac{32}{20}$, i.e. $n < \frac{32}{11}$. Condition (H1) is seen to be satisfied for any $n \in [1,3)$. Condition (H2) is satisfied for $n \in (\frac{12}{11}, 12)$. Condition (H5) is also satisfied for some $\tau = \tau(n)$ since for $\gamma = -2$ it is equivalent to $\frac{5}{4}(2b - \frac{1}{2}n) - (b+1) > 0$ which in turn is equivalent to $5(\frac{8}{20}n + \frac{24}{20}) - 4(\frac{9}{20}n + \frac{32}{20}) > 0$, i.e. $40n + 120 - 36n - 128 > 0$. The latter condition reduces to $4n > 8$, i.e. $n > 2$. It remains to check condition (H4). This condition now reads

$$\begin{aligned}
&\left(\frac{11}{20}n - \frac{12}{20}\right)\left(-\frac{1}{20}n + \frac{12}{20}\right)\left(\frac{9}{20}n - \frac{8}{20}\right)\left(\frac{28}{20} - \frac{9}{20}n\right) \\
\geq &\frac{1}{4}\left[\left(\frac{14}{20}n - \frac{48}{20}\right)\left(\frac{9}{20}n - \frac{8}{20}\right)\right]^2
\end{aligned}$$

Using a computer algebra program (or doing the calculations by hand), one can check that (H5) is therefore equivalent to

$$\frac{3(n-2)(9n-8)(n(188-57n)-48)}{80\ 000} \geq 0$$

Calculating the roots of the third polynomial factor in this expression, Condition (H3) is therefore satisfied as long as $n \in [2, \frac{2(47+5\sqrt{61})}{57})$, i.e. especially for $n \in [2,3)$.

Now that we have checked the assumptions of Lemma 20, to finish the proof of assertion a) we apply this Lemma with $x_1 := x_0 - \epsilon$, where ϵ is the parameter of the lemma. Evaluating the first integral in the estimate from Lemma 20, we obtain that

$$
\begin{aligned}
&\inf\{t \geq 0 : \operatorname{supp} u(.,t) \cap (-\infty, x_0) \neq \emptyset\}\\
\leq &\inf\{t \geq 0 : \operatorname{supp} u(.,t) \cap (x_1 - \epsilon, x_1 + \epsilon) \neq \emptyset\}\\
\leq &\frac{1}{n\tau}\left[\int_{\mathbb{R}\setminus(x_1-\epsilon,x_1+\epsilon)} |x - x_1|^{\gamma+4\frac{1+\alpha}{n}}\,dx\right]^{\frac{n}{1+\alpha}} \left[\int_{\mathbb{R}} u_0^{1+\alpha}|x - x_1|^{\gamma}\,dx\right]^{-\frac{n}{1+\alpha}}\\
\leq &C(n)\left[\int_{\mathbb{R}\setminus(x_1-\epsilon,x_1+\epsilon)} |x - x_1|^{\gamma+4\frac{1+\alpha}{n}}\,dx\right]^{\frac{n}{1+\alpha}} \left[\int_{\mathbb{R}} u_0^{1+\alpha}|x - x_1|^{\gamma}\,dx\right]^{-\frac{n}{1+\alpha}}\\
= &C(n)\left[-\frac{2}{1+\gamma+4\frac{1+\alpha}{n}}|\epsilon|^{1+\gamma+4\frac{1+\alpha}{n}}\right]^{\frac{n}{1+\alpha}} \left[\int_{\mathbb{R}} u_0^{1+\alpha}|x - x_0 + \epsilon|^{\gamma}\,dx\right]^{-\frac{n}{1+\alpha}} ,
\end{aligned}
$$

where we have used the fact that $\alpha < -\frac{1}{2}$, $\gamma = -2$, $n > 2$, which implies $\gamma + 4\frac{1+\alpha}{n} < -1$. This proves assertion a) since $\epsilon > 0$ was arbitrary.

Note that one could prove the theorem for the slightly larger range $n \in (2, \frac{2}{9}(10+\sqrt{10}))$ using e.g. a computer algebra system to solve the inequalities $(H1)$ to $(H5)$. See Section 3.5 below for a plot of the set of admissible pairs (n, b).

Assertion b) is shown just as assertion a), the only difference being that we estimate $\int_{\mathbb{R}} u_0^{1+\alpha}|x - x_0 + \epsilon|^{-2}\,dx \geq \frac{1}{4}\epsilon^{-2}\int_{(x_0,x_0+\epsilon)} u_0^{1+\alpha}\,dx$ and pass to the limit $\epsilon \to 0$. Note that the waiting time T^* of u at x_0 is bounded by $T^* \leq \limsup_{\epsilon\to 0} \inf\{t \geq 0 : \operatorname{supp} u(.,t) \cap (x_0 - 2\epsilon, x_0) \neq \emptyset\}$ and that $(x_0 - 2\epsilon, x_0) = (x_1 - \epsilon, x_1 + \epsilon)$.

Assertion c) is also a consequence of the previous lemma: for $n = 2$ and $\alpha = -\frac{1}{2}$, conditions (H1) to (H4) (see Lemma 18) are readily verified. Inserting $n = 2$ and $\alpha = -\frac{1}{2}$ into (H5) and multiplying the resulting inequality by $(b+1)^2$, we see that for $\gamma < 0$ the condition (H5) is equivalent to

$$4(\gamma - 3) - 5(\gamma - 2) \leq \frac{25}{2}\tau(\gamma - 2) .$$

Thus, for $\gamma \geq -2$ the condition (H5) is satisfied for sure if

$$-2 - \gamma \leq -50\tau .$$

This implies that we can choose $\tau := \frac{2+\gamma}{50}$. Fix some $\tilde{T} > 0$. By the finite speed of support propagation property which holds for strong energy solutions [39], for $t \leq \tilde{T}$ we have $\operatorname{supp} u(.,t) \subset B_{R_1}(x_0)$ for some R_1 depending on u_0 and $\tilde{T}$. Set $x_1 := x_0 - \epsilon$ with $\epsilon < \min(R_1, \frac{\delta}{2})$. Lemma 20 now asserts that $(x_0 - 2\epsilon, x_0) \cap \operatorname{supp} u(.,t) \neq \emptyset$ for some $t \in (0,T)$, where

$$T \leq \frac{1}{2\tau}\left[\int_{B_{2R_1}(x_1)\setminus B_\epsilon(x_1)} |x-x_1|^{\gamma+1}\, dx\right]^4 \left(\int_{\mathbb{R}} u_0^{\frac{1}{2}}|x-x_1|^\gamma\, dx\right)^{-4} ,$$

if the expression on the right-hand side does not exceed $\tilde{T}$ (for $t > \tilde{T}$ the assumption $\operatorname{supp} u(.,t) \subset B_{R_1}(x_0)$ which we used may be invalid). Using the estimate $\int_{\mathbb{R}} u_0^{\frac{1}{2}}|x-x_1|^\gamma\, dx \geq \frac{1}{4}\epsilon^\gamma \int_{(x_0,x_0+\epsilon)} u_0^{\frac{1}{2}}\, dx$ (since $-2 \leq \gamma \leq -1$) and the fact that

$$\int_{B_{2R_1}(x_1)\setminus B_\epsilon(x_1)} |x-x_1|^{\gamma+1}\, dx \leq (2R_1)^{2+\gamma} \int_{B_{2R_1}(x_1)\setminus B_\epsilon(x_1)} |x-x_1|^{-1}\, dx ,$$

we obtain $(x_0 - 2\epsilon, x_0) \cap \operatorname{supp} u(.,t) \neq \emptyset$ for some $t \in (0,T)$, where

$$T \leq \frac{1}{2\tau}\left[(2R_1)^{2+\gamma}\log\frac{2R_1}{\epsilon}\right]^4 \left(\frac{1}{4}\epsilon^\gamma \int_{(x_0,x_0+\epsilon)} u_0^{\frac{1}{2}}\, dx\right)^{-4} ,$$

if the expression on the right-hand side does not exceed $\tilde{T}$. We now set $\gamma := -2 - \frac{50}{\log\epsilon}$ which implies $\tau = -\frac{1}{\log\epsilon}$ and obtain $(x_0 - 2\epsilon, x_0) \cap \operatorname{supp} u(.,t) \neq \emptyset$ for some $t \in (0,T)$, where

$$T \leq C(2R_1)^{8+4\gamma}\left(\frac{\epsilon^{-2-\frac{50}{\log\epsilon}}}{|\log(2R_1)|\cdot|\log\epsilon|^{\frac{1}{4}} + |\log\epsilon|^{\frac{5}{4}}} \int_{(x_0,x_0+\epsilon)} u_0^{\frac{1}{2}}\, dx\right)^{-4} ,$$

if the expression on the right again does not exceed $\tilde{T}$.

Noting that $\epsilon^{-\frac{50}{\log\epsilon}} = e^{-50}$ and letting $\epsilon \to 0$, we obtain using $8 + 4\gamma = -\frac{200}{\log\epsilon} \to 0$ and $|\log R_1| \leq |\log\epsilon|$ for ϵ small enough

$$T^* \leq C \liminf_{\epsilon\to 0}\left(\frac{\epsilon^{-2}}{|\log\epsilon|^{\frac{5}{4}}} \int_{(x_0,x_0+\epsilon)} u_0^{\frac{1}{2}}\, dx\right)^{-4}$$

if the expression on the right-hand side does not exceed $\tilde{T}$. As $\tilde{T} > 0$ was arbitrary, by choosing $\tilde{T}$ large enough the assertion c) of the theorem is obtained. □

3.4 The case of several spatial dimensions

We now derive upper bounds on waiting times for the thin-film equation in the case of several spatial dimensions. If $d > 1$, an additional difficulty arises: The attempt to use $|x|^\gamma$ as a weight function fails as the constant in front of the positive terms in the weighted entropy estimate is no longer large enough to ensure that the positive terms dominate the negative term, at least for those γ which would allow for the derivation of optimal upper

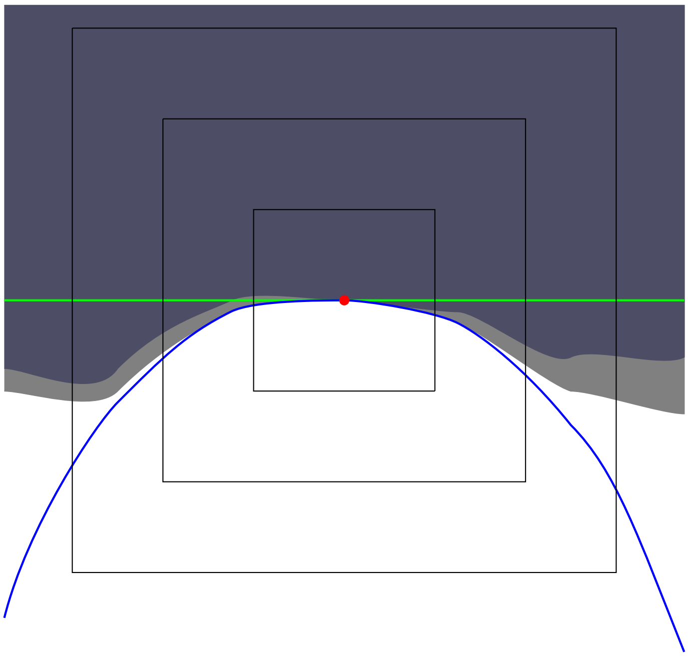


Figure 1: A sketch of the situation of Lemma 21. The deep blue area corresponds to $\operatorname{supp} u_0$; the union of the grey and deep blue areas represents the set M. The boundary of the grey area corresponds to the graph of ξ.
The three boxes represent the sets Z_{3r}, Z_{2r}, Z_r. The red dot denotes the point 0. The green line marks the tangent plane H to ∂M in 0 (i.e. $\mathbb{R}^{d-1}$).
The blue curve corresponds to the graph of $\tilde{\xi}$. It is clearly visible that the graph of $\tilde{\xi}$ coincides with ∂M (i.e. the graph of ξ) in Z_r, but moves away from ∂M as one moves away from 0; in $Z_{3r} \setminus Z_{2r}$ the graph of $\tilde{\xi}$ lies at least Kr^2 below the set M.

bounds on waiting times. This problem is resolved using a localized test function adapted to the shape of the initial support, which approximately reduces the situation to the one-dimensional one.

For the next lemma, we assume that we are given a point $x_0 \in \partial \operatorname{supp} u_0$ such that there exists a C^4 domain, whose closure we denote by M, with the property that in some neighbourhood of x_0 the set $\operatorname{supp} u_0$ is contained in M; moreover, we require $x_0 \in \partial M$. The tangent plane of the manifold ∂M in x_0 will be denoted by H. Without loss of generality, we may assume that $x_0 = 0$ and $H = \{x \in \mathbb{R}^d : x_d = 0\}$. In this case, ∂M is locally given as the graph of a function $\xi : H \to \mathbb{R}$. We define another function $\tilde{\xi}$ to be equal to ξ in some (cylindric) neighbourhood Z_r of x_0, but require the graph of $\tilde{\xi}$ to move away from M as one moves away from x_0.

Our test function ψ takes the form $|x_d - \tilde{\xi}(x_1, \ldots, x_{d-1}) + \delta|^\gamma \cdot \phi$, where ϕ is some cutoff. The singularities of our test function ψ lie on a curve which corresponds to the graph of $\tilde{\xi}$ shifted downwards by δ.

As $\tilde{\xi}$ is nonconstant, we shall see that additional terms involving derivatives of $\tilde{\xi}$ arise during the derivation of our differential inequality. If r^{-1} is large enough (in comparison to these derivatives of $\tilde{\xi}$), our lemma gives an upper bound on the waiting time in the neighbourhood Z_{3r} of x_0. By decreasing r one can always enforce this condition; however, by decreasing r the additional condition (22) becomes stricter.

Due to the cutoff present in our test function, an additional inhomogeneity appears on the right-hand side of our differential inequality for $\int_\Omega u^{1+\alpha}\psi \, dx$. In order to nevertheless prove blow-up of $\int_\Omega u^{1+\alpha}\psi \, dx$, we need to assume that the inhomogeneity is smaller than $\int_\Omega u_0^{1+\alpha}\psi \, dx$. This condition is precisely (22). As we shall see this inhomogeneity becomes irrelevant as we "zoom in" on the free boundary, at least for $n > 2$; for $n = 2$, the inhomogeneity gives rise to a stronger condition on the initial data.

Lemma 21. *Let u be a strong energy solution of the thin-film equation on a domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$, with nonnegative initial data $u_0 \in H^1(\Omega)$ with bounded support and let $n \in (1, 3)$, $\alpha \in (-1, 0)$. Suppose that u satisfies the α entropy estimate. Setting $b := n + \alpha$, assume that the conditions (H1), (H2), (H3), (H4) preceding Lemma 18 are satisfied. Given $\gamma \in [-20; -1]$, suppose furthermore that*

(H5) The condition

$$\left(2b - \frac{1}{2}n\right) \frac{\gamma - 3}{(b+1)^2(\gamma - 2)} - \frac{1}{b+1} \geq \tau$$

is satisfied for some $\tau \in (0, 1)$.

Let M be the closure of a C^4 domain and let $x_0 \in \partial M$; w.l.o.g. we may assume that $x_0 = 0$. Denote the tangent plane to ∂M in 0 by H; w.l.o.g. (i.e. possibly after a rotation and reflection) we may assume that $H = \{x \in$

$\mathbb{R}^d : x_d = 0\}$ *and that* $x_0 + \mu \vec{e_d} \in M$ *for any* $\mu > 0$ *small enough. Denote the projection onto* H *by* P. *Define*

$$Z_\rho := \{x : |Px| < \rho, |x_d| < \rho\} \ . \tag{20}$$

Let $R > 0$ *and let* $\xi : H \to \mathbb{R}$, $\xi \in C^4$, *be a function such that*

$$Z_R \cap M = Z_R \cap \{x \in \mathbb{R}^d : x_d \geq \xi(Px)\} \tag{21}$$

holds (for R *small enough such a function exists by the implicit function theorem). Note that* $\xi(0) = 0$ *and that* $\nabla\xi(0) = 0$ *as* H *is tangent to* ∂M *at* 0.

Assume that $Z_R \subset\subset \Omega$.

Take any $r \in \left(0, \frac{R}{3}\right)$ *and any* $K \in \mathbb{R}_0^+$ *such that*

(P1) $\operatorname{supp} u_0 \cap Z_{3r} \subset M$, *i.e. locally near* x_0 *the support of* u_0 *is contained in* M.

(P2) $|D^2\xi(Px)| \leq K$, $|D^3\xi(Px)| \leq \frac{K}{r}$, *and* $|D^4\xi(Px)| \leq \frac{K}{r^2}$ *for any* $x \in \mathbb{R}^d$ *with* $|Px| \leq 3r$.

(P3) The inequality $Kr < \epsilon(d,n)\tau$ *holds for some small constant* $\epsilon(d,n) < \frac{1}{10}$ *which is to be determined in the course of the proof below.*

Then there exist constants $C_0(d,n) > 0$, $C(d,n) > 0$ *such that for any* $\delta \in (0,r)$ *the following statement holds: Setting*

$$T := \inf\{t > 0 : \operatorname{supp} u(.,t) \cap (\mathbb{R}^d \setminus M) \cap Z_{3r} \neq \emptyset\}$$

and assuming that the estimate

$$\begin{aligned} &\int_{\{x:|Px|<r,|x_d|<2r\}} u_0^{1+\alpha} \cdot |x_d - \xi(Px) + \delta|^\gamma \ dx \\ \geq & C_0(d,n) r^4 (Kr^2)^{\gamma-4} \int_0^{\tilde{T}} \int_\Omega |\nabla u^{\frac{1+n+\alpha}{4}}(.,t)|^4 \ dx \ dt \end{aligned} \tag{22}$$

is satisfied for some $\tilde{T} > 0$, *we have*

$$\begin{aligned} T \leq & \frac{C(d,n,\alpha)}{\tau} \cdot \left(r^{d-1} \int_\delta^{C(d)r} z^{\gamma + 4\frac{1+\alpha}{n}} \ dz \right)^{\frac{n}{1+\alpha}} \\ & \cdot \left(\int_{\{x:|Px|<r,|x_d|<2r\}} u_0^{1+\alpha} |x_d - \xi(Px) + \delta|^\gamma \ dx \right)^{-\frac{n}{1+\alpha}} \end{aligned} \tag{23}$$

if the expression on the right-hand side does not exceed $\tilde{T}$.

Proof. The proof is somewhat analogous to the proof of Lemma 20; however, we additionally make heavy use of cutoff arguments.

Set

$$\epsilon(d,n) := \min\left(\epsilon_0, \epsilon_1, \frac{1}{10}\right) \tag{24}$$

where ϵ_0 and ϵ_1 are to be chosen below depending only on d and n. From now on, to simplify notation we write ϵ instead of $\epsilon(d,n)$.

Take a smooth cutoff $\phi : \mathbb{R}^d \to \mathbb{R}$ with $0 \leq \phi \leq 1$, $\phi \equiv 1$ on Z_{2r}, $\operatorname{supp}\phi \subset Z_{3r}$, and $|\nabla\phi| \leq \frac{C(d)}{r}$, $|D^2\phi| \leq \frac{C(d)}{r^2}$, $|D^3\phi| \leq \frac{C(d)}{r^3}$, $|D^4\phi| \leq \frac{C(d)}{r^4}$.

Define $\tilde{\xi} : H \to \mathbb{R}$ by

$$\tilde{\xi}(x) := \xi(x) - Kr^{-3}(|x| - r)_+^5 \ . \tag{25}$$

It is immediate that $\tilde{\xi} \in C^4$. The function $\tilde{\xi}$ satisfies some estimates similar to (P2), namely:

(P2') We have $|D^2\tilde{\xi}(Px)| \leq C(d)K$, $|D^3\tilde{\xi}(Px)| \leq \frac{C(d)K}{r}$, and $|D^4\tilde{\xi}(Px)| \leq \frac{C(d)K}{r^2}$ for any $x \in \mathbb{R}^d$ with $|Px| \leq 3r$.

We set $\psi(x) := |x_d - \tilde{\xi}(Px) + \delta|^\gamma \phi^2(x)$. This function obviously satisfies $\psi \in C^4(M)$ (as the points at which the function has singularities do not belong to M). In Lemma 22 below, additional properties of this test function which we shall need in the sequel are proved.

By the assumptions of our Lemma, Lemma 18 is applicable. Assuming that $t_1, t_2 < T$, we use ψ as a test function in Lemma 18 (this is possible by the definition of T and the definition of ψ: for $t < T$ we have $\operatorname{supp} u(.,t) \cap Z_{3r} \setminus M = \emptyset$ and $\operatorname{supp}\psi \subset Z_{3r}$). Making use of the estimates (30) and (31) from the lemma below, we obtain for a.e. $t_1, t_2 \in [0,T)$ with $t_2 > t_1$ and a.e.

$t_2 \in [0,T)$ in case $t_1 = 0$

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi\ dx \\
\geq &\left(\frac{2}{3}b - \frac{1}{6}n\right) \int_{t_1}^{t_2} \int_\Omega u^{b-1} |\nabla u|^2 \Delta\psi + \left(\frac{4}{3}b - \frac{1}{3}n\right) \int_{t_1}^{t_2} \int_\Omega u^{b-1} \nabla u \cdot D^2\psi \cdot \nabla u \\
&- \frac{1}{b+1} \int_{t_1}^{t_2} \int_\Omega u^{b+1} \Delta^2 \psi \\
= &\frac{\frac{8}{3}b - \frac{4}{6}n}{(b+1)^2} \int_{t_1}^{t_2} \int_\Omega |\nabla u^{\frac{b+1}{2}}|^2 \Delta\psi + \frac{\frac{16}{3}b - \frac{4}{3}n}{(b+1)^2} \int_{t_1}^{t_2} \int_\Omega \nabla u^{\frac{b+1}{2}} \cdot D^2\psi \cdot \nabla u^{\frac{b+1}{2}} \\
&- \frac{1}{b+1} \int_{t_1}^{t_2} \int_\Omega u^{b+1} \Delta^2 \psi \\
\geq &\left(\gamma(\gamma-1) \frac{\frac{8}{3}b - \frac{4}{6}n}{(b+1)^2} - C(d,n)Kr\right) \qquad (26) \\
&\qquad \cdot \int_{t_1}^{t_2} \int_\Omega |\nabla u^{\frac{b+1}{2}}|^2 \cdot \phi^2(x) |x_d - \tilde{\xi}(x) + \delta|^{\gamma-2} \\
&+ \gamma(\gamma-1) \frac{\frac{16}{3}b - \frac{4}{3}n}{(b+1)^2} \int_{t_1}^{t_2} \int_\Omega |\partial_d u^{\frac{b+1}{2}}|^2 \cdot \phi^2(x) |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \\
&- \left(\frac{\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{b+1} + C(d,n)Kr\right) \\
&\qquad \cdot \int_{t_1}^{t_2} \int_\Omega u^{b+1} \cdot \phi^2(x) |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4} \\
&- C(d,n)(Kr^2)^{\gamma-2} \int_{t_1}^{t_2} \int_{Z_{3r}} |\nabla u^{\frac{b+1}{2}}|^2 - C(d,n)(Kr^2)^{\gamma-4} \int_{t_1}^{t_2} \int_{Z_{3r}} u^{b+1}
\end{aligned}
$$

where we have used the fact that $\left|\frac{\frac{8}{3}b - \frac{4}{6}n}{(b+1)^2}\right| \leq C(d,n)$ by assumption (H1), the fact that $\left|\frac{1}{b+1}\right| \leq C(d,n)$ again by (H1), and the fact that $\operatorname{supp}\psi \subset Z_{3r}$.

By the assumption $Kr \leq \tau\epsilon$ (see (P3)), assumptions (H1) and (H3) and the condition $-20 \leq \gamma \leq -1$, we see that

$$
\begin{aligned}
&\left(\gamma(\gamma-1) \frac{\frac{8}{3}b - \frac{4}{6}n}{(b+1)^2} - C(d,n)Kr\right) \geq \left(\gamma(\gamma-1) \frac{\frac{8}{3}b - \frac{4}{6}n}{(b+1)^2} - C(d,n)\epsilon\tau\right) \\
\geq &\left(2 \frac{\frac{6}{3}b - \frac{2}{3}}{9} - C(d,n)\epsilon\tau\right) \geq \left(\frac{8}{27} - C(d,n)\epsilon\tau\right) \qquad (27)
\end{aligned}
$$

Thus, by $\tau < 1$ (see (H5)) we see that the prefactor of the first term on the right-hand side of (26) is nonnegative if we choose ϵ_1 small enough depending only on d and n. Thus we can estimate this term from below by dropping the derivatives in directions perpendicular to $\vec{e}_d$. Additionally taking into account our assumption $Kr \le \epsilon\tau$, we obtain

$$\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi\ dx \\
\ge& \left(\gamma(\gamma-1)\frac{8b-2n}{(b+1)^2} - C(d,n)\epsilon\tau\right) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega |\partial_d u^{\frac{b+1}{2}}|^2 \cdot \phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \\
&- \left(\frac{\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{b+1} + C(d,n)\epsilon\tau\right) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1} \cdot \phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4} \\
&- C(d,n)(Kr^2)^{\gamma-2}\int_{t_1}^{t_2}\int_{Z_{3r}} |\nabla u^{\frac{b+1}{2}}|^2 - C(d,n)(Kr^2)^{\gamma-4}\int_{t_1}^{t_2}\int_{Z_{3r}} u^{b+1}\ ,
\end{aligned}$$

where the prefactor of the first term on the right-hand side is still nonnegative (the term $\gamma(\gamma-1)\frac{\frac{16}{3}b-\frac{4}{3}n}{(b+1)^2}$, by which the prefactor has increased, is nonnegative as shown in (27)).

We now put ϕ under the derivative in the first term on the right-hand side and (in the second inequality below) use the first assertion of Lemma 22 below, the fact that $\operatorname{supp} u(.,t) \cap Z_{3r} \subset M$ for $t < T$ (recall also $\operatorname{supp} \phi \subset Z_{3r}$), as well as the estimate $|\nabla\phi| \le \frac{C(d)}{r} \le \frac{C(d)}{Kr^2}$ (recall that $Kr \le \epsilon\tau \le 1$) and

Young's inequality to obtain

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi\ dx \\
\geq &\left(\gamma(\gamma-1)\frac{8b-2n}{(b+1)^2} - C(d,n)\epsilon\tau\right)\int_{t_1}^{t_2}\int_\Omega |\partial_d(u^{\frac{b+1}{2}}\phi)|^2 \cdot |x_d - \tilde{\xi}(Px)+\delta|^{\gamma-2} \\
&- C(d,n)\int_{t_1}^{t_2}\int_\Omega |\partial_d u^{\frac{b+1}{2}}|\ u^{\frac{b+1}{2}}\ |\nabla\phi|\ \phi \cdot |x_d - \tilde{\xi}(Px)+\delta|^{\gamma-2} \\
&- C(d,n)\int_{t_1}^{t_2}\int_\Omega u^{b+1}|\nabla\phi|^2 \cdot |x_d - \tilde{\xi}(Px)+\delta|^{\gamma-2} \\
&- \left(\frac{\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{b+1} + C(d,n)\epsilon\tau\right) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d - \tilde{\xi}(Px)+\delta|^{\gamma-4} \\
&- C(d,n)(Kr^2)^{\gamma-2}\int_{t_1}^{t_2}\int_{Z_{3r}} |\nabla u^{\frac{b+1}{2}}|^2 - C(d,n)(Kr^2)^{\gamma-4}\int_{t_1}^{t_2}\int_{Z_{3r}} u^{b+1} \\
\geq &\left(\gamma(\gamma-1)\frac{8b-2n}{(b+1)^2} - C(d,n)\epsilon\tau\right)\int_{t_1}^{t_2}\int_\Omega |\partial_d(u^{\frac{b+1}{2}}\phi)|^2 \cdot |x_d - \tilde{\xi}(Px)+\delta|^{\gamma-2} \\
&- \left(\frac{\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{b+1} + C(d,n)\epsilon\tau\right) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d - \tilde{\xi}(Px)+\delta|^{\gamma-4} \\
&- C(d,n)(Kr^2)^{\gamma-2}\int_{t_1}^{t_2}\int_{Z_{3r}} |\nabla u^{\frac{b+1}{2}}|^2 \\
&- C(d,n)(Kr^2)^{\gamma-4}\int_{t_1}^{t_2}\int_{Z_{3r}} u^{b+1}\ .
\end{aligned}
$$

The prefactor of the first term on the right-hand side did not change and so is still nonnegative.

An application of Fubini's theorem and the one-dimensional Hardy inequality (see Lemma 19) on all the lines $\{x : Px = y\}$, $y \in H$, with the weight $\psi = |x_d - \tilde{\xi}(y)+\delta|^{\gamma-2}$ (note that this function has its singularity at $x_d = \tilde{\xi}(y)-\delta$, so we must check that $(u^{\frac{b+1}{2}}\phi)(x_1,\ldots,x_{d-1},\cdot,t)$ is zero on some neighbourhood of $\tilde{\xi}(y)-\delta$; this check is performed easily since $\operatorname{supp} u(.,t) \subset M$ and since $x_d \geq \xi(Px) \geq \tilde{\xi}(Px)$ for $x \in M \cap Z_{3r}$), yields (recall that the prefactor of the first term on the right-hand side is nonnegative)

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi\ dx \\
&\geq \left(\gamma(\gamma-1)\frac{8b-2n}{(b+1)^2} - C(d,n)\epsilon\tau\right) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_H\int_{\mathbb{R}} |\partial_d(u^{\frac{b+1}{2}}\phi)|^2 \cdot |x_d - \tilde{\xi}(Px)+\delta|^{\gamma-2}\ dx_d\ d(x_1,\dots,x_{d-1})\ dt \\
&\quad - \left(\frac{\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{b+1} + C(d,n)\epsilon\tau\right) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d-\tilde{\xi}(Px)+\delta|^{\gamma-4} \\
&\quad - C(d,n)(Kr^2)^{\gamma-2}\int_{t_1}^{t_2}\int_{Z_{3r}} u^{\frac{b+1}{2}}|\nabla u^{\frac{b+1}{4}}|^2 \\
&\quad - C(d,n)(Kr^2)^{\gamma-4}\int_{t_1}^{t_2}\int_{Z_{3r}} u^{b+1} \\
&\geq \left(\gamma(\gamma-1)(\gamma-3)^2\frac{2b-\frac{1}{2}n}{(b+1)^2} - C(d,n)\epsilon\tau\right) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d-\tilde{\xi}(Px)+\delta|^{\gamma-4} \\
&\quad - \left(\frac{\gamma(\gamma-1)(\gamma-2)(\gamma-3)}{b+1} + C(d,n)\epsilon\tau\right) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d-\tilde{\xi}(Px)+\delta|^{\gamma-4} \\
&\quad - C(d,n)(Kr^2)^{\gamma}\int_{t_1}^{t_2}\int_{Z_{3r}} |\nabla u^{\frac{b+1}{4}}|^4 \\
&\quad - C(d,n)(Kr^2)^{\gamma-4}\int_{t_1}^{t_2}\int_{Z_{3r}} u^{b+1}
\end{aligned}
$$

where we have used the fact that $-20 \leq \gamma \leq -1$ and applied Young's inequality to the penultimate term. Assumption (H5) now gives

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi\ dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi\ dx \\
&\geq (\gamma(\gamma-1)(\gamma-2)(\gamma-3)\tau - C(d,n)\epsilon\tau) \qquad (28) \\
&\qquad \cdot \int_{t_1}^{t_2}\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d-\tilde{\xi}(Px)+\delta|^{\gamma-4} \\
&\quad - C(d,n)(Kr^2)^{\gamma}\int_{t_1}^{t_2}\int_{Z_{3r}} |\nabla u^{\frac{b+1}{4}}|^4 \\
&\quad - C(d,n)(Kr^2)^{\gamma-4}\int_{t_1}^{t_2}\int_{Z_{3r}} u^{b+1}\ .
\end{aligned}
$$

Using $\gamma \leq -1$, we see that choosing ϵ_0 small enough depending only on n and d we can enforce that $\gamma(\gamma-1)(\gamma-2)(\gamma-3)\tau - C(d,n)\epsilon\tau > \tau$.

Recall that $\operatorname{supp} u(.,t) \cap Z_{3r} \cap \Omega \setminus M = \emptyset$ for any $0 \leq t < T$. Since $Kr \leq \epsilon\tau \leq \frac{1}{10}$ (by our choice of ϵ in (24) and by $\tau \leq 1$) and $|\xi(Px)| \leq 9Kr^2 < r$ in case $|Px| \leq 3r$ (due to $\xi(0) = 0$, $D\xi(0) = 0$, and $|D^2\xi| \leq K$) by (21) we see that $\operatorname{supp} u(.,t) \cap \{x : |Px| < 3r, x_d \in (-3r,-r)\} = \emptyset$ for any $t \in [0,T)$. Therefore we may apply Fubini's theorem and the one-dimensional Poincare inequality on the one-dimensional segments $\{x : Px = y, x_d \in (-3r,3r)\}$, $y \in H$, to estimate

$$
\begin{aligned}
&\int_{t_1}^{t_2} \int_{Z_{3r}} u^{b+1} \; dx \; dt \\
=&\int_{t_1}^{t_2} \int_{PZ_{3r}} \int_{\{x_d:|x_d|<3r\}} u^{b+1} \; dx_d \; d(x_1,\ldots,x_{d-1}) \; dt \\
\leq&\int_{t_1}^{t_2} \int_{PZ_{3r}} C(d)(4r)^4 \int_{\{x_d:|x_d|<3r\}} |\nabla u^{\frac{b+1}{4}}|^4 \; dx_d \; d(x_1,\ldots,x_{d-1}) \; dt \\
=&C(d)r^4 \int_{t_1}^{t_2} \int_{Z_{3r}} |\nabla u^{\frac{b+1}{4}}|^4 \; dx \; dt \; .
\end{aligned}
$$

Putting these considerations together, we obtain from (28)

$$
\begin{aligned}
&\int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi \; dx - \int_\Omega \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi \; dx \\
\geq&\tau \int_{t_1}^{t_2} \int_\Omega u^{b+1} \cdot \phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4} \\
&- C(d,n)(r^4(Kr^2)^{\gamma-4} + (Kr^2)^\gamma) \int_{t_1}^{t_2} \int_{Z_{3r}} |\nabla u^{\frac{b+1}{4}}|^4 \; .
\end{aligned}
\tag{29}
$$

Hölder's inequality implies (since $\operatorname{supp} u(.,t) \cap \operatorname{supp} \phi \subset M$ for $t < T$)

$$
\begin{aligned}
&\int_\Omega u^{1+\alpha} \cdot \phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^\gamma \; dx \\
\leq&\left(\int_\Omega u^{b+1} \cdot \phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4} \; dx\right)^{\frac{1+\alpha}{b+1}} \\
&\cdot \left(\int_M \phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma+4\frac{1+\alpha}{n}} \; dx\right)^{\frac{n}{b+1}}
\end{aligned}
$$

Estimating the second integral on the right-hand side using Fubini's theorem, the local representation of M (see (21)), the definition of $\tilde{\xi}$ (see (25)), the fact that $\delta \leq r$ (see the assumptions of the lemma), the estimate $\xi(Px) \geq \tilde{\xi}(Px)$, the fact that $\operatorname{supp} \phi \subset Z_{3r}$, and the estimate $|\tilde{\xi}(Px)| \leq C(d)Kr^2 \leq C(d)r$ for $x \in Z_{3r}$ (recall $\tilde{\xi}(0) = 0$, $D\tilde{\xi}(0) = 0$, and $|D^2\tilde{\xi}(Px)| \leq C(d)K$ for $x \in Z_{3r}$), we get

$$\begin{aligned}
&\left(\int_\Omega u^{1+\alpha}\cdot\phi^2(x)|x_d-\tilde{\xi}(Px)+\delta|^\gamma\ dx\right)^{\frac{b+1}{1+\alpha}}\\
\leq&\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d-\tilde{\xi}(Px)+\delta|^{\gamma-4}\ dx\\
&\cdot\left(\int_{PZ_{3r}}\int_{\xi(y)}^{3r}|z-\tilde{\xi}(y)+\delta|^{\gamma+4\frac{1+\alpha}{n}}\ dz\ dy\right)^{\frac{n}{1+\alpha}}\\
=&\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d-\tilde{\xi}(Px)+\delta|^{\gamma-4}\ dx\\
&\cdot\left(\int_{PZ_{3r}}\int_{\xi(y)-\tilde{\xi}(y)+\delta}^{3r-\tilde{\xi}(y)+\delta}|z|^{\gamma+4\frac{1+\alpha}{n}}\ dz\ dy\right)^{\frac{n}{1+\alpha}}\\
\leq&\int_\Omega u^{b+1}\cdot\phi^2(x)|x_d-\tilde{\xi}(Px)+\delta|^{\gamma-4}\ dx\\
&\cdot\left(C(d)r^{d-1}\int_{\delta}^{C_3(d)r}|z|^{\gamma+4\frac{1+\alpha}{n}}\ dz\right)^{\frac{n}{1+\alpha}}\ .
\end{aligned}$$

Plugging this estimate into (29), multiplying by $1+\alpha$ and using $\delta\leq r$ we see that

$$\begin{aligned}
&\int_\Omega u^{1+\alpha}(.,t_2)\psi\ dx-\int_\Omega u^{1+\alpha}(.,t_1)\psi\ dx\\
\geq&c_1(d,n,\alpha)\tau\left(r^{d-1}\int_\delta^{C_3(d)r}z^{\gamma+4\frac{1+\alpha}{n}}\ dz\right)^{-\frac{n}{1+\alpha}}\cdot\int_{t_1}^{t_2}\left(\int_\Omega u^{1+\alpha}\psi\ dx\right)^{\frac{b+1}{1+\alpha}}\ dt\\
&-C_2(d,n)(r^4(Kr^2)^{\gamma-4}+(Kr^2)^\gamma)\int_{t_1}^{t_2}\int_\Omega|\nabla u^{\frac{b+1}{4}}|^4\ dx\ dt
\end{aligned}$$

holds for a.e. $t_1,t_2\in[0,T)$ with $t_2>t_1$ and a.e. $t_2\in[0,T)$ in case $t_1=0$.

We therefore have derived a differential inequality for $\int_\Omega u^{1+\alpha}(.,t)\psi\ dx$. By the comparison principle, the solution of the corresponding differential equation yields a lower bound on $\int_\Omega u^{1+\alpha}(.,t)\psi\ dx$ (as the right-hand side of our differential inequality is locally Lipschitz with respect to the solution). The corresponding differential equation reads

$$\begin{aligned}
\frac{d}{dt}f=&c_1(d,n,\alpha)\tau\left(r^{d-1}\int_\delta^{C_3(d)r}z^{\gamma+4\frac{1+\alpha}{n}}\ dz\right)^{-\frac{n}{1+\alpha}}\cdot f^{\frac{b+1}{1+\alpha}}\\
&-C_2(d,n)(r^4(Kr^2)^{\gamma-4}+(Kr^2)^\gamma)\int_\Omega|\nabla u^{\frac{b+1}{4}}|^4\ dx
\end{aligned}$$

and the initial condition is $f(0)=\int_\Omega u_0^{1+\alpha}\psi\ dx$.

Fixing some $\tilde{T}\in(0,T)$ we can show that the solution f is bounded from below at all $t \in [0, \tilde{T}]$ by the solution g of

$$\frac{d}{dt} g = c_1(d,n,\alpha)\tau \left(r^{d-1} \int_\delta^{C_3(d)r} z^{\gamma+4\frac{1+\alpha}{n}} \, dz \right)^{-\frac{n}{1+\alpha}} \cdot g^{\frac{b+1}{1+\alpha}}$$

with initial data

$$\begin{aligned} g(0) :=& \int_\Omega u_0^{1+\alpha} \psi \, dx \\ & - C_2(d,n)(r^4 (Kr^2)^{\gamma-4} + (Kr^2)^\gamma) \int_0^{\tilde{T}} \int_\Omega |\nabla u^{\frac{b+1}{4}}|^4 \, dx \, dt \ , \end{aligned}$$

provided that we have $g(0) > 0$.

It suffices to prove $f \geq g_\mu$ in $[0, \tilde{T}]$ for all g_μ solving the same differential equation as g, but with initial data $g_\mu(0) := g(0) - \mu > 0$: We know that $g_\mu(t)$ converges to $g(t)$ as $\mu \to 0$ for any fixed $t \geq 0$. To prove $f \geq g_\mu$ in $[0, \tilde{T}]$, we argue by contradiction and assume that $t_\mu := \inf\{t \in [0, \tilde{T}] : g_\mu(t) > f(t)\} < \infty$. This gives

$$\begin{aligned} & f(t_\mu) - g_\mu(t_\mu) \\ =& f(0) - g_\mu(0) - C_2(d,n)(r^4(Kr^2)^{\gamma-4} + (Kr^2)^\gamma) \int_0^{t_\mu} \int_\Omega |\nabla u^{\frac{b+1}{4}}|^4 \, dx \, dt \\ & + c_1(d,n,\alpha)\tau \left(r^{d-1} \int_\delta^{C_3(d)r} z^{\gamma+4\frac{1+\alpha}{n}} \, dz \right)^{-\frac{n}{1+\alpha}} \cdot \int_0^{t_\mu} f^{\frac{b+1}{1+\alpha}}(t) - g_\mu^{\frac{b+1}{1+\alpha}}(t) \, dt \\ \geq& f(0) - g_\mu(0) - C_2(d,n)(r^4(Kr^2)^{\gamma-4} + (Kr^2)^\gamma) \int_0^{\tilde{T}} \int_\Omega |\nabla u^{\frac{b+1}{4}}|^4 \, dx \, dt \\ \geq& \mu \end{aligned}$$

where we have used the fact that $f(t) \geq g_\mu(t)$ for $t < t_\mu$ and the definition of $g_\mu(0)$ to obtain the desired contradiction (due to continuity of f and g_μ, the definition of t_μ would imply that $g_\mu(t_\mu) \geq f(t_\mu)$).

We now choose $C_0(d,n) := 4C_2(d,n)$ in condition (22). Using the estimate $(Kr^2)^\gamma \leq r^4(Kr^2)^{\gamma-4}$ (which holds since $Kr \leq \tau\epsilon \leq 1$ by the conditions on τ and ϵ) as well as the fact that $\xi(Px) = \tilde{\xi}(Px)$ for $|Px| \leq r$ and the fact that $\phi \equiv 1$ on Z_{2r}, we see that (22) then implies

$$\begin{aligned} g(0) \geq& \int_{\{x:|Px|<r,|x_d|<2r\}} u_0^{1+\alpha} |x_d - \xi(Px) + \delta|^\gamma \, dx \\ & - 2C_2(d,n) r^4 (Kr^2)^{\gamma-4} \int_0^{\tilde{T}} \int_\Omega |\nabla u^{\frac{b+1}{4}}|^4 \, dx \, dt \\ \geq& \frac{1}{2} \int_{\{x:|Px|<r,|x_d|<2r\}} u_0^{1+\alpha} |x_d - \xi(Px) + \delta|^\gamma \, dx \ . \end{aligned}$$

Since the equation for g can be solved explicitly (the solution of $\frac{d}{dt}g(t) = q \cdot [g(t)]^m$ is $g(t) = [g(0)^{1-m} - (m-1) \cdot q \cdot t]^{\frac{1}{1-m}}$), this implies that g and therefore f and therefore also $\int_\Omega u^{1+\alpha}(.,t)\psi\, dx$ needs to blow up before time

$$\frac{1+\alpha}{c_1(d,n,\alpha)\cdot n \cdot \tau} \cdot \left(r^{d-1} \int_\delta^{C_3(d)r} z^{\gamma+4\frac{1+\alpha}{n}}\, dz \right)^{\frac{n}{1+\alpha}} \cdot \left(\frac{1}{2} \int_{\{x:|Px|<r, |x_d|<2r\}} u_0^{1+\alpha} |x_d - \xi(Px) + \delta|^\gamma \, dx \right)^{-\frac{n}{1+\alpha}}$$

if this quantity does not exceed $\tilde{T}$.

This yields an upper bound on T: we know that $\phi^2(x)|x_d - \tilde{\xi}(Px) + \delta|^\gamma$ is compactly supported and bounded from above by δ^γ on M; moreover, for $t < T$ we have $\operatorname{supp} u(.,t) \cap \operatorname{supp} \phi \subset M$. As $\int_\Omega u(.,t)\, dx = \int_\Omega u_0\, dx < \infty$ by conservation of mass, by Hölder's inequality $\int_\Omega u^{1+\alpha}(.,t)\psi\, dx$ must remain bounded for $t < T$. Thus, if this quantity blows up at some time T' we necessarily have $T' \geq T$.

This finishes the proof of the lemma. □

Lemma 22. *With ϕ defined as at the beginning of the proof of the previous lemma, for any $x \in M \cap \operatorname{supp} \nabla\phi$ we have $x_d - \tilde{\xi}(Px) \geq Kr^2$.*

Moreover, with ψ defined as in the proof of the previous lemma, the following estimate holds for the second derivative of ψ for any $x \in M$:

$$\left| D^2\psi(x) - \gamma(\gamma-1)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \cdot \phi^2(x) \cdot \vec{e}_d \otimes \vec{e}_d \right| \leq C(d)Kr|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \cdot \phi^2(x) + C(d)[Kr^2]^{\gamma-2} \tag{30}$$

For the fourth derivative, the following estimate is satisfied for any $x \in M$:

$$\left| \Delta^2\psi(x) - \gamma(\gamma-1)(\gamma-2)(\gamma-3)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4} \cdot \phi^2(x) \right| \leq C(d)Kr|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4} \cdot \phi^2(x) + C(d)[Kr^2]^{\gamma-4} \tag{31}$$

Proof. For $x \in Z_{2r}$, we have $\phi(x) = 1$; additionally we have $\operatorname{supp} \phi \subset Z_{3r}$. Thus for $x \in M \cap \operatorname{supp} \nabla\phi$ we know $x \in Z_{3r}$; moreover, we either have $|x_d| \geq 2r$ or $|Px| \geq 2r$.

- In the latter case, by definition of $\tilde{\xi}$ (see (25)) we obtain $\xi(Px) - \tilde{\xi}(Px) \geq Kr^2$ which implies $x_d - \tilde{\xi}(Px) \geq Kr^2$ (since $x_d \geq \xi(Px)$ due to $x \in M \cap Z_{3r}$ and (21)).

- To deal with the former case, we observe that $|D\xi(Px)| \leq K \cdot 3r$ for $x \in Z_{3r}$ by our assumption (P2) since $|D^2\xi(Px)| \leq K$ for $x \in Z_{3r}$ and $D\xi(0) = 0$; using $\xi(0) = 0$ this implies $|\xi(Px)| \leq K(3r)^2 = 9Kr^2 \leq$

$9\epsilon(d,n)r \le r$ for $x \in Z_{3r}$ by (P3), our choice of ϵ (see (24)), and $0 < \tau < 1$. Thus $|x_d| \ge 2r$ and $x \in M \cap Z_{3r}$ imply $x_d \ge 2r$, the case $x_d \le -2r$ being impossible (as $x \in M \cap Z_{3r}$ implies $x_d \ge \xi(Px) \ge -r$). This gives $x_d - \xi(Px) \ge 2r - r = r \ge Kr^2$ by condition (P3), our choice of ϵ (24), and $0 < \tau < 1$. Since we have $\tilde{\xi}(Px) \le \xi(Px)$ by (25), we deduce $x_d - \tilde{\xi}(Px) \ge Kr^2$.

This finishes the proof of the first assertion.

We calculate for $x \in M \cap Z_{3r}$ (which implies $x_d \ge \xi(Px) \ge \tilde{\xi}(Px)$)

$$\begin{aligned} &D^2(|x_d - \tilde{\xi}(Px) + \delta|^\gamma) \\ =&\gamma(\gamma-1)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \cdot (\vec{e}_d - D\tilde{\xi}(Px)) \otimes (\vec{e}_d - D\tilde{\xi}(Px)) \\ &- \gamma|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1} \cdot D^2\tilde{\xi}(Px) \end{aligned} \tag{32}$$

(where we think of $D\tilde{\xi}$ as taking values in $\mathbb{R}^d$, $\mathbb{R}^d$ being a superspace of the tangent space of H; we also think of $D^2\tilde{\xi}$ as taking values in $\mathbb{R}^{d\times d}$) and using $-20 \le \gamma \le -1$ we obtain

$$\begin{aligned} &\left|\Delta^2(|x_d - \tilde{\xi}(Px) + \delta|^\gamma) - \gamma(\gamma-1)(\gamma-2)(\gamma-3)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4}\right| \\ \le\ & C(d)|D\tilde{\xi}(Px)| \cdot (1 + |D\tilde{\xi}(Px)|)^3 \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4} \\ &+ C(d)|D^2\tilde{\xi}(Px)| \cdot (1 + |D\tilde{\xi}(Px)|)^2 \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-3} \\ &+ C(d)\left(|D^3\tilde{\xi}(Px)| \cdot (1 + |D\tilde{\xi}(Px)|) + |D^2\tilde{\xi}(Px)|^2\right) \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \\ &+ C(d)|D^4\tilde{\xi}(Px)| \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1} . \end{aligned} \tag{33}$$

From (32), for $x \in M$ it follows that

$$\begin{aligned} &\Big|D^2(|x_d - \tilde{\xi}(Px) + \delta|^\gamma \phi^2(x)) \\ &\quad - \gamma(\gamma-1)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \cdot \phi^2(x) \cdot \vec{e}_d \otimes \vec{e}_d\Big| \\ \le\ & C(d)(|D\tilde{\xi}(Px)| + |D\tilde{\xi}(Px)|^2)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2}\phi^2(x) \\ &+ C(d)|D^2\tilde{\xi}(Px)|\ |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1}\phi^2(x) \\ &+ C(d)\sup_x |D\phi(x)| \sup_{x\in \operatorname{supp} D\phi \cap M} \left[|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1}|\vec{e}_d - D\tilde{\xi}(Px)|\right] \\ &+ C(d)\sup_x(|D^2\phi(x)| + |D\phi(x)|^2) \sup_{x\in \operatorname{supp} D\phi \cap M} |x_d - \tilde{\xi}(Px) + \delta|^\gamma \\ \le\ & C(d)(Kr + K^2r^2)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2}\phi^2(x) \\ &+ C(d)K|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1}\phi^2(x) \\ &+ C(d)\sup_x |D\phi(x)| \sup_{x\in \operatorname{supp} D\phi \cap M} \left[|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1}|\vec{e}_d - D\tilde{\xi}(Px)|\right] \\ &+ C(d)\sup_x(|D^2\phi(x)| + |D\phi(x)|^2) \sup_{x\in \operatorname{supp} D\phi \cap M} |x_d - \tilde{\xi}(Px) + \delta|^\gamma \end{aligned}$$

where we have used the fact that $|D\tilde{\xi}(Px)| \leq |D\tilde{\xi}(0)| + C(d)K|Px| \leq C(d)Kr$ for $x \in Z_{3r}$ (by (P2') and $D\tilde{\xi}(0) = 0$) and that $\operatorname{supp}\phi \subset Z_{3r}$; moreover, we have made use of the estimate $|D^2\tilde{\xi}| \leq C(d)K$ (by (P2')).

We have $|x_d - \tilde{\xi}(Px) + \delta| \leq C(d)r$ for any $x \in \operatorname{supp}\phi$: it holds that $\operatorname{supp}\phi \subset Z_{3r}$; moreover we have $0 < \delta < r$ and $|\tilde{\xi}(Px)| \leq C(d)Kr^2 \leq C(d)r$ (by (P2'), by $\tilde{\xi}(0) = 0$, $D\tilde{\xi}(0) = 0$ and since $Kr \leq \epsilon\tau \leq 1$) for any $x \in Z_{3r}$.

Thus, the second term on the right-hand side in the previous inequality can be estimated from above by a constant times the first term on the right-hand side. Using the estimate on $D\tilde{\xi}$ and the bounds $|D\phi| \leq C(d)r^{-1}$ and $|D^2\phi| \leq C(d)r^{-2}$, we therefore obtain

$$\begin{aligned}
&\Big| D^2(|x_d - \tilde{\xi}(Px) + \delta|^\gamma \phi^2(x)) \\
&\qquad - \gamma(\gamma-1)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \cdot \phi^2(x) \cdot \vec{e}_d \otimes \vec{e}_d \Big| \\
&\leq C(d)(Kr + K^2r^2)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2}\phi^2(x) \\
&\quad + \frac{C(d)}{r}(1 + Kr) \sup_{x \in \operatorname{supp} D\phi \cap M} |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1} \\
&\quad + \frac{C(d)}{r^2} \sup_{x \in \operatorname{supp} D\phi \cap M} |x_d - \tilde{\xi}(Px) + \delta|^{\gamma} \ .
\end{aligned}$$

By this estimate, the inequality $Kr \leq \epsilon \cdot \tau \leq 1$ (the latter inequality holds due to our conditions on τ and our choice of ϵ), and the first assertion of the present lemma, we obtain (30).

We now derive a similar estimate for the fourth derivative. Using the estimates on the derivatives of $\tilde{\xi}$ (see (P2')), the estimate $|D\tilde{\xi}(Px)| \leq C(d)Kr$ for $x \in Z_{3r}$ (see the proof of (30)), the fact that $|x_d - \tilde{\xi}(Px) + \delta| \leq C(d)r$ for any $x \in \operatorname{supp}\phi$ (see the proof of (30)), and the fact that $Kr \leq 1$, inequality (33) implies for any $x \in M$

$$\begin{aligned}
&\Big| \phi^2(x)\Delta^2(|x_d - \tilde{\xi}(Px) + \delta|^\gamma) \\
&\qquad - \gamma(\gamma-1)(\gamma-2)(\gamma-3)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4}\phi^2(x) \Big| \\
&\leq C(d)Kr \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-4}\phi^2(x) \ .
\end{aligned}$$

Thus, by the Leibniz formula and the estimates on the derivatives of ϕ, we obtain for $x \in M$

$$
\begin{aligned}
&\Big|\Delta^2(|x_d-\tilde\xi(Px)+\delta|^\gamma\phi^2(x))\\
&\quad -\gamma(\gamma-1)(\gamma-2)(\gamma-3)|x_d-\tilde\xi(Px)+\delta|^{\gamma-4}\phi^2(x)\Big|\\
&\leq C(d)Kr\cdot|x_d-\tilde\xi(Px)+\delta|^{\gamma-4}\phi^2(x)\\
&\quad+\sum_{j=0}^{3}C(d)r^{-4+j}\sup_{x\in\operatorname{supp}D\phi\cap M}\Big|D^j|x_d-\tilde\xi(Px)+\delta|^\gamma\Big|\\
&\leq C(d)Kr\cdot|x_d-\tilde\xi(Px)+\delta|^{\gamma-4}\phi^2(x)\\
&\quad+\sum_{j=0}^{3}C(d)r^{-4+j}\sup_{x\in\operatorname{supp}D\phi\cap M}C(d)|x_d-\tilde\xi(Px)+\delta|^{\gamma-j}\\
&\leq C(d)Kr\cdot|x_d-\tilde\xi(Px)+\delta|^{\gamma-4}\phi^2(x) \qquad (34)\\
&\quad+C(d)\sum_{j=0}^{3}r^{-4+j}(Kr^2)^{\gamma-j}\ ,
\end{aligned}
$$

where in the third step we have used the first assertion of the lemma and in the second step we have used the estimate

$$
\begin{aligned}
\Big|D|x_d-\tilde\xi(Px)+\delta|^\gamma\Big| &\leq C(d)(1+Kr)|x_d-\tilde\xi(Px)+\delta|^{\gamma-1}\\
&\leq C(d)|x_d-\tilde\xi(Px)+\delta|^{\gamma-1}
\end{aligned}
$$

which one easily verifies using $|D\tilde\xi(Px)|\leq C(d)Kr$ for $x\in Z_{3r}$ and $Kr\leq 1$, the estimate

$$
\begin{aligned}
&\Big|D^2|x_d-\tilde\xi(Px)+\delta|^\gamma\Big|\\
\leq& C(d)(1+Kr+K^2r^2)|x_d-\tilde\xi(Px)+\delta|^{\gamma-2}+C(d)K|x_d-\tilde\xi(Px)+\delta|^{\gamma-1}\\
\leq& C(d)(1+Kr+K^2r^2)|x_d-\tilde\xi(Px)+\delta|^{\gamma-2}\\
\leq& C(d)|x_d-\tilde\xi(Px)+\delta|^{\gamma-2}
\end{aligned}
$$

which follows from (32) in connection with the bound $|D\tilde\xi(Px)|\leq C(d)Kr$ and the bounds (P2') as well as the bound $|x_d-\tilde\xi(Px)+\delta|\leq C(d)r$ for

$x \in Z_{3r}$ (see above) and the inequality $Kr \leq 1$, and the estimate

$$\begin{aligned}
&\left|D^3|x_d - \tilde{\xi}(Px) + \delta|^\gamma\right| \\
&\leq C(d)|D^3\tilde{\xi}(Px)| \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1} \\
&\quad + C(d)|D^2\tilde{\xi}(Px)| \cdot |\vec{e}_d - D\tilde{\xi}(Px)| \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \\
&\quad + C(d)|\vec{e}_d - D\tilde{\xi}(Px)|^3 |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-3} \\
&\leq C(d)\frac{K}{r}|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-1} \\
&\quad + C(d)K \cdot (1 + Kr) \cdot |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-2} \\
&\quad + C(d)(1 + Kr)^3 |x_d - \tilde{\xi}(Px) + \delta|^{\gamma-3} \\
&\leq C(d)|x_d - \tilde{\xi}(Px) + \delta|^{\gamma-3}
\end{aligned}$$

which is obtained by differentiating (32) and using $|D\tilde{\xi}(Px)| \leq C(d)Kr$ for $x \in Z_{3r}$ as well as (P2') and $|x_d - \tilde{\xi}(Px) + \delta| \leq C(d)r$ for $x \in Z_{3r}$ and the inequality $Kr \leq 1$.

Applying $Kr \leq 1$ to (34) we obtain (31).

□

We are now in position to prove our main theorem in the multidimensional case.

Proof of Theorem 5. Assertion a) is a consequence of Lemma 21: We set $b := \frac{9}{20}n + \frac{12}{20}$, $\gamma := -2$. For these choices, conditions (H1) to (H5) have already been checked in the proof of Theorem 3 (in case of (H5) for $\tau = \tau(n)$ sufficiently small).

W.l.o.g. we may assume that $x_0 = 0$, that $H = \{x \in \mathbb{R}^d : x_d = 0\}$, and that $x_0 + \mu\vec{e_d} \in M$ for any $\mu > 0$ small enough. Under the assumptions of Theorem 5, we can then find $R > 0$ such that in Z_R (as defined in (20)) our set M is the supergraph of a C^4 function $\xi : H \to \mathbb{R}$ with $D\xi(0) = 0$ and $\xi(0) = 0$; i.e. (21) holds. Set

$$K := \sup_{x \in Z_R} \max\left(|D^2\xi(Px)|, \frac{|D^3\xi(Px)|}{R}, \frac{|D^4\xi(Px)|}{R^2}\right) .$$

Then there exists $\tilde{R} \in (0, \frac{R}{3})$ such that $Z_{3\tilde{R}} \subset\subset \Omega$ and such that for any $r \in (0, \tilde{R}]$, the assumptions (P1), (P2) and (P3) of Lemma 21 are fulfilled.

Possibly decreasing $\tilde{R}$, we may enforce

$$|x_d - \xi(Px)| \leq 2\,\mathrm{dist}(x, \partial M) \tag{35}$$

for any $x \in Z_{\tilde{R}}$: if $\tilde{R}$ is small enough, we know that for $x \in Z_{\tilde{R}}$ we have $\mathrm{dist}(x, \partial M \setminus Z_R) > \mathrm{dist}(x, \partial M)$. In this case, as $Z_R \cap \partial M$ is given by the graph of ξ over $Z_R \cap H$, for $x \in Z_{\tilde{R}}$ we obtain

$$
\begin{aligned}
&\operatorname{dist}(x, \partial M)\\
=&\inf_{y \in H \cap Z_R} \sqrt{|Px-y|^2 + |x_d - \xi(y)|^2}\\
=&\inf_{y \in H \cap Z_R : |Px-y| \le |x_d - \xi(Px)|} \sqrt{|Px-y|^2 + |x_d - \xi(y)|^2}\ ,
\end{aligned}
$$

where in the second step we have used the fact $\operatorname{dist}(x, \partial M) \le |x_d - \xi(Px)|$. By the triangle inequality, we obtain

$$
\begin{aligned}
&\operatorname{dist}(x, \partial M)\\
\ge&\inf_{y \in H \cap Z_R : |Px-y| \le |x_d - \xi(Px)|} \left[\sqrt{|Px-y|^2 + |x_d - \xi(Px)|^2} - |\xi(Px) - \xi(y)|\right]\\
\ge&|x_d - \xi(Px)| - \sup_{y \in H \cap Z_R : |Px-y| \le |x_d - \xi(Px)|} |\xi(Px) - \xi(y)|\\
\ge&|x_d - \xi(Px)| - 3K\tilde{R}|x_d - \xi(Px)|\ ,
\end{aligned}
$$

where in the last step we have used the fact that $|D\xi(Pz)| \le 3K\tilde{R}$ for $z \in Z_{3\tilde{R}}$ (which follows from $D\xi(0) = 0$ and $|D^2\xi(z)| \le K$ for $z \in Z_{3\tilde{R}}$); note that $y \in Z_{3\tilde{R}}$ since otherwise $|Px - y| \le |x_d - \xi(Px)|$ could not hold: we have $x \in Z_{\tilde{R}}$ which implies $|x_d| \le \tilde{R}$, $|Px| < \tilde{R}$ as well as $|\xi(Px)| \le K\tilde{R}^2 \le \tilde{R}$ (since $\xi(0) = 0$, $D\xi(0) = 0$, $|D^2\xi| \le K$, $K\tilde{R} \le 1$). Thus we obtain by (P2) and (P3) (recall that we have already checked (P2) and (P3) for any $r \in (0, \tilde{R}]$)

$$
\operatorname{dist}(x, \partial M) \ge (1 - 3\epsilon\tau)|x_d - \xi(Px)|
$$

which finishes the proof of (35) since $\tau < 1$ and $\epsilon < \frac{1}{10}$.

From now on, let $r \in (0, \frac{\tilde{R}}{3})$.

It remains to check (22). Using $|\xi(Px)| \le Kr^2 \le \frac{r}{2}$ for $x \in H \cap Z_r$ (which follows from (P2), the fact that $D\xi(0) = 0$ and $\xi(0) = 0$, and the fact that $Kr \le \epsilon\tau \le \frac{1}{10}$), in case $\delta < \frac{r}{4}$ we have

$$
\begin{aligned}
&\int_{\{x:|Px|<r, |x_d|<2r\}} u_0^{1+\alpha} \cdot |x_d - \xi(Px) + \delta|^\gamma \, dx\\
\ge&\int_{\{x:|Px|<r, |x_d - \xi(Px)|<2\delta\}} u_0^{1+\alpha} \cdot |x_d - \xi(Px) + \delta|^{-2} \, dx\\
\ge&c(d)\delta^{\frac{4(1+\alpha)}{n} - 1} r^{d-1} \fint_{\{x:|Px|<r, |x_d - \xi(Px)|<2\delta\}} \left|\frac{1}{\delta^{\frac{4}{n}}} u_0\right|^{1+\alpha} dx\\
\ge&c(d)\delta^{\frac{4(1+\alpha)}{n} - 1} r^{d-1} \fint_{\{x:|Px|<r, |x_d|<r, \operatorname{dist}(x,\partial M)<\delta\}} \left|\frac{1}{\delta^{\frac{4}{n}}} u_0\right|^{1+\alpha} dx \qquad (36)
\end{aligned}
$$

where in the third step we have used (35) and the estimate

$$
\begin{aligned}
&\mathcal{L}^d(\{x : |Px| < r, |x_d - \xi(Px)| < 2\delta\})\\
\le&2\mathcal{L}^d(\{x : |Px| < r, |x_d| < r, \operatorname{dist}(x, \partial M) < \delta\}) \qquad (37)
\end{aligned}
$$

which holds since

$$\{x : |Px| < r, |x_d - \xi(Px)| < \delta\} \subset \{x : |Px| < r, |x_d| < r, \operatorname{dist}(x, \partial M) < \delta\} \ .$$

We now fix $\tilde{T} > 0$. Denote by $(r_i)_{i\in\mathbb{N}}$ a sequence for which the outer lim sup in the definition of W in Theorem 5 is approached. Denote by $(\delta_j^r)_j$ for fixed $r > 0$ a sequence for which the inner lim sup in the definition of W in Theorem 5 is approached.

Note that $\frac{4(1+\alpha)}{n} - 1 < 0$ since $\alpha < -\frac{1}{2}$ and $n > 2$. Using (36), we see that by our definition of W (note that $\operatorname{dist}_C(x, x_0) = \max(|Px|, |x_d|)$ since $x_0 = 0$ and $H = \{x \in \mathbb{R}^d : x_d = 0\}$) we have

$$\lim_{j\to\infty} \int_{\{x:|Px|<r_i,|x_d|<2r_i\}} u_0^{1+\alpha} \cdot |x_d - \xi(Px) + \delta_j^{r_i}|^\gamma \, dx = \infty$$

for any i for which the inner lim sup in the definition of W is nonzero for $r = r_i$, in particular for any i large enough. Thus for any i large enough there exists $j_0(i, \tilde{T})$ such that for any $j \geq j_0(i, \tilde{T})$ condition (22) is satisfied for our r_i, $\delta_j^{r_i}$ and our fixed $\tilde{T}$ (as u is assumed to satisfy the α entropy estimate).

Utilizing formula (36) to estimate the second integral on the right-hand side of (23) and estimating the first integral on the right-hand side of (23) (note that $-1 + 4\frac{1+\alpha}{n} < 0$ since $\alpha < -\frac{1}{2}$ and $n > 2$), we see that the waiting time T^* of u at x_0 is bounded from above by

$$\begin{aligned} T^* \leq & \\ & \liminf_{r\to 0} \liminf_{\delta\to 0} \left[C(d,n) \cdot \left(r^{d-1} \frac{-1}{-1 + 4\frac{1+\alpha}{n}} \delta^{-1+4\frac{1+\alpha}{n}} \right)^{\frac{n}{1+\alpha}} \right. \\ & \qquad \left. \cdot \left(\delta^{\frac{4(1+\alpha)}{n}-1} r^{d-1} \fint_{\{x:|Px|<r,|x_d|<r,\operatorname{dist}(x,\partial M)<\delta\}} \left[\frac{1}{\delta^{\frac{4}{n}}} u_0 \right]^{1+\alpha} dx \right)^{-\frac{n}{1+\alpha}} \right] \\ = & C(d,n) W^{-\frac{n}{1+\alpha}} \end{aligned}$$

if the expression on the right-hand side does not exceed $\tilde{T}$. However, $\tilde{T} > 0$ was arbitrary and the expression does not depend on $\tilde{T}$. Choosing $\tilde{T}$ to be larger than this expression, this finishes the proof of assertion a).

Assertion b) is shown similarly. Again, w.l.o.g. we may assume that $x_0 = 0$, that $H = \{x \in \mathbb{R}^d : x_d = 0\}$, and that $x_0 + \mu\vec{e_d} \in M$ for any $\mu > 0$ small enough. Define ξ, R, K, $\tilde{R}$ as in the case of assertion a). Thus for $r \in (0, \frac{\tilde{R}}{3})$ conditions (P1) and (P2) are fulfilled and (35) holds.

However, to prove assertion b) we now let δ and r tend to zero simultaneously. Set $\alpha := -\frac{1}{2}$. Conditions (H1) to (H4) are readily verified. Condition (H5) is seen to be equivalent to

$$2\frac{\gamma - 3}{\gamma - 2} - \frac{5}{2} \geq \frac{25}{4}\tau$$

which in turn (due to $\gamma < 0$) is equivalent to

$$4(-\gamma + 3) - 5(-\gamma + 2) \geq \frac{25}{2}(-\gamma + 2)\tau$$

which in particular is satisfied if

$$\gamma \geq -2 + C\tau$$

holds for $C = 22 \cdot \frac{25}{2}$ (since we assume $\gamma \in [-20, -1]$).

We now set $r := \frac{1}{|\log\delta|}$ and $\tau := \frac{2K}{\epsilon|\log\delta|}$, i.e. $\gamma := -2 + \frac{CK}{\epsilon|\log\delta|}$ (with $\epsilon = \epsilon(d, n)$ from condition (P3)). By our choice of τ and r, the condition (P3) of Lemma 21 is satisfied.

Let δ_i be a sequence converging to zero for which the lim sup in the assumptions of Theorem 5 b) is approached (with h replaced by δ).

It remains to check (22). We know $|\xi(Px)| \leq Kr^2 \leq \epsilon\tau r \leq \frac{r}{2}$ for $x \in Z_r$ (since $\xi(0) = 0$, $D\xi(0) = 0$, $|D^2\xi(Px)| \leq K$). For δ small enough, we have $r = |\log\delta|^{-1} > 4\delta$. Thus for δ small enough we see that condition (22) is satisfied for sure if

$$\begin{aligned}&\int_{\{x:|Px|<|\log\delta|^{-1},|x_d-\xi(Px)|<2\delta\}} u_0^{1+\alpha} \cdot |x_d - \xi(Px) + \delta|^\gamma \, dx \\ \geq & C_0(d,n) K^{-6+\frac{CK}{\epsilon|\log\delta|}} |\log\delta|^{8-\frac{2CK}{\epsilon|\log\delta|}} \int_0^{\tilde{T}} \int_\Omega |\nabla u^{\frac{1+n+\alpha}{4}}(.,t)|^4 \, dx \, dt \, .\end{aligned}$$

The previous inequality in turn is implied by the condition

$$\begin{aligned}&\frac{c(d)|\log\delta|^{1-d}\delta}{3^{2-\frac{CK}{\epsilon|\log\delta|}}} \fint_{\{x:|Px|<\frac{1}{|\log\delta|},|x_d-\xi(Px)|<2\delta\}} u_0^{\frac{1}{2}} \delta^{-2+\frac{CK}{\epsilon|\log\delta|}} \, dx \\ \geq & C_0(d,n) K^{-6+\frac{CK}{\epsilon|\log\delta|}} |\log\delta|^{8-\frac{2CK}{\epsilon|\log\delta|}} \int_0^{\tilde{T}} \int_\Omega |\nabla u^{\frac{1+n+\alpha}{4}}(.,t)|^4 \, dx \, dt \, ,\end{aligned}$$

which due to (35) and (37) in turn is implied by

$$\begin{aligned}&c(d) e^{\frac{CK\log\delta}{\epsilon|\log\delta|}} \fint_{\{x:|Px|<\frac{1}{|\log\delta|},|x_d|<\frac{1}{|\log\delta|},\mathrm{dist}(x,\partial M)<\delta\}} \left[\frac{1}{\delta^2|\log\delta|^{14+2d}} u_0\right]^{\frac{1}{2}} \, dx \\ \geq & C_0(d,n) K^{-6+\frac{CK}{\epsilon|\log\delta|}} e^{-\frac{2CK\log|\log\delta|}{|\log\delta|\epsilon}} \int_0^{\tilde{T}} \int_\Omega |\nabla u^{\frac{1+n+\alpha}{4}}(.,t)|^4 \, dx \, dt \, .\end{aligned}$$

Thus, evaluating at δ_i and passing to the limit $i \to \infty$, we see that the condition (22) is satisfied for any δ_i with i large enough if we have chosen $\tilde{T} > 0$ small enough (as the integral on the right-hand side of the present formula tends to zero as $\tilde{T} \to 0$).

Using $|\xi(Px)| \leq r$ for $x \in Z_{3r}$ and using (35), for δ so small that $r = |\log\delta|^{-1} > 4\delta$ the estimate (23) in connection with (35) and (37) yields

$$
\begin{aligned}
&\inf\{t \geq 0 : \operatorname{supp} u(.,t) \cap Z_{\frac{3}{|\log\delta|}} \not\subset \operatorname{supp} u_0\} \\
\leq & \frac{C(d)\epsilon}{K} |\log\delta| \cdot \left(|\log\delta|^{-d+1} (C(d)|\log\delta|^{-1})^{\frac{CK}{\epsilon|\log\delta|}} \int_\delta^{\frac{C(d)}{|\log\delta|}} z^{-1}\, dz \right)^4 \\
& \cdot \left(\int_{\{x:|Px|<|\log\delta|^{-1}, |x_d - \xi(Px)|<2\delta\}} u_0^{\frac{1}{2}} |x_d - \xi(Px) + \delta|^\gamma \, dx \right)^{-4} \\
\leq & \frac{C(d)\epsilon}{K} |\log\delta| \cdot \left(|\log\delta|^{-d+1} (C(d)|\log\delta|^{-1})^{\frac{CK}{\epsilon|\log\delta|}} \int_\delta^{\frac{C(d)}{|\log\delta|}} z^{-1}\, dz \right)^4 \\
& \cdot \left(\delta |\log\delta|^{-d+1} \fint_{\{x:|Px|<|\log\delta|^{-1}, |x_d|<|\log\delta|^{-1}, \operatorname{dist}(x,\partial M)<\delta\}} u_0^{\frac{1}{2}} \delta^{-2+\frac{CK}{\epsilon|\log\delta|}} \, dx \right)^{-4}
\end{aligned}
$$

if the expression on the right-hand side does not exceed $\tilde{T}$. Rearranging, setting $\delta := \delta_i$, evaluating the first integral and letting $i \to \infty$, we obtain (since for i large enough we have $|\log|\log\delta_i|| + |\log C(d)| \leq |\log\delta_i|$)

$$
\begin{aligned}
T^* \leq & \\
\liminf_{i\to\infty} & \left[\frac{C(d)\epsilon}{K} |\log\delta_i| \right. \\
& \cdot \left(|\log\delta_i|^{-d+1} (C(d)|\log\delta_i|^{-1})^{\frac{CK}{\epsilon|\log\delta_i|}} \left(\log\frac{C(d)}{|\log\delta_i|} - \log\delta_i \right) \right)^4 \\
& \left. \cdot \left(\delta_i |\log\delta_i|^{-d+1} \fint_{\{x:|Px|<|\log\delta_i|^{-1}, |x_d|<|\log\delta_i|^{-1}, \operatorname{dist}(x,\partial M)<\delta_i\}} u_0^{\frac{1}{2}} \delta_i^{-2+\frac{CK}{\epsilon|\log\delta_i|}} \, dx \right)^{-4} \right] \\
\leq & \frac{C(d)\epsilon}{K} \lim_{i\to\infty} (C(d)|\log\delta_i|^{-1})^{\frac{4CK}{\epsilon|\log\delta_i|}} \\
& \cdot \liminf_{i\to\infty} \left(|\log\delta_i|^{-\frac{5}{4}} \fint_{\{x:|Px|<|\log\delta_i|^{-1}, |x_d|<|\log\delta_i|^{-1}, \operatorname{dist}(x,\partial M)<\delta_i\}} u_0^{\frac{1}{2}} \delta_i^{-1} e^{-\frac{CK}{\epsilon}} \, dx \right)^{-4}
\end{aligned}
$$

if the expression on the right-hand side is smaller than $\tilde{T}$. However, the first limit on the right-hand side is equal to 1, while the second limit on the right-hand side is zero by the assumptions in Theorem 5 b). This proves the second assertion of the theorem. □

Proof of Corollary 6. The assertion of a) follows using Theorem 5 a). First we need to construct an appropriate set M since $\operatorname{supp} u_0$ is only locally the closure of a C^4 domain.

After translation, rotation and (possibly) reflection, we may assume that $x_0 = 0$, that the tangent plane H of $\partial \operatorname{supp} u_0$ in x_0 is equal to $\{x \in \mathbb{R}^d : x_d = 0\}$, and that for any $\mu > 0$ small enough we have $x_0 + \mu\vec{e_d} \in \operatorname{supp} u_0$.

Denote the orthogonal projection onto H by P. Define $Z_\rho := \{x \in \mathbb{R}^d : |Px| < \rho, |x_d| < \rho\}$. Then for $\tilde{R} > 0$ small enough the set $\operatorname{supp} u_0 \cap Z_{5\tilde{R}}$ corresponds to the supergraph of a C^4 function $\xi : H \to \mathbb{R}$; more precisely, we have $\operatorname{supp} u_0 \cap Z_{5\tilde{R}} = Z_{5\tilde{R}} \cap \{x \in \mathbb{R}^d : x_d \geq \xi(Px)\}$. Decreasing $\tilde{R}$ if necessary, we may assume that $Z_{5\tilde{R}} \subset \Omega$. Using the fact that $\xi(0) = 0$ and that $\nabla\xi(0) = 0$ (since $\{x \in \mathbb{R}^d : x_d = 0\}$ is the tangent plane to $\partial \operatorname{supp} u_0$ in 0), by decreasing $\tilde{R}$ we may enforce that $|\xi(Px)| \leq \frac{\tilde{R}}{2}$ for $|Px| < 5\tilde{R}$.

Take a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ with $0 \leq \phi \leq 1$, $\phi(s) = 1$ for $s < \tilde{R}$, $\phi(s) = 0$ for $s > \frac{4}{3}\tilde{R}$. We now define $\hat{\xi} : H \cap Z_{2\tilde{R}} \to \mathbb{R}$,

$$\hat{\xi}(Px) := \phi(|Px|)\xi(Px) + (1 - \phi(|Px|)) \left(\sqrt{(2\tilde{R})^2 - |Px|^2} - 3\tilde{R} \right) .$$

Then the graph of $\hat{\xi}$ on $\{x \in H : \frac{4}{3}\tilde{R} < |x| < 2\tilde{R}\}$ coincides with the set $\partial B_{2\tilde{R}}(-3\tilde{R}\vec{e_d}) \cap \{x \in \mathbb{R}^d : \frac{4}{3}\tilde{R} < |Px| < 2\tilde{R}, -3\tilde{R} < x_d\}$. Note also that $\hat{\xi} \leq \xi$ on $H \cap Z_{2\tilde{R}}$ and that we have $\hat{\xi} \in C^4$ on $H \cap Z_{\frac{5}{3}\tilde{R}}$. Therefore

$$D := \{x \in \mathbb{R}^d : |Px| < 2\tilde{R}, -3\tilde{R} < x_d < \hat{\xi}(Px)\} \cup B_{2\tilde{R}}(-3\tilde{R}\vec{e_d})$$

is a domain with C^4 boundary; setting $M := D^c$, we see that $\operatorname{supp} u_0 \subset M$ since $D \subset Z_{5\tilde{R}}$ and since $\hat{\xi} \leq \xi$. Moreover, in some neighbourhood of x_0 we have $\partial M = \partial \operatorname{supp} u_0$ (since $\phi(|Px|) = 1$ for $|Px| < \tilde{R}$, i.e. $\hat{\xi}(Px) = \xi(Px)$ for $|Px| < \tilde{R}$). Thus M satisfies the conditions of Theorem 5.

For any sufficiently small $r > 0$, $h > 0$, we now obtain the estimate

$$\begin{aligned}
&\int_{\{x:|Px|<r,|x_d|<r,\operatorname{dist}(x,\partial M)<h\}} \left[\frac{1}{h^{\frac{4}{n}}} u_0\right]^{1+\alpha} dx \\
=&\int_{\{x:|Px|<r,|x_d|<r,\operatorname{dist}(x,\partial \operatorname{supp} u_0)<h\}} \left[\frac{1}{h^{\frac{4}{n}}} u_0\right]^{1+\alpha} dx \\
\geq&\int_{\{x:|Px|<r,|x_d|<r,\frac{h}{3}<\operatorname{dist}(x,\partial \operatorname{supp} u_0)<h\}} \left[\frac{1}{h^{\frac{4}{n}}} u_0\right]^{1+\alpha} dx \ .
\end{aligned}$$

This implies for $r > 0$ small enough and $h \in (0, r)$ (recall that ∂M is a C^4 manifold with tangent plane $H = \{x \in \mathbb{R}^d : x_d = 0\}$ in $x_0 = 0$; recall also that $\partial \operatorname{supp} u_0$ coincides with ∂M in some neighbourhood of x_0)

$$\begin{aligned}
&\fint_{\{x:|Px|<r,|x_d|<r,\operatorname{dist}(x,\partial M)<h\}} \left[\frac{1}{h^{\frac{4}{n}}} u_0\right]^{1+\alpha} dx \\
\geq&\frac{c(d)}{hr^{d-1}} \int_{\{x:|Px|<r,|x_d|<r,\operatorname{dist}(x,\partial M)<h\}} \left[\frac{1}{h^{\frac{4}{n}}} u_0\right]^{1+\alpha} dx \\
\geq&\frac{c(d)}{hr^{d-1}} \int_{\{x:|Px|<r,|x_d|<r,\frac{h}{3}<\operatorname{dist}(x,\partial \operatorname{supp} u_0)<h\}} \left[\frac{1}{h^{\frac{4}{n}}} u_0\right]^{1+\alpha} dx \\
\geq&c(d)\fint_{\{x:|Px|<r,|x_d|<r,\frac{h}{3}<\operatorname{dist}(x,\partial \operatorname{supp} u_0)<h\}} \left[\frac{1}{h^{\frac{4}{n}}} u_0\right]^{1+\alpha} dx \\
\geq&c(n,d,\alpha) S^{1+\alpha} \ .
\end{aligned}$$

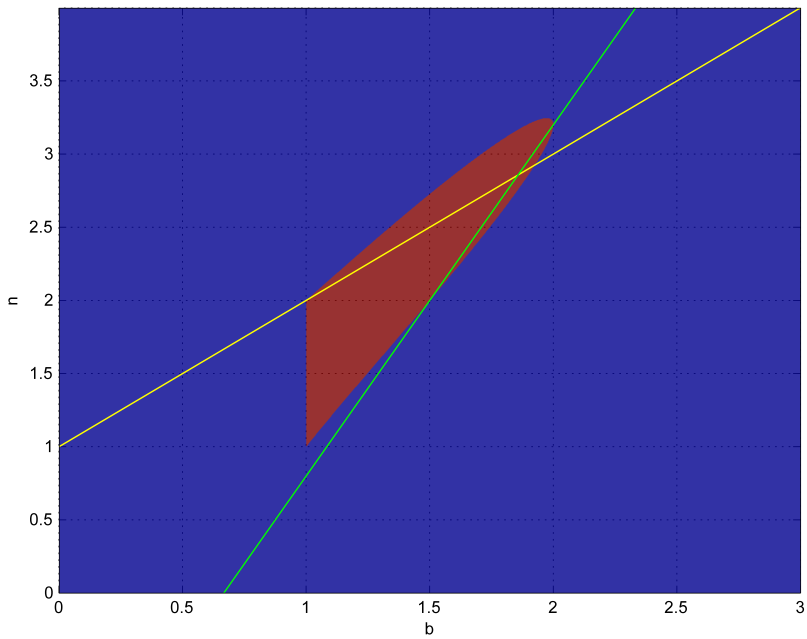
Passing to the limit $h \to 0$, then $r \to 0$, our assertion is established.

Assertion b) is proven using the same construction for M as well as Theorem 5 b) and analogous estimates for the integrals. $\square$

3.5 Admissible values for n and b and limitations of our approach

As a last point, we would like to discuss the results our approach yields for $n \in [1, 2)$.

In the figure below, the red area marks the set of pairs (b, n) for which conditions (H1), (H2) and (H4) of Lemma 18 are satisfied. All pairs below the yellow line satisfy condition (H3). For all pairs below the green line, $\gamma = -2$ is an admissible value in condition (H5) of Lemma 20 and Lemma 21. The green line intersects the boundary of the red area at $n = 2$, $b = \frac{3}{2}$. The yellow line intersects the boundary of the red area at $n = \frac{2}{9}(10 + \sqrt{10}) \approx 2.92495$, $b = \frac{1}{9}(11 + 2\sqrt{10}) \approx 1.92495$.



Starting at $n = 2$, $b = \frac{3}{2}$ and tracking the boundary of the red area as n decreases, we see that for $n < 2$ the minimal values of γ which are admissible become larger until for $n = 1$ only values in $(-1, 0)$ are admissible. At the same time, $\alpha = b - n$ also increases until for $n = 1$ we have $\alpha = 0$. In particular, for $n < 2$ we have $1 + \gamma + 4\frac{1+\alpha}{n} > 0$.

Considering the case $d = 1$, let $x_0 \in \partial \operatorname{supp} u_0$ be a point with $\operatorname{supp} u_0 \cap (-\infty, x_0) = \emptyset$. Applying Lemma 20 with $x_1 := x_0 - \epsilon$, we see that the estimate on T^* provided by the lemma converges to zero as $\epsilon \to 0$ if near the free boundary the growth condition $u_0(x) \geq \tilde{S} \cdot (x-x_0)_+^{\frac{-\gamma-1}{1+\alpha}}$ is satisfied for some $\tilde{S} > 0$. Thus for $n < 2$ we only obtain immediate support spreading if u_0 grows steeper than $(x-x_0)_+^\beta$ at the free boundary for some $\beta = \beta(n) < 2$; this β tends to zero as n tends to 1. Note that for $\beta \leq \frac{1}{2}$ the condition $u_0 \in H^1(\mathbb{R})$ can no longer be satisfied; thus we have to work with the notion of solutions with weak initial trace.

Proof of Theorem 8. Dal Passo and Garcke [50] approximate u_0 by regularized initial data $u_{0\delta}$, e.g. by mollified versions $u_{0\delta} := \rho_\delta * u_0$, and consider the strong solution u_δ of the thin-film equation with initial data $u_{0\delta}$ (in the sense of Definition 9) constructed by a procedure like in [12]; then they pass to the limit $\delta \to 0$ and construct the solution u to be the limit of an appropriate subsequence.

First, observe that for such a subsequence we have strong convergence of $u_\delta^{1+\alpha+n}$ to $u^{1+\alpha+n}$ in $L^1_{loc}(\mathbb{R}\times[0,\infty))$ as $\delta \to 0$ if $n \in (1,2)$ and $\alpha \in (-1,0)$: We know strong convergence of u_δ to u in $L^1_{loc}(\mathbb{R}\times[0,\infty))$ (see Lemma 6 in [50]). The estimate

$$\begin{aligned}||u_\delta(.,t)||_{L^\infty(\mathbb{R})} \leq& C||\nabla u_\delta(.,t)||_{L^2(\mathbb{R})} + C||u_\delta(.,t)||_{L^1(\mathbb{R})}\\ \leq& C(n)||u_0||_{L^1(\mathbb{R})}^{\frac{8-n}{8+2n}} t^{-\frac{3}{8+2n}} + C||u_0||_{L^1(\mathbb{R})}\end{aligned}$$

(the first estimate corresponds to a Sobolev inequality; for the second inequality see Theorem 2 in [50]) implies that u_δ^3 is bounded uniformly (with respect to δ) in $L^1(K\times[0,T))$ for every $T > 0$ and every $K \subset\subset \mathbb{R}$ (note that $-\frac{3}{8+2n}\cdot 3 > -1$). Putting these considerations together, we obtain strong convergence of $u_\delta^{1+\alpha+n}$ to $u^{1+\alpha+n}$ in $L^1_{loc}(\mathbb{R}\times[0,\infty))$ since $1+\alpha+n < 3$.

We now notice that the assertion of Lemma 23 survives the approximation procedure: For the solutions u_δ, Lemma 23 applies (provided that we can find b subject to conditions (H1) to (H4); see below), i.e. the formula in Lemma 18 holds for u_δ. We then want to pass to the limit $\delta \to 0$. Convergence of the terms on the left-hand side in the formula of Lemma 18 for a.e. $t_2 > t_1 > 0$ follows using convergence of u_δ in $L^1_{loc}(\mathbb{R}\times I)$ (passing if necessary to a further subsequence). For $t_1 = 0$, convergence of the term on the left-hand side follows by the assumption of our theorem that $u_\delta(.,0) \to u_0$ in $L^1(\mathbb{R})$. Convergence of the third term on the right-hand side is a consequence of convergence of $u_\delta^{1+\alpha+n}$ in $L^1_{loc}(\mathbb{R}\times[0,\infty))$. It remains to deal with the first two terms on the right-hand side. Note that $\frac{n}{2} \leq b$ by (H2), thus the prefactors of these terms are nonnegative.

Since $\operatorname{supp} u_0$ is assumed to be bounded, we have $\operatorname{supp} u_0 \subset B_{R_0}(0)$ for some $R_0 > 0$. The approximation procedure by Dal Passo and Garcke (see Section 3 in [50]) then guarantees that $\operatorname{supp} u_\delta(.,0) \subset B_{R_0+\delta}(0)$. By the finite speed of propagation result Theorem 10, we thus obtain $\operatorname{supp} u_\delta(.,t) \subset B_{R_\delta(t)}(0)$, where $R_\delta(t) := R_0+\delta+C(n)||u_{0\delta}||_{L^1(\mathbb{R})}^{\frac{n}{4+n}} t^{\frac{1}{4+n}}$. Note that $||u_{0\delta}||_{L^1(\mathbb{R})} = ||u_0||_{L^1(\mathbb{R})}$ by the choice of $u_{0\delta}$ by Dal Passo and Garcke. Fix $T > 0$ and assume that $\psi_{xx} \geq 0$ on $B_{R_1(T)}(0)$. By the lower semicontinuity of L^2 norms with respect to convergence in the sense of distributions, we obtain

$$\liminf_{\delta\to 0} \int_{t_1}^{t_2} \int_A |(u_\delta^{\frac{b+1}{2}})_x|^2 \, dx \, dt \geq \int_{t_1}^{t_2} \int_A |(u^{\frac{b+1}{2}})_x|^2 \, dx \, dt$$

for any open set $A \subset \mathbb{R}$ and any $0 \leq t_1 \leq t_2$. Using Fubini's theorem we see that

$$\begin{aligned}&\int_{t_1}^{t_2} \int_{\mathbb{R}} u_\delta^{b-1} |(u_\delta)_x|^2 \psi_{xx} \, dx \, dt \\ =&\frac{4}{(b+1)^2} \int_{(0,\infty)} \int_{t_1}^{t_2} \int_{\{\psi_{xx}>s\}} |(u_\delta^{\frac{b+1}{2}})_x|^2 \, dx \, dt \, ds\end{aligned}$$

for $0 \leq t_1 < t_2 \leq T$ and $\delta < 1$ (note that by our assumption we have $\psi_{xx} \geq 0$ on $\operatorname{supp} u_\delta(.,t)$ for $t \leq T$ and $\delta < 1$, thus the present equation indeed holds). By Fatou's Lemma, we therefore get (due to continuity of ψ_{xx} the sets $\{x \in \mathbb{R} : \psi_{xx}(x) > s\}$ are indeed open)

$$\begin{aligned}&\liminf_{\delta\to 0} \int_{t_1}^{t_2} \int_{\mathbb{R}} u_\delta^{b-1} |(u_\delta)_x|^2 \psi_{xx} \, dx \, dt \\ \geq&\frac{4}{(b+1)^2} \int_{(0,\infty)} \liminf_{\delta\to 0} \int_{t_1}^{t_2} \int_{\{\psi_{xx}>s\}} |(u_\delta^{\frac{b+1}{2}})_x|^2 \, dx \, dt \, ds \\ \geq&\int_{t_1}^{t_2} \int_{\mathbb{R}} u^{b-1} |u_x|^2 \psi_{xx} \, dx \, dt\end{aligned}$$

for $0 \leq t_1 < t_2 \leq T$. Thus the inequality in Lemma 18 also holds for the limit u, at least if we have $\psi_{xx} \geq 0$ on $B_{R_1(T)}(0)$ and $t_2 \leq T$ for some $T > 0$ and if b is chosen such that (H1) to (H4) are satisfied.

Let $\epsilon > 0$ and $T > 0$. Set $x_1 := x_0 - \epsilon$. Consider the function $|x - x_1|^\gamma$ (with $\gamma < -1$) on the interval $[x_1 + \epsilon, R_1(T))$. This function can be extended to a nonnegative function $\tilde{\psi} \in C_c^4(\mathbb{R})$ which satisfies $\tilde{\psi}_{xx} \geq 0$ on $(-R_1(T), R_1(T))$. Thus by the previous considerations, the formula of Lemma 18 holds for the solution u and the test function $\tilde{\psi}$ as long as $t_2 \leq T$ (and provided that (H1) to (H4) are satisfied). Define

$$\tilde{T} := \inf\{t \geq 0 : \operatorname{supp} u(.,t) \cap (-\infty, x_0) \neq \emptyset\}$$

and choose $T > \tilde{T}$. Then for $t < \tilde{T}$ we have $\operatorname{supp} u(.,t) \subset [x_0, R_1(T))$; in particular, $\tilde{\psi}$ coincides with $|x - x_1|^\gamma$ on $\operatorname{supp} u(.,t)$ for $t < \tilde{T}$. Therefore we see that the formula of Lemma 18 holds for $\psi(x) := |x - x_1|^\gamma$ and a.e. $t_2, t_1 \in (0, \tilde{T})$ with $t_1 \leq t_2$ as well as a.e. $t_2 \in (0, \tilde{T})$ in case $t_1 = 0$ (provided that b satisfies (H1) to (H4)).

Since the proof of Lemma 20 used the inequality from Lemma 18 only for the test function $|x-x_1|^\gamma$, we see that Lemma 20 also applies to our limit u (at least if the parameter T of Lemma 20 is chosen to satisfy $T<\tilde{T}$). Suppose now that we can find b and γ such that (H1) to (H5) are satisfied. Using the assertion of Lemma 20 for all $T<\min(\tilde{T},1)$, we therefore have the upper bound

$$\begin{aligned}\tilde{T} \leq& \inf\{t\geq 0 : \operatorname{supp} u(.,t)\cap(x_1-\epsilon,x_1+\epsilon)\neq\emptyset\}\\ \leq& \frac{1}{n\tau}\left[\int_{\overline{\bigcup_{t\in[0,1]}\operatorname{supp} u(.,t)}\setminus(x_1-\epsilon,x_1+\epsilon)} |x-x_1|^{\gamma+4\frac{1+\alpha}{n}}\,dx\right]^{\frac{n}{1+\alpha}}\\ &\cdot\left[\int_{\mathbb{R}} u_0^{1+\alpha}|x-x_1|^{\gamma}\,dx\right]^{-\frac{n}{1+\alpha}}\end{aligned}$$

if the expression on the right-hand side is smaller than 1. Thus, we get

$$\begin{aligned}\tilde{T} \leq& C(n,\alpha,\gamma)\left[\int_{\overline{\bigcup_{t\in[0,1]}\operatorname{supp} u(.,t)}} |x-x_1|^{\gamma+4\frac{1+\alpha}{n}}\,dx\right]^{\frac{n}{1+\alpha}}\\ &\cdot\left[\fint_{(x_0,x_0+\epsilon)} u_0^{1+\alpha}\epsilon^{\gamma+1}\,dx\right]^{-\frac{n}{1+\alpha}}\end{aligned} \tag{38}$$

if the expression on the right-hand side is smaller than 1.

We now choose b and γ such that (H1) to (H5) are satisfied. For $n\in(1,1.5]$ we choose $b:=\frac{9}{20}n+\frac{22}{40}$; in case $n\in(1.5,2)$ we choose $b:=\frac{11}{20}n+\frac{16}{40}$. For these choices, conditions (H1) to (H4) are verified in the proof of Theorem 11 below. It remains to choose $\gamma<-1$ such that (H5) holds. Note that for $n\in(1,1.5]$ condition (H5) holds for some $\tau>0$ if $(2b-\frac{n}{2})\cdot\frac{\gamma-3}{\gamma-2}-(b+1)>0$, i.e. if $\frac{16n+44}{40}\cdot\frac{\gamma-3}{\gamma-2}>\frac{18n+62}{40}$; this is equivalent to $\gamma>-\frac{2(3n+2)}{n+9}=:\gamma_{inf}(n)$. For $n\in(1.5,2)$, (H5) holds for some $\tau>0$ if $\frac{24n+32}{40}\cdot\frac{\gamma-3}{\gamma-2}>\frac{22n+56}{40}$ which is equivalent to $\gamma>-\frac{2(7n-4)}{12-n}=:\gamma_{inf}(n)$.

Choose $\gamma\in(\gamma_{\text{inf}},-1)$. Note that by our choice of b and the definition of γ_{inf} we have $\gamma+4\frac{1+\alpha}{n}>-1$ (since $\gamma_{inf}>-2$ and $\alpha\in(-\frac{1}{2},0)$ as $\alpha=b-n$). Thus by the finite speed of propagation estimate the first integral on the right-hand side of (38) converges to some finite value as $\epsilon\to 0$. The second integral tends to infinity as $\epsilon\to 0$ (at least for a subsequence) if the growth condition from our theorem is satisfied for $\beta:=\frac{-\gamma_{inf}-1}{1+\alpha}$ and if γ has been chosen close enough to γ_{inf} (depending on the τ from our growth condition in our theorem). Thus, the main assertion of the theorem is established.

Note that since $\gamma_{inf}<-1$ we have $\beta>0$. Moreover, we have $\beta<2$ since $\alpha>-\frac{1}{2}$ and $\gamma_{inf}>-2$. □

Most probably our results in the regime of strong slippage are not optimal; we expect that at least for initial data with growth steeper than $(x-x_0)_+^2$

one should have immediate support spreading. However, the derivation of such an improved result is currently out of reach.

4 Proof of the optimal lower bounds on asymptotic support propagation rates for the thin-film equation

4.1 Derivation of a simplified entropy estimate

Recall the notation $b := n+\alpha$ and the following definitions from the previous chapter (see Lemma 18):

(H1) Assume that $1 \leq b \leq 2$.

(H2) Suppose that $\frac{n}{2} \leq b \leq n$.

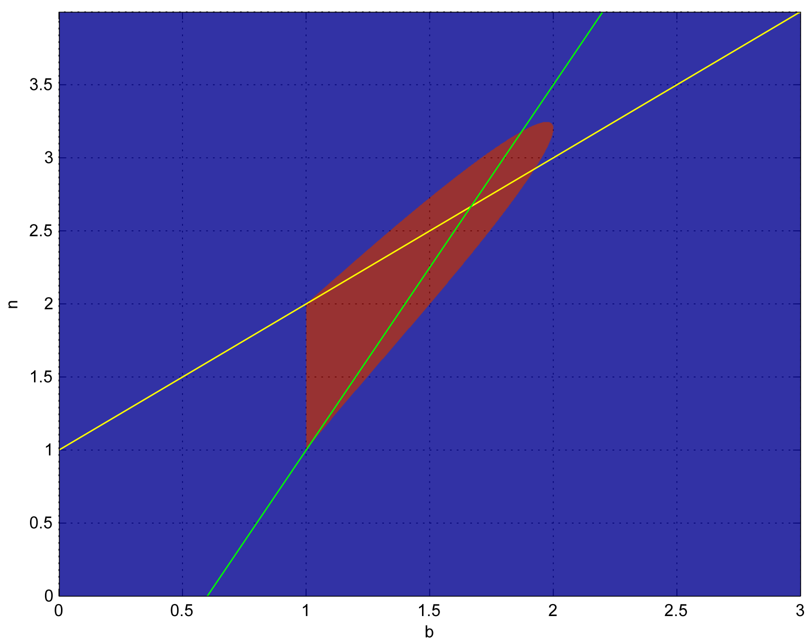
(H3) Assume that $n-1 < b$.

(H4) Suppose that the inequality

$$(n-b)\left(b-\frac{n}{2}\right)(b-1)(2-b) \geq \frac{1}{4}\left[\left(\frac{5n}{2}-4b\right)(b-1)\right]^2$$

is satified.

The set of $(b,n) \in \mathbb{R}\times\mathbb{R}$ for which (H1) to (H4) are satisfied is depicted below. The set of points for which (H1), (H2) and (H4) hold at the same time corresponds to the red area. All points below the yellow line satisfy (H3). The yellow line intersects the boundary of the red area at $b \approx 1.92$, $n \approx 2.92$. For all points below the green line, $\gamma = -d$ is an admissible choice in condition (H5a) of Lemma 25 below. The green line intersects the boundary of the red area at $b = 1$, $n = 1$.



In the previous chapter, Lemma 18 has been seen to be a consequence of Lemma 16, Lemma 17, and Young's inequality. Recall that in the case of the Cauchy problem Lemma 18 reads as follows:

Lemma 18. *Let $n \in (1,3)$, $\alpha \in (-1,0)$, and let u be a strong energy solution of the thin-film equation on $\mathbb{R}^d$, $d \leq 3$, with nonnegative initial data $u_0 \in H^1(\mathbb{R}^d)$. Assume that* $\operatorname{supp} u_0$ *is bounded. Suppose that u satisfies the α entropy estimate. Set $b := n+\alpha$ and assume that (H1) to (H4) are satisfied.*

Let $\psi \in C_c^4(\mathbb{R}^d)$; assume that $\psi \geq 0$. Then for a.e. $t_1, t_2 \in [0,\infty)$ with $t_2 \geq t_1$ and for a.e. $t_2 \in [0,\infty)$ in case $t_1 = 0$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)\psi(.)\ dx - \int_{\mathbb{R}^d} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)\psi(.)\ dx \\
&\geq \left(\frac{2}{3}b - \frac{1}{6}n\right) \int_{t_1}^{t_2} \int_{\mathbb{R}^d} u^{b-1} |\nabla u|^2 \Delta\psi\ dx\ dt \\
&\quad + \left(\frac{4}{3}b - \frac{1}{3}n\right) \int_{t_1}^{t_2} \int_{\mathbb{R}^d} u^{b-1} \nabla u \cdot D^2\psi \cdot \nabla u\ dx\ dt \\
&\quad - \frac{1}{b+1} \int_{t_1}^{t_2} \int_{\mathbb{R}^d} u^{b+1} \Delta^2\psi\ dx\ dt\ .
\end{aligned}$$

However, for $d > 1$ existence of strong energy solutions is only known for $n \in \left(2 - \sqrt{\frac{8}{8+d}}, 3\right)$. Thus, for $d > 1$ and n slightly larger than 1 only existence of strong solutions is guaranteed. We therefore need to extend Lemma 18 to the case of strong solutions:

Lemma 23. *Let $d \leq 3$ and $n \in (1,2)$ as well as $\alpha \in (-1,0)$. Suppose that (H1) to (H4) are satisfied. Let $u_0 \in H^1(\mathbb{R}^d)$ be nonnegative and have bounded support. Then the assertion of Lemma 18 also holds if u is not a strong energy solution, but a strong solution to the Cauchy problem for the thin-film equation constructed by the procedure in [12].*

Recall that the regularity of solutions $u \in L^\infty(I; H^1(\mathbb{R}^d)) \cap H^1_{loc}(I; [W^{1,p}(\mathbb{R}^d)]')$ implies $u \in C^0_{loc}(I; L^2(V))$ for any bounded open set $V \subset \mathbb{R}^d$ with smooth boundary (see e.g. Corollary 4 in [53]). Thus by approximation, the formula in our lemma again holds for all $t_2 \geq t_1 \geq 0$, not just a.e..

The remainder of the current section is dedicated to the approximation argument necessary for establishing Lemma 23; it may be skipped on first reading.

Proof. We use the notation from [12]. In [12], the solution to the Cauchy problem is obtained as follows: First a solution u of the thin-film equation on the bounded domain $\Omega_W := B_W(0)$ is constructed as the limit of the solutions $u_{\delta\sigma}$ of the problem

$$\begin{aligned}\frac{d}{dt}u_{\delta\sigma} &= -\nabla\cdot\left(m_{\delta\sigma}(u_{\delta\sigma})\nabla\Delta u_{\delta\sigma}\right) \ \text{ in } \Omega_W\times[0,\infty), \qquad (39)\\ \vec{n}\cdot\nabla u_{\delta\sigma} &= \vec{n}\cdot\nabla\Delta u_{\delta\sigma} = 0 \text{ on } \partial\Omega_W\times[0,\infty)\ ,\\ u_{\delta\sigma}(.,0) &= u_0+\delta^{\theta_1}+\sigma^{\theta_2} \text{ in } \Omega_W\ ,\end{aligned}$$

(where W is chosen so large that $\operatorname{supp} u_0 \subset\subset \Omega_W$) with $\theta_1,\theta_2>0$ and

$$m_{\delta\sigma}(v) := \frac{v^{n+s}}{\delta v^n + v^s + \sigma v^{n+s}}\ .$$

Let $T>0$. By the finite speed of propagation result (see [12]), if W has been chosen large enough (depending on u_0 and T), it is seen that $\operatorname{supp} u(.,t)$ cannot touch $\partial\Omega_W$ for $t\le T$. Therefore extending u to $\mathbb{R}^d\times[0,T]$ by setting $u=0$ outside of $\Omega_W\times[0,T]$, a solution to the Cauchy problem for the thin-film equation on the time interval $[0,T]$ is obtained. This process is repeated starting at time T with initial data $u(.,T)$, resulting in a solution to the Cauchy problem on the time interval $[T,2T]$. Stitching together the two solutions, a solution to the Cauchy problem on the time interval $[0,2T]$ is obtained. By an inductive construction, the global solution to the Cauchy problem is constructed.

It is therefore sufficient to prove the assertion of our lemma for all strong solutions u on some Ω_W which have been constructed as the limit of some sequence $u_{\delta\sigma}$ ($u_{\delta\sigma}$ satisfying the auxiliary problem (39)). From now on we abbreviate $\Omega := \Omega_W$. Let $\delta,\sigma>0$.

Then, as shown in [34], choosing $s>8$ if $d=3$ or $s>4$ if $d=1$ or $d=2$, there exists a solution $u_{\delta\sigma}$ which is strictly positive for a.e. $t>0$ and satisfies the energy estimate

$$\int_\Omega |\nabla u_{\delta\sigma}(.,t)|^2\,dx + \int_0^t\int_\Omega m_{\delta\sigma}(u_{\delta\sigma})|\nabla\Delta u_{\delta\sigma}|^2\,dx\,dt \le \int_\Omega |\nabla u_0|^2\,dx\ .$$

Moreover, this solution has the property $u_{\delta\sigma}\in H^1_{loc}(I;(H^1(\Omega))')$; for a.e. $t>0$ we know that $\nabla\Delta u_{\delta\sigma}\in L^2(\Omega)$ holds and for any $\phi\in L^2(I;H^1(\Omega))$ any any $T>0$ the solution satisfies

$$\int_0^T \left\langle \frac{d}{dt}u_{\delta\sigma},\phi\right\rangle\,dt = \int_0^T\int_\Omega m_{\delta\sigma}(u_{\delta\sigma})\nabla\Delta u_{\delta\sigma}\cdot\nabla\phi\,dx\,dt\ . \qquad (40)$$

For a proof of these claims see also [34].

We now proceed as in the derivation of the entropy inequalities in [12]. Set

$$g_{\delta\sigma}(v) := \frac{\delta}{\alpha+n-s}v^{\alpha+n-s} + \frac{1}{\alpha}v^\alpha + \frac{\sigma}{\alpha+n}v^{\alpha+n}$$

and

$$G_{\delta\sigma}(v) := \frac{\delta}{(\alpha+n-s)(\alpha+n-s+1)} v^{\alpha+n-s+1} + \frac{1}{\alpha(\alpha+1)} v^{\alpha+1} + \frac{\sigma}{(\alpha+n)(\alpha+n+1)} v^{\alpha+n+1} \ . \tag{41}$$

We have $G'_{\delta\sigma}(v) = g_{\delta\sigma}(v)$ and $g'_{\delta\sigma}(v) = \frac{v^{n+\alpha-1}}{m_{\delta\sigma}(v)}$. Knowing that $u_{\delta\sigma}^{\frac{1+n+\alpha}{4}} \in L^4_{loc}(I; W^{1,4}(\Omega))$ (by the entropy estimate; see [51]) which by the Sobolev embedding implies $u_{\delta\sigma}^{\frac{1+n+\alpha}{2}} \in L^2_{loc}(I; L^\infty(\Omega))$ and knowing that we have $u_{\delta\sigma} \in L^\infty(I; H^1(\Omega))$, due to $n+\alpha-1 \leq \frac{1+n+\alpha}{2}$ (by (H1)) it is easily seen that $g^\alpha_{\delta\sigma}(u_{\delta\theta}+\epsilon) \in L^2_{loc}(I; H^1(\Omega))$. Thus we may test the equation (40) with $\psi \cdot g^\alpha_{\delta\sigma}(u_{\delta\theta}+\epsilon)$, where $\psi \in C^\infty_c(\Omega)$, to obtain for a.e. $t_2 > t_1 > 0$ and a.e. $t_2 > 0$ in case $t_1 = 0$ (for the rearrangements involving the term with the time derivative, see the proof of Theorem 3.1 in [12])

$$\begin{aligned}
&\int_\Omega G_{\delta\sigma}(u_{\delta\sigma}+\epsilon)\cdot\psi \ dx\Big|_{t_1}^{t_2} \\
&= \int_{t_1}^{t_2}\int_\Omega \psi \cdot \frac{(u_{\delta\sigma}+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(u_{\delta\sigma}+\epsilon)} \cdot m_{\delta\sigma}(u_{\delta\sigma})\nabla\Delta u_{\delta\sigma}\cdot\nabla u_{\delta\sigma} \ dx \ dt \\
&\quad + \int_{t_1}^{t_2}\int_\Omega g_{\delta\sigma}(u_{\delta\sigma}+\epsilon)\cdot m_{\delta\sigma}(u_{\delta\sigma})\nabla\Delta u_{\delta\sigma}\cdot\nabla\psi \ dx \ dt \ .
\end{aligned}$$

Integrating by parts yields (recall that for a.e. $t > 0$ we have $\inf_x u(x,t) > 0$

and $u(.,t) \in H^3_{loc}(\Omega)$; thus integration by parts is possible)

$$\begin{aligned}
&\int_\Omega G_{\delta\sigma}(u_{\delta\sigma}+\epsilon)\cdot\psi\,dx\Bigg|_{t_1}^{t_2}\\
=&-\int_{t_1}^{t_2}\int_\Omega \psi\cdot\frac{(u_{\delta\sigma}+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(u_{\delta\sigma}+\epsilon)}\cdot m_{\delta\sigma}(u_{\delta\sigma})D^2u_{\delta\sigma}:D^2u_{\delta\sigma}\,dx\,dt\\
&-\int_{t_1}^{t_2}\int_\Omega \frac{(u_{\delta\sigma}+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(u_{\delta\sigma}+\epsilon)}\cdot m_{\delta\sigma}(u_{\delta\sigma})\nabla\psi\cdot D^2u_{\delta\sigma}\cdot\nabla u_{\delta\sigma}\,dx\,dt\\
&-\int_{t_1}^{t_2}\int_\Omega \psi\cdot\Bigg[\frac{(u_{\delta\sigma}+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(u_{\delta\sigma}+\epsilon)}\cdot m'_{\delta\sigma}(u_{\delta\sigma})\\
&\qquad\qquad+(n+\alpha-1)\frac{(u_{\delta\sigma}+\epsilon)^{n+\alpha-2}}{m_{\delta\sigma}(u_{\delta\sigma}+\epsilon)}\cdot m_{\delta\sigma}(u_{\delta\sigma})\\
&\qquad\qquad-\frac{(u_{\delta\sigma}+\epsilon)^{n+\alpha-1}}{[m_{\delta\sigma}(u_{\delta\sigma}+\epsilon)]^2}\cdot m'_{\delta\sigma}(u_{\delta\sigma}+\epsilon)\cdot m_{\delta\sigma}(u_{\delta\sigma})\Bigg]\\
&\qquad\nabla u_{\delta\sigma}\cdot D^2u_{\delta\sigma}\cdot\nabla u_{\delta\sigma}\,dx\,dt\\
&-\int_{t_1}^{t_2}\int_\Omega\Bigg[\frac{(u_{\delta\sigma}+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(u_{\delta\sigma}+\epsilon)}\cdot m_{\delta\sigma}(u_{\delta\sigma})\\
&\qquad\qquad+g_{\delta\sigma}(u_{\delta\sigma}+\epsilon)\cdot m'_{\delta\sigma}(u_{\delta\sigma})\Bigg]\nabla u_{\delta\sigma}\cdot D^2u_{\delta\sigma}\cdot\nabla\psi\,dx\,dt\\
&-\int_{t_1}^{t_2}\int_\Omega g_{\delta\sigma}(u_{\delta\sigma}+\epsilon)\cdot m_{\delta\sigma}(u_{\delta\sigma})D^2u_{\delta\sigma}:D^2\psi\,dx\,dt\\
=:&\,I+II+III+IV+V\ .
\end{aligned}$$

Now we can pass to the limit $\epsilon\to 0$. Note that the strict positivity of $u_{\delta\sigma}(.,t)$ for a.e. $t>0$ and for $t=0$ implies that the left-hand side converges for a.e. $t_1,t_2>0$ and a.e. $t_2>0$ in case $t_1=0$.

To prove convergence of the right-hand side, first recall that we have $u_{\delta\sigma}^{\frac{n+\alpha+1}{2}}\in L^2_{loc}(I;H^2(\Omega))$ and $u_{\delta\sigma}^{\frac{n+\alpha+1}{4}}\in L^4_{loc}(I;W^{1,4}(\Omega))$ (see [51]) which implies that $u_{\delta\sigma}^{\frac{n+\alpha-1}{2}}D^2u_{\delta\sigma}\in L^2_{loc}(I;L^2(\Omega))$ and $u_{\delta\sigma}^{\frac{n+\alpha-3}{4}}\nabla u_{\delta\sigma}\in L^4_{loc}(I;L^4(\Omega))$.

Convergence of term I can be shown as follows: Suppose $\epsilon<1$. Note that we have $m_{\delta\sigma}(v)\le\delta^{-1}v^s$; moreover, for $v\le 2$ we have $m_{\delta\sigma}(v)\ge c(n,s,\delta,\sigma)v^s$. Thus, in case $v\le\epsilon$ we have

$$\frac{(v+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(v+\epsilon)}\cdot m_{\delta\sigma}(v)\le C(n,s,\alpha,\delta,\sigma)\frac{\epsilon^{n+\alpha-1}}{\epsilon^s}\cdot v^s\le C(n,s,\alpha,\delta,\sigma)v^{n+\alpha-1}$$

where we have used the fact that $s>4$ (in particular s is larger than n and larger than $n+\alpha-1$). In case $v>\epsilon$ we obtain

$$\begin{aligned}
\frac{(v+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(v+\epsilon)}\cdot m_{\delta\sigma}(v)\le&C(n,s,\alpha,\delta,\sigma)\frac{v^{n+\alpha-1}}{m_{\delta\sigma}(v)}m_{\delta\sigma}(v)\\
\le&C(n,s,\alpha,\delta,\sigma)v^{n+\alpha-1}\ .
\end{aligned}$$

Thus, we get convergence of term I as $\epsilon \to 0$ using dominated convergence (since pointwise convergence holds a.e. due to the strict positivity of $u_{\delta\sigma}$ a.e.).

Term II can be treated similarly.

To show convergence of term IV, we only need to show the bound

$$|g_{\delta\sigma}(v+\epsilon) \cdot m'_{\delta\sigma}(v)| \leq C(n,s,\alpha,\delta,\sigma)v^{n+\alpha-1}$$

(as a corresponding bound for the first term in brackets in term IV has already been derived); then again convergence follows by the dominated convergence theorem. Let $\epsilon < 1$. We obtain in case $v \leq \epsilon$

$$|g_{\delta\sigma}(v+\epsilon) \cdot m'_{\delta\sigma}(v)| \leq C(n,s,\alpha,\delta,\sigma)\epsilon^{\alpha+n-s} \cdot v^{s-1} \leq C(n,s,\alpha,\delta,\sigma)v^{\alpha+n-1}$$

since $\alpha + n - s \leq \alpha \leq 0$ (as $s > 4$) and since $m'_{\delta\sigma}(v) \leq C(n,s,\sigma,\delta)v^{s-1}$ for $v \leq 1$. For $v > \epsilon$ we get

$$\begin{aligned}|g_{\delta\sigma}(v+\epsilon) \cdot m'_{\delta\sigma}(v)| \leq & C(n,s,\alpha,\delta,\sigma)(v^{\alpha+n-s} + v^{\alpha+n})\frac{v^{n+s-1}}{v^n + v^{n+s}} \\ \leq & C(n,s,\alpha,\delta,\sigma)v^{\alpha+n-1} \ .\end{aligned}$$

Regarding term V, for $\epsilon \leq 1$ we deduce in case $v \leq \epsilon$ that

$$|g_{\delta\sigma}(v+\epsilon) \cdot m_{\delta\sigma}(v)| \leq C(n,s,\alpha,\delta,\sigma)\epsilon^{\alpha+n-s}v^s \leq C(n,s,\alpha,\delta,\sigma)v^{\alpha+n} \ .$$

In case $v > \epsilon$ we have

$$\begin{aligned}|g_{\delta\sigma}(v+\epsilon) \cdot m_{\delta\sigma}(v)| \leq & C(n,s,\alpha,\delta,\sigma)(v^{\alpha+n-s} + v^{\alpha+n})\frac{v^{n+s}}{v^n + v^{n+s}} \\ \leq & C(n,s,\alpha,\delta,\sigma)v^{\alpha+n} \ .\end{aligned}$$

Again by dominated convergence, the term V converges.

It remains to prove convergence of term III. Using dominated convergence, the convergence of this term is established as soon as we have shown the appropriate estimates. Assume $\epsilon \leq 1$. For $v \leq \epsilon$ we get

$$\begin{aligned}&\left| \frac{(v+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(v+\epsilon)} \cdot m'_{\delta\sigma}(v) + (n+\alpha-1)\frac{(v+\epsilon)^{n+\alpha-2}}{m_{\delta\sigma}(v+\epsilon)} \cdot m_{\delta\sigma}(v) \right. \\ &\quad \left. - \frac{(v+\epsilon)^{n+\alpha-1}}{[m_{\delta\sigma}(v+\epsilon)]^2} \cdot m'_{\delta\sigma}(v+\epsilon) \cdot m_{\delta\sigma}(v) \right| \\ \leq & C(n,s,\alpha,\delta,\sigma)\frac{\epsilon^{n+\alpha-1}}{\epsilon^s} \cdot v^{s-1} + C(n,s,\alpha,\delta,\sigma)\frac{\epsilon^{n+\alpha-2}}{\epsilon^s} \cdot v^s \\ &\quad + C(n,s,\alpha,\delta,\sigma)\frac{\epsilon^{n+\alpha-1}}{\epsilon^{2s}} \cdot \epsilon^{s-1} \cdot v^s \\ \leq & C(n,s,\alpha,\delta,\sigma)v^{n+\alpha-2} \ .\end{aligned}$$

On the other hand, for $v > \epsilon$ we obtain

$$\begin{aligned}
&\left| \frac{(v+\epsilon)^{n+\alpha-1}}{m_{\delta\sigma}(v+\epsilon)} \cdot m'_{\delta\sigma}(v) + (n+\alpha-1)\frac{(v+\epsilon)^{n+\alpha-2}}{m_{\delta\sigma}(v+\epsilon)} \cdot m_{\delta\sigma}(v) \right. \\
&\quad \left. - \frac{(v+\epsilon)^{n+\alpha-1}}{[m_{\delta\sigma}(v+\epsilon)]^2} \cdot m'_{\delta\sigma}(v+\epsilon) \cdot m_{\delta\sigma}(v) \right| \\
\leq & C(n,s,\alpha,\delta,\sigma)\frac{v^{n+\alpha-1}}{m_{\delta\sigma}(v)} \cdot \frac{v^{n+s-1}}{v^n + v^s + v^{n+s}} + C(n,s,\alpha,\delta,\sigma)\frac{v^{n+\alpha-2}}{m_{\delta\sigma}(v)} \cdot m_{\delta\sigma}(v) \\
&\quad + C(n,s,\alpha,\delta,\sigma)\frac{v^{n+\alpha-1}}{[m_{\delta\sigma}(v)]^2} \cdot \frac{v^{n+s-1}}{v^n + v^s + v^{n+s}} \cdot m_{\delta\sigma}(v) \\
\leq & C(n,s,\alpha,\delta,\sigma) v^{n+\alpha-2} \ .
\end{aligned}$$

Summing up, we have shown that (recall that $b := n + \alpha$)

$$\begin{aligned}
&\int_\Omega G_{\delta\sigma}(u_{\delta\sigma}) \cdot \psi \ dx \Big|_{t_1}^{t_2} \\
=& - \int_{t_1}^{t_2} \int_\Omega \psi \cdot u_{\delta\sigma}^{n+\alpha-1} D^2 u_{\delta\sigma} : D^2 u_{\delta\sigma} \ dx \ dt \\
&- \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{n+\alpha-1} \nabla\psi \cdot D^2 u_{\delta\sigma} \cdot \nabla u_{\delta\sigma} \ dx \ dt \\
&- \int_{t_1}^{t_2} \int_\Omega \psi \cdot (n+\alpha-1) u_{\delta\sigma}^{n+\alpha-2} \nabla u_{\delta\sigma} \cdot D^2 u_{\delta\sigma} \cdot \nabla u_{\delta\sigma} \ dx \ dt \\
&- \int_{t_1}^{t_2} \int_\Omega \left[u_{\delta\sigma}^{n+\alpha-1} + g_{\delta\sigma}(u_{\delta\sigma}) \cdot m'_{\delta\sigma}(u_{\delta\sigma}) \right] \nabla u_{\delta\sigma} \cdot D^2 u_{\delta\sigma} \cdot \nabla\psi \ dx \ dt \\
&- \int_{t_1}^{t_2} \int_\Omega g_{\delta\sigma}(u_{\delta\sigma}) \cdot m_{\delta\sigma}(u_{\delta\sigma}) D^2 u_{\delta\sigma} : D^2\psi \ dx \ dt \\
=& - \int_{t_1}^{t_2} \int_\Omega \psi \cdot u_{\delta\sigma}^{b-1} D^2 u_{\delta\sigma} : D^2 u_{\delta\sigma} \ dx \ dt \\
&- \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b-1} \nabla\psi \cdot D^2 u_{\delta\sigma} \cdot \nabla u_{\delta\sigma} \ dx \ dt \\
&- \int_{t_1}^{t_2} \int_\Omega \psi \cdot (n+\alpha-1) u_{\delta\sigma}^{b-2} \nabla u_{\delta\sigma} \cdot D^2 u_{\delta\sigma} \cdot \nabla u_{\delta\sigma} \ dx \ dt \\
&- \int_{t_1}^{t_2} \int_\Omega \left[u_{\delta\sigma}^{b-1} + \frac{n}{\alpha} u_{\delta\sigma}^{b-1} \right] \nabla u_{\delta\sigma} \cdot D^2 u_{\delta\sigma} \cdot \nabla\psi \ dx \ dt \\
&- \int_{t_1}^{t_2} \int_\Omega \frac{1}{\alpha} u_{\delta\sigma}^{b} D^2 u_{\delta\sigma} : D^2\psi \ dx \ dt \\
&- \int_{t_1}^{t_2} \int_\Omega \left[g_{\delta\sigma}(u_{\delta\sigma}) \cdot m'_{\delta\sigma}(u_{\delta\sigma}) - \frac{n}{\alpha} u_{\delta\sigma}^{b-1} \right] \nabla u_{\delta\sigma} \cdot D^2 u_{\delta\sigma} \cdot \nabla\psi \ dx \ dt \\
&- \int_{t_1}^{t_2} \int_\Omega \left[g_{\delta\sigma}(u_{\delta\sigma}) \cdot m_{\delta\sigma}(u_{\delta\sigma}) - \frac{1}{\alpha} u_{\delta\sigma}^{b} \right] D^2 u_{\delta\sigma} : D^2\psi \ dx \ dt \ .
\end{aligned}$$

Recall that u is strictly positive and bounded for a.e. $t > 0$ and that $u(.,t) \in H^3_{loc}(\Omega)$ for a.e. $t > 0$. Several integrations by parts therefore yield

$$
\begin{aligned}
&\int_\Omega G_{\delta\sigma}(u_{\delta\sigma}) \cdot \psi \, dx \Bigg|_{t_1}^{t_2} \\
=& - \int_{t_1}^{t_2} \int_\Omega \psi \cdot u_{\delta\sigma}^{b-1} |D^2 u_{\delta\sigma}|^2 \, dx \, dt \\
&+ \frac{1}{2} \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b-1} |\nabla u_{\delta\sigma}|^2 \Delta\psi \, dx \, dt \\
&+ \frac{b-1}{2} \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b-2} |\nabla u_{\delta\sigma}|^2 \nabla u_{\delta\sigma} \cdot \nabla\psi \, dx \, dt \\
&- (b-1) \int_{t_1}^{t_2} \int_\Omega \psi \cdot u_{\delta\sigma}^{b-2} \nabla u_{\delta\sigma} \cdot D^2 u_{\delta\sigma} \cdot \nabla u_{\delta\sigma} \, dx \, dt \\
&+ \frac{b}{2\alpha} \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b-1} |\nabla u_{\delta\sigma}|^2 \Delta\psi \, dx \, dt \\
&+ \frac{b(b-1)}{2\alpha} \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b-2} |\nabla u_{\delta\sigma}|^2 \nabla u_{\delta\sigma} \cdot \nabla\psi \, dx \, dt \\
&+ \frac{b}{\alpha} \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b-1} \nabla u_{\delta\sigma} \cdot D^2\psi \cdot \nabla u_{\delta\sigma} \, dx \, dt \\
&+ \frac{1}{\alpha} \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b} \nabla u_{\delta\sigma} \cdot \nabla\Delta\psi \, dx \, dt \\
&- \int_{t_1}^{t_2} \int_\Omega \left[g_{\delta\sigma}(u_{\delta\sigma}) \cdot m'_{\delta\sigma}(u_{\delta\sigma}) - \frac{n}{\alpha} u_{\delta\sigma}^{b-1} \right] \nabla u_{\delta\sigma} \cdot D^2 u_{\delta\sigma} \cdot \nabla\psi \, dx \, dt \\
&- \int_{t_1}^{t_2} \int_\Omega \left[g_{\delta\sigma}(u_{\delta\sigma}) \cdot m_{\delta\sigma}(u_{\delta\sigma}) - \frac{1}{\alpha} u_{\delta\sigma}^{b} \right] D^2 u_{\delta\sigma} : D^2\psi \, dx \, dt \, .
\end{aligned}
$$

Further integrations by parts yield

$$\begin{aligned}
&\int_\Omega G_{\delta\sigma}(u_{\delta\sigma}) \cdot \psi \; dx \Big|_{t_1}^{t_2} \\
=&\frac{n-b}{\alpha} \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b-1} |D^2 u_{\delta\sigma}|^2 \psi \; dx \; dt \\
&+ \frac{1}{\alpha}\left(b - \frac{1}{2}n\right) \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b-1} |\nabla u_{\delta\sigma}|^2 \Delta\psi \; dx \; dt \\
&+ \frac{b}{\alpha} \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b-1} \nabla u_{\delta\sigma} \cdot D^2\psi \cdot \nabla u_{\delta\sigma} \; dx \; dt \\
&- \frac{1}{\alpha(b+1)} \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b+1} \Delta^2\psi \; dx \; dt \\
&+ \frac{1}{\alpha}\left(b - \frac{n}{2}\right)(b-1)(2-b) \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b-3} |\nabla u_{\delta\sigma}|^4 \psi \; dx \; dt \\
&+ \frac{1}{\alpha}(2n-3b)(b-1) \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b-2} \nabla u_{\delta\sigma} \cdot D^2 u_{\delta\sigma} \cdot \nabla u_{\delta\sigma} \; \psi \; dx \; dt \\
&+ \frac{1}{\alpha}\left(\frac{n}{2} - b\right)(b-1) \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b-2} |\nabla u_{\delta\sigma}|^2 \Delta u_{\delta\sigma} \; \psi \; dx \; dt \\
&- \int_{t_1}^{t_2} \int_\Omega \left[g_{\delta\sigma}(u_{\delta\sigma}) \cdot m'_{\delta\sigma}(u_{\delta\sigma}) - \frac{n}{\alpha} u_{\delta\sigma}^{b-1} \right] \nabla u_{\delta\sigma} \cdot D^2 u_{\delta\sigma} \cdot \nabla\psi \; dx \; dt \\
&- \int_{t_1}^{t_2} \int_\Omega \left[g_{\delta\sigma}(u_{\delta\sigma}) \cdot m_{\delta\sigma}(u_{\delta\sigma}) - \frac{1}{\alpha} u_{\delta\sigma}^{b} \right] D^2 u_{\delta\sigma} : D^2\psi \; dx \; dt \; .
\end{aligned}$$

Multiplying the equation by α, conditions (H1) to (H4) in connection with Lemma 17 now imply (for details see the proof of Lemma 18) that

$$\begin{aligned}
&\alpha \int_\Omega G_{\delta\sigma}(u_{\delta\sigma}(.,t_2)) \cdot \psi \; dx - \alpha \int_\Omega G_{\delta\sigma}(u_{\delta\sigma}(.,t_1)) \cdot \psi \; dx \\
\geq &\left(\frac{2}{3}b - \frac{1}{6}n\right) \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b-1} |\nabla u_{\delta\sigma}|^2 \Delta\psi \; dx \; dt \\
&+ \left(\frac{4}{3}b - \frac{1}{3}n\right) \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b-1} \nabla u_{\delta\sigma} \cdot D^2\psi \cdot \nabla u_{\delta\sigma} \; dx \; dt \\
&- \frac{1}{b+1} \int_{t_1}^{t_2} \int_\Omega u_{\delta\sigma}^{b+1} \Delta^2\psi \; dx \; dt \\
&- \int_{t_1}^{t_2} \int_\Omega \left[\alpha g_{\delta\sigma}(u_{\delta\sigma}) \cdot m'_{\delta\sigma}(u_{\delta\sigma}) - n u_{\delta\sigma}^{b-1} \right] \nabla u_{\delta\sigma} \cdot D^2 u_{\delta\sigma} \cdot \nabla\psi \; dx \; dt \\
&- \int_{t_1}^{t_2} \int_\Omega \left[\alpha g_{\delta\sigma}(u_{\delta\sigma}) \cdot m_{\delta\sigma}(u_{\delta\sigma}) - u_{\delta\sigma}^{b} \right] D^2 u_{\delta\sigma} : D^2\psi \; dx \; dt
\end{aligned} \tag{42}$$

for a.e. $t_2 > t_1 > 0$ and a.e. $t_2 > 0$ in case $t_1 = 0$.

We now pass to the limit $\delta \to 0$, then to the limit $\sigma \to 0$. The first three terms on the right-hand side are seen to converge since $u_{\delta\sigma}^{\frac{\alpha+n+1}{2}} \to u_\sigma^{\frac{\alpha+n+1}{2}}$ and $u_\sigma^{\frac{\alpha+n+1}{2}} \to u^{\frac{\alpha+n+1}{2}}$ strongly in $L^2_{loc}(I; H^1(\Omega))$ (see Proposition 1.6 in [51]).

We now show that for a.e. $t_1, t_2 > 0$ the terms on the left-hand side converge. Recall (41). Using the strong convergence of $u_{\delta\sigma}^{n+\alpha+1}$ and $u_\sigma^{n+\alpha+1}$ in $L^1_{loc}(I; L^1(\Omega))$, we first deduce for a subsequence convergence of $u_{\delta\sigma}^{n+\alpha+1}(.,t)$ and $u_\sigma^{n+\alpha+1}(.,t)$ for a.e. $t > 0$ in $L^1(\Omega)$. This implies convergence of the integral of the second term in the definition of $G_{\delta\sigma}(u_{\delta\sigma}(.,t)) \cdot \psi$ (i.e. convergence of $\int_\Omega u_{\delta\sigma}^{1+\alpha}(.,t)\psi \; dx$) to the desired limit for a.e. $t > 0$ and for $t = 0$ since $1 + \alpha \in (0,1)$. Moreover, by the uniform boundedness of $u_{\delta\sigma}$ (with respect to $\delta \in (0,1)$, $\sigma \in (0,1)$) in $L^\infty(I; H^1(\Omega))$ and the Sobolev embedding, we see that the integral of the third term in the definition of $G_{\delta\sigma}(u_{\delta\sigma}(.,t)) \cdot \psi$ (i.e. $\sigma \int_\Omega u_{\delta\sigma}^{1+n+\alpha}(.,t)\psi \; dx$) is bounded by $C(d,\Omega)\sigma ||\psi||_{L^\infty(\Omega\times I)} ||u_{\delta\sigma}||^{1+n+\alpha}_{L^\infty(I;H^1(\Omega))}$; therefore this term vanishes in the limit $\sigma \to 0$. It remains to prove convergence to zero of the integral of the first term in the definition of $G_{\delta\sigma}(u_{\delta\sigma}(.,t)) \cdot \psi$ (i.e. convergence to zero of $\delta \int_\Omega u_{\delta\sigma}^{\alpha+n-s+1}(.,t)\psi \; dx$); this is more involved since $\alpha + n - s + 1 < 0$ due to $s > 4$ and $n < 3$, $\alpha < 0$.

Considering the $\tilde\alpha$ entropy inequality (see Proposition 1.2 in [51] and relation (4) in [12]) for $u_{\delta\sigma}$ for $\tilde\alpha := \alpha - \nu$ with $\nu > 0$ sufficiently small, we deduce using $(\alpha - \nu + n - s + 1) < 0$ and $(\alpha - \nu + n) > 0$ as well as $\alpha - \nu \in (-1, 0)$ (these inequalities hold since $\nu > 0$ is small) that for a.e. $t > 0$

$$
\begin{aligned}
&\int_\Omega \delta u_{\delta\sigma}^{\alpha-\nu+n-s+1}(x,t) \; dx \\
\leq & C(n,\alpha,\nu,s) \int_\Omega u_{\delta\sigma}^{1+\alpha-\nu}(x,t) \; dx \\
&+ C(n,\alpha,\nu,s) \int_\Omega \delta \cdot (u_0 + \delta^{\theta_1} + \sigma^{\theta_2})^{\alpha-\nu+n-s+1} \; dx \\
&+ C(n,\alpha,\nu,s) \int_\Omega \sigma \cdot (u_0 + \delta^{\theta_1} + \sigma^{\theta_2})^{n+\alpha-\nu+1} \; dx \; .
\end{aligned}
$$

We know that mass is conserved; in connection with Hölder's inequality this gives a uniform (with respect to t and $\delta \in (0,1)$ and $\sigma \in (0,1)$) bound on $\int_\Omega u_{\delta\sigma}^{1+\alpha-\nu}(x,t) \; dx$ (since $0 < 1 + \alpha - \nu < 1$). The third term on the right-hand side is bounded by $C(d,n,\alpha,\nu,s,\Omega)\sigma ||u_{\delta\sigma}(.,0)||^{1+n+\alpha-\nu}_{H^1(\Omega)}$; this yields a uniform (in $\delta \in (0,1)$, $\sigma \in (0,1)$) bound. The second term on the right-hand side is bounded by $C(n,\alpha,\nu,s)\delta|\Omega|\sigma^{\theta_2\cdot(n-s+\alpha-\nu+1)}$ (note that the exponent is negative). For $\nu > 0$ small enough and $\sigma > 0$ fixed we therefore obtain a uniform (with respect to $\delta \in (0,1)$ and $t > 0$) bound for $\int_\Omega \delta u_{\delta\sigma}^{\alpha-\nu+n-s+1}(x,t) \; dx$. For every $\mu > 0$ we now have

$$
\int_\Omega \delta u_{\delta\sigma}^{\alpha+n-s+1}(x,t) \; dx \leq \delta\mu^{\alpha+n-s+1}|\Omega| + \mu^\nu \int_\Omega \delta u_{\delta\sigma}^{\alpha-\nu+n-s+1}(x,t) \; dx \; .
$$

The latter integral being bounded uniformly with respect to $\delta \in (0,1)$ and t, setting $\mu := \delta^\beta$ for $\beta > 0$ small enough the convergence to zero of the integral of the first term in the definition of $G_{\delta\sigma}(u_{\delta\sigma}) \cdot \psi$ (i.e. of $\delta \int_\Omega u_{\delta\sigma}^{\alpha+n-s+1}(.,t)\psi \; dx$; note that ψ is bounded) as $\delta \to 0$ follows.

It remains to prove that the last two terms on the right-hand side in (42) converge to zero. We compute

$$
\begin{aligned}
&|\alpha g_{\delta\sigma}(v) m_{\delta\sigma}(v) - v^{n+\alpha}| \\
\leq &\left| v^{\alpha} \frac{v^{n+s}}{\delta v^n + v^s + \sigma v^{n+s}} - v^{n+\alpha} \right| + \delta C(n,\alpha,s) v^{n+\alpha-s} \frac{v^{n+s}}{\delta v^n + v^s + \sigma v^{n+s}} \\
&+ \sigma C(n,\alpha,s) v^{n+\alpha} \frac{v^{n+s}}{\delta v^n + v^s + \sigma v^{n+s}} \\
\leq &C(n,\alpha,s) \frac{\delta v^n + \sigma v^{n+s}}{\delta v^n + v^s + \sigma v^{n+s}} v^{n+\alpha} \ .
\end{aligned}
$$

Using this estimate and Young's inequality we get for all $\mu > 0$

$$
\begin{aligned}
&\left| \int_{t_1}^{t_2} \int_{\Omega} \left[\alpha g_{\delta\sigma}(u_{\delta\sigma}) \cdot m_{\delta\sigma}(u_{\delta\sigma}) - u_{\delta\sigma}^b \right] D^2 u_{\delta\sigma} : D^2 \psi \ dx \ dt \right| \\
\leq &\frac{\mu}{2} \int_{t_1}^{t_2} \int_{\Omega} u_{\delta\sigma}^{b-1} |D^2 u_{\delta\sigma}|^2 \ dx \ dt \\
&+ \frac{1}{2\mu} \int_{t_1}^{t_2} \int_{\Omega} C(n,\alpha,s) \left(\frac{\delta u_{\delta\sigma}^n + \sigma u_{\delta\sigma}^{n+s}}{\delta u_{\delta\sigma}^n + u_{\delta\sigma}^s + \sigma u_{\delta\sigma}^{n+s}} \right)^2 u_{\delta\sigma}^{n+\alpha+1} |D^2 \psi|^2 \ dx \ dt \ .
\end{aligned}
$$

By Vitali's convergence theorem and the convergence $u_{\delta\sigma}^{\frac{\alpha+n+1}{2}} \to u_{\sigma}^{\frac{\alpha+n+1}{2}}$ and $u_{\sigma}^{\frac{\alpha+n+1}{2}} \to u^{\frac{\alpha+n+1}{2}}$ strongly in $L^2_{loc}(I; L^2(\Omega))$, the second integral on the right-hand side tends to zero when passing to the limits $\delta \to 0$ and $\sigma \to 0$; the first integral is known to be bounded uniformly. Since $\mu > 0$ is arbitrary, the term on the left-hand side converges to zero.

We have

$$
\begin{aligned}
&|\alpha g_{\delta\sigma}(v) m'_{\delta\sigma}(v) - n v^{n+\alpha-1}| \\
\leq &\left| v^{\alpha} \frac{(\delta v^n + v^s + \sigma v^{n+s})(n+s) v^{n+s-1}}{(\delta v^n + v^s + \sigma v^{n+s})^2} \right. \\
&\quad \left. + v^{\alpha} \frac{-v^{n+s}(n\delta v^{n-1} + s v^{s-1} + (n+s)\sigma v^{n+s-1})}{(\delta v^n + v^s + \sigma v^{n+s})^2} - n v^{n+\alpha-1} \right| \\
&+ C(n,s,\alpha)(\delta v^{n+\alpha-s} + \sigma v^{n+\alpha}) \\
&\quad \cdot \left[\frac{(\delta v^n + v^s + \sigma v^{n+s})(n+s) v^{n+s-1}}{(\delta v^n + v^s + \sigma v^{n+s})^2} \right. \\
&\qquad \left. + \frac{v^{n+s}(n\delta v^{n-1} + s v^{s-1} + (n+s)\sigma v^{n+s-1})}{(\delta v^n + v^s + \sigma v^{n+s})^2} \right] \\
\leq &C(n,s,\alpha) \frac{\delta v^n + \sigma v^{n+s}}{\delta v^n + v^s + \sigma v^{n+s}} v^{n+\alpha-1} \ .
\end{aligned}
$$

We therefore obtain by Young's inequality

$$\begin{aligned}&\left|\int_{t_1}^{t_2}\int_\Omega \left[\alpha g_{\delta\sigma}(u_{\delta\sigma})\cdot m'_{\delta\sigma}(u_{\delta\sigma}) - n u_{\delta\sigma}^{b-1}\right]\nabla u_{\delta\sigma}\cdot D^2 u_{\delta\sigma}\cdot\nabla\psi \; dx\; dt\right|\\
\leq&\mu\int_{t_1}^{t_2}\int_\Omega u_{\delta\sigma}^{n+\alpha-1}|D^2u_{\delta\sigma}|^2\; dx\; dt + \mu\int_{t_1}^{t_2}\int_\Omega u_{\delta\sigma}^{n+\alpha-3}|\nabla u_{\delta\sigma}|^4\; dx\; dt\\
&+\frac{C(n,s,\alpha)}{\mu^3}\int_{t_1}^{t_2}\int_\Omega\left(\frac{\delta u_{\delta\sigma}^n+\sigma u_{\delta\sigma}^{n+s}}{\delta u_{\delta\sigma}^n + u_{\delta\sigma}^s+\sigma u_{\delta\sigma}^{n+s}}\right)^4 u_{\delta\sigma}^{n+\alpha+1}|\nabla\psi|^4\; dx\; dt\end{aligned}$$

and conclude that the term on the left-hand side tends to zero using Vitali's theorem, the uniform bounds on $u_{\delta\sigma}$, and the arbitrariness of μ. □

4.2 Suboptimal estimates on asymptotic support propagation rates

In this section, we derive a differential inequality for the weighted entropy $\int_{\mathbb{R}^d} u^{1+\alpha}(.,t)|x-x_0|^\gamma \; dx$ in order to obtain a first lower bound on support propagation, which however is not yet optimal.

We need the following version of Hardy's inequality:

Lemma 24 (Hardy's inequality)**.** *For any $v\in H^1(\mathbb{R}^d)$ with* $\operatorname{supp} v \subset\subset \mathbb{R}^d\setminus\{0\}$ *and any $\psi\in C^\infty(\mathbb{R}^d\setminus\{0\})$ with $\Delta\psi>0$ on $\mathbb{R}^d\setminus\{0\}$ the inequality*

$$\int_{\mathbb{R}^d} v^2\Delta\psi\; dx \leq 4\int_{\mathbb{R}^d}\left|\frac{\nabla\psi}{|\nabla\psi|}\cdot\nabla v\right|^2\frac{|\nabla\psi|^2}{\Delta\psi}\; dx$$

holds.

Proof. Integration by parts and Hölder's inequality give

$$\begin{aligned}&\int_{\mathbb{R}^d} v^2\Delta\psi\; dx = -2\int_{\mathbb{R}^d} v\nabla v\cdot\nabla\psi\; dx\\
\leq&2\left(\int_{\mathbb{R}^d} v^2\Delta\psi\; dx\right)^{\frac12}\left(\int_{\mathbb{R}^d}\frac{1}{\Delta\psi}|\nabla v\cdot\nabla\psi|^2\; dx\right)^{\frac12}.\end{aligned}$$

The inequality now follows easily. □

Combining the results of the previous section with Hardy's inequality, we shall prove the following lemma:

Lemma 25. *Let $u_0\in H^1(\mathbb{R}^d)$, $1\leq d\leq 3$, be nonnegative and compactly supported. Let u be a strong energy solution of the Cauchy problem for the thin-film equation with initial data u_0 and $n\in\left(2-\sqrt{\frac{8}{8+d}},3\right)$ or let u be a strong solution of the thin-film equation constructed as in [12] and $n\in(1,2)$.*

Suppose that conditions (H1) to (H4) of Lemma 18 are satisfied and assume that u satisfies the α entropy estimate. Given $\gamma\leq-\frac12$, suppose furthermore that

(H5a) The condition

$$\frac{2b-\frac{1}{2}n}{(b+1)^2}\cdot\frac{(\gamma-4+d)(\gamma+\frac{d-4}{3})}{(\gamma-2)(\gamma-2+d)}-\frac{1}{b+1}\geq\tau$$

is satisfied for some $\tau>0$.

(H6) We have $\gamma\leq -d$.

(H7) It holds that $\gamma+d+4\frac{1+\alpha}{n}>0$.

Let $x_0\in\mathbb{R}^d\setminus\operatorname{supp}u_0$ be some point.

Let $t_0>0$. Suppose that

$$t_0\geq\epsilon[\operatorname{diam}(\operatorname{supp}u_0)+\operatorname{dist}(x_0,\operatorname{supp}u_0)]^{4+n\cdot d}||u_0||_{L^1(\mathbb{R}^d)}^{-n}\tag{43}$$

holds for some $\epsilon>0$. Define

$$T^*:=\inf\left\{T\geq 0:\inf_{0\leq t\leq T}\operatorname{dist}(x_0,\operatorname{supp}u(.,t))=0\right\}.$$

Then there exists a constant $C(d,n,\alpha,\gamma,\epsilon)>0$ such that the estimate

$$T^*\leq\max\Bigg(2t_0,\quad C(d,n,\alpha,\gamma,\epsilon)||u_0||_{L^1(\mathbb{R}^d)}^{\frac{4+4\alpha+n(d+\gamma)}{-\gamma+\alpha\cdot d}}\cdot\left[\int_{\mathbb{R}^d}u^{1+\alpha}(.,t_0)|x-x_0|^\gamma\,dx\right]^{-\frac{4+n\cdot d}{-\gamma+\alpha\cdot d}}\Bigg)$$

holds.

Proof. By our assumptions, Lemma 18 and/or Lemma 23 are applicable. We may assume $t_0<T^*$ as otherwise the assertion of our lemma is trivial.

Let $T<T^*$. The function $|x-x_0|^\gamma$ is smooth in some neighbourhood of the set $\bigcup_{t\in[0,T]}\operatorname{supp}u(.,t)$. By the FSOP estimate Theorem 10, $|x-x_0|^\gamma$ coincides on $\bigcup_{t\in[0,T]}\operatorname{supp}u(.,t)$ with a nonnegative smooth compactly supported function.

Thus, we may use $|x-x_0|^\gamma$ as a test function in Lemma 18 or Lemma 23. This yields for $t_0\leq t_1\leq t_2<T^*$

$$\begin{aligned}&\int_{\mathbb{R}^d}\frac{1}{1+\alpha}u^{1+\alpha}(.,t_2)|x-x_0|^\gamma\,dx-\int_{\mathbb{R}^d}\frac{1}{1+\alpha}u^{1+\alpha}(.,t_1)|x-x_0|^\gamma\,dx\\ &\geq\left(\frac{2}{3}b-\frac{1}{6}n\right)\int_{t_1}^{t_2}\int_{\mathbb{R}^d}u^{b-1}|\nabla u|^2\Delta|x-x_0|^\gamma\,dx\,dt\\ &\quad+\left(\frac{4}{3}b-\frac{1}{3}n\right)\int_{t_1}^{t_2}\int_{\mathbb{R}^d}u^{b-1}\nabla u\cdot D^2|x-x_0|^\gamma\cdot\nabla u\,dx\,dt\\ &\quad-\frac{1}{b+1}\int_{t_1}^{t_2}\int_{\mathbb{R}^d}u^{b+1}\Delta^2|x-x_0|^\gamma\,dx\,dt\,.\end{aligned}$$

We calculate

$$
\begin{aligned}
D^2|x-x_0|^\gamma &= D(\gamma|x-x_0|^{\gamma-2}(x-x_0)) \\
=&\gamma(\gamma-2)|x-x_0|^{\gamma-4}(x-x_0)\otimes(x-x_0)+\gamma|x-x_0|^{\gamma-2}Id\ .
\end{aligned}
$$

Combining the last two formulas, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\frac{1}{1+\alpha}u^{1+\alpha}(.,t_2)|x-x_0|^\gamma\ dx-\int_{\mathbb{R}^d}\frac{1}{1+\alpha}u^{1+\alpha}(.,t_1)|x-x_0|^\gamma\ dx \\
\geq&\gamma\cdot(\gamma-2+d)\cdot\left(\frac{2}{3}b-\frac{1}{6}n\right)\int_{t_1}^{t_2}\int_{\mathbb{R}^d}u^{b-1}|\nabla u|^2|x-x_0|^{\gamma-2}\ dx\ dt \\
&+\gamma(\gamma-1)\left(\frac{4}{3}b-\frac{1}{3}n\right)\int_{t_1}^{t_2}\int_{\mathbb{R}^d}u^{b-1}\left|\frac{x-x_0}{|x-x_0|}\cdot\nabla u\right|^2|x-x_0|^{\gamma-2}\ dx\ dt \\
&+\gamma\left(\frac{4}{3}b-\frac{1}{3}n\right) \\
&\quad\cdot\int_{t_1}^{t_2}\int_{\mathbb{R}^d}u^{b-1}\nabla u\cdot\left(Id-\frac{x-x_0}{|x-x_0|}\otimes\frac{x-x_0}{|x-x_0|}\right)\cdot\nabla u\ |x-x_0|^{\gamma-2}\ dx\ dt \\
&-\gamma(\gamma-2+d)(\gamma-2)(\gamma-4+d)\frac{1}{b+1}\int_{t_1}^{t_2}\int_{\mathbb{R}^d}u^{b+1}|x-x_0|^{\gamma-4}\ dx\ dt\ .
\end{aligned}
$$

We now rewrite

$$
|\nabla u|^2=\nabla u\cdot\left(Id-\frac{x-x_0}{|x-x_0|}\otimes\frac{x-x_0}{|x-x_0|}\right)\cdot\nabla u+\left|\frac{x-x_0}{|x-x_0|}\cdot\nabla u\right|^2\ .
$$

Thus we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\frac{1}{1+\alpha}u^{1+\alpha}(.,t_2)|x-x_0|^\gamma\ dx-\int_{\mathbb{R}^d}\frac{1}{1+\alpha}u^{1+\alpha}(.,t_1)|x-x_0|^\gamma\ dx \\
\geq&\gamma(3\gamma-4+d)\left(\frac{2}{3}b-\frac{1}{6}n\right)\int_{t_1}^{t_2}\int_{\mathbb{R}^d}u^{b-1}\left|\frac{x-x_0}{|x-x_0|}\cdot\nabla u\right|^2|x-x_0|^{\gamma-2}\ dx\ dt \\
&+\gamma(\gamma+d)\left(\frac{2}{3}b-\frac{1}{6}n\right) \\
&\quad\cdot\int_{t_1}^{t_2}\int_{\mathbb{R}^d}u^{b-1}\nabla u\cdot\left(Id-\frac{x-x_0}{|x-x_0|}\otimes\frac{x-x_0}{|x-x_0|}\right)\cdot\nabla u\ |x-x_0|^{\gamma-2}\ dx\ dt \\
&-\gamma(\gamma-2+d)(\gamma-2)(\gamma-4+d)\frac{1}{b+1}\int_{t_1}^{t_2}\int_{\mathbb{R}^d}u^{b+1}|x-x_0|^{\gamma-4}\ dx\ dt\ .
\end{aligned}
$$

We may drop the second term on the right-hand side since it is nonnegative by (H6) and (H2) . Thus we get

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\frac{1}{1+\alpha}u^{1+\alpha}(.,t_2)|x-x_0|^\gamma\ dx-\int_{\mathbb{R}^d}\frac{1}{1+\alpha}u^{1+\alpha}(.,t_1)|x-x_0|^\gamma\ dx \\
\geq&4\frac{\gamma(\gamma+\frac{d-4}{3})}{(b+1)^2}\left(2b-\frac{1}{2}n\right)\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\left|\frac{x-x_0}{|x-x_0|}\cdot\nabla u^{\frac{b+1}{2}}\right|^2|x-x_0|^{\gamma-2}\ dx\ dt \\
&-\gamma(\gamma-2+d)(\gamma-2)(\gamma-4+d)\frac{1}{b+1}\int_{t_1}^{t_2}\int_{\mathbb{R}^d}u^{b+1}|x-x_0|^{\gamma-4}\ dx\ dt\ .
\end{aligned}
$$

An application of Hardy's inequality (Lemma 24) with $\psi = |x-x_0|^{\gamma-2}$ yields

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^\gamma \, dx - \int_{\mathbb{R}^d} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^\gamma \, dx \\
\geq & \gamma(\gamma-2+d)(\gamma-2)(\gamma-4+d) \\
& \cdot \left[\frac{2b-\frac{1}{2}n}{(b+1)^2} \cdot \frac{(\gamma-4+d)(\gamma+\frac{d-4}{3})}{(\gamma-2)(\gamma-2+d)} - \frac{1}{b+1} \right] \\
& \cdot \int_{t_1}^{t_2} \int_{\mathbb{R}^d} u^{b+1} |x-x_0|^{\gamma-4} \, dx \, dt \, .
\end{aligned}$$

By (H5a) we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_2)|x-x_0|^\gamma \, dx - \int_{\mathbb{R}^d} \frac{1}{1+\alpha} u^{1+\alpha}(.,t_1)|x-x_0|^\gamma \, dx \\
\geq & \tau \int_{t_1}^{t_2} \int_{\mathbb{R}^d} u^{b+1} |x-x_0|^{\gamma-4} \, dx \, dt \, .
\end{aligned} \tag{44}$$

Hölder's inequality gives

$$\begin{aligned}
&\int_{\mathbb{R}^d} u^{1+\alpha}(.,t)|x-x_0|^\gamma \, dx \\
\leq & \left(\int_{\operatorname{supp} u(.,t)} |x-x_0|^{\gamma+4\frac{1+\alpha}{n}} \, dx \right)^{\frac{n}{b+1}} \left(\int_{\mathbb{R}^d} u^{b+1}(.,t)|x-x_0|^{\gamma-4} \, dx \right)^{\frac{1+\alpha}{b+1}} .
\end{aligned}$$

Let $y \in \operatorname{supp} u_0$. By Theorem 10 we know that $\operatorname{supp} u(.,t) \subset B_{R(t)}(y)$, where $R(t) = \operatorname{diam}(\operatorname{supp} u_0) + C(n,d)||u_0||_{L^1(\mathbb{R}^d)}^{\frac{n}{4+d\cdot n}} t^{\frac{1}{4+d\cdot n}}$. Rearranging the previous inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} u^{b+1}(.,t)|x-x_0|^{\gamma-4} \, dx \\
\geq & \left(\int_{B_{R(t)+\operatorname{dist}(y,x_0)}(x_0)} |x-x_0|^{\gamma+4\frac{1+\alpha}{n}} \, dx \right)^{-\frac{n}{1+\alpha}} \\
& \cdot \left(\int_{\mathbb{R}^d} u^{1+\alpha}(.,t)|x-x_0|^\gamma \, dx \right)^{\frac{b+1}{1+\alpha}} .
\end{aligned}$$

Using (H7) and (44), we therefore arrive at the differential inequality

$$\begin{aligned}
&\frac{1}{1+\alpha} \int_{\mathbb{R}^d} u^{1+\alpha}(.,t_2)|x-x_0|^\gamma \, dx - \frac{1}{1+\alpha} \int_{\mathbb{R}^d} u^{1+\alpha}(.,t_1)|x-x_0|^\gamma \, dx \\
\geq & c(n,\alpha,d,\gamma) \int_{t_1}^{t_2} [R(t)+\operatorname{dist}(y,x_0)]^{-\left(\gamma+4\frac{1+\alpha}{n}+d\right)\cdot\frac{n}{1+\alpha}} \\
& \cdot \left(\int_{\mathbb{R}^d} u^{1+\alpha}(.,t)|x-x_0|^\gamma \, dx \right)^{\frac{b+1}{1+\alpha}} \, dt \, .
\end{aligned}$$

Taking into account our assumption on t_0 (43) and using the fact that $y \in \operatorname{supp} u_0$ was arbitrary, we obtain for all t_2, t_1 with $t_2 \geq t_1$ and $t_1, t_2 \in [t_0, T)$

$$\begin{aligned}
&\frac{1}{1+\alpha}\int_{\mathbb{R}^d} u^{1+\alpha}(.,t_2)|x-x_0|^\gamma \, dx - \frac{1}{1+\alpha}\int_{\mathbb{R}^d} u^{1+\alpha}(.,t_1)|x-x_0|^\gamma \, dx \\
\geq & c(n,\alpha,d,\gamma,\epsilon)\int_{t_1}^{t_2} \left[||u_0||_{L^1(\mathbb{R}^d)}^{\frac{n}{4+n\cdot d}} t^{\frac{1}{4+n\cdot d}}\right]^{-\left(\gamma+4\frac{1+\alpha}{n}+d\right)\cdot\frac{n}{1+\alpha}} \\
&\cdot\left(\int_{\mathbb{R}^d} u^{1+\alpha}(.,t)|x-x_0|^\gamma \, dx\right)^{\frac{b+1}{1+\alpha}} \, dt \, .
\end{aligned}$$

The solution of the differential equation $\frac{d}{dt}z(t) = b(t)\cdot(z(t))^p$, $z(t_0) = a$, is given by $z(t) = (a^{1-p} - (p-1)\int_{t_0}^t b(t) \, dt)^{\frac{1}{1-p}}$; in particular the solution blows up as soon as the term in brackets becomes zero. Set $p := \frac{b+1}{1+\alpha}$. Using the comparison principle, we see that blowup of the weighted entropy $\int_{\mathbb{R}^d} u^{1+\alpha}(.,t)|x-x_0|^\gamma \, dx$ occurs before or at time T if the condition

$$\begin{aligned}
&c(n,\alpha,d,\gamma,\epsilon)\int_{t_0}^{T} \left[||u_0||_{L^1(\mathbb{R}^d)}^{\frac{n}{4+n\cdot d}} t^{\frac{1}{4+n\cdot d}}\right]^{-\left(\gamma+4\frac{1+\alpha}{n}+d\right)\cdot\frac{n}{1+\alpha}} \, dt \\
\geq & \left(\int_{\mathbb{R}^d} u^{1+\alpha}(.,t_0)|x-x_0|^\gamma \, dx\right)^{-\frac{n}{1+\alpha}}
\end{aligned}$$

is satisfied. This condition is implied by the condition

$$\begin{aligned}
&c(n,\alpha,d,\gamma,\epsilon)||u_0||_{L^1(\mathbb{R}^d)}^{-\frac{n}{4+n\cdot d}\cdot\left(\gamma+4\frac{1+\alpha}{n}+d\right)\cdot\frac{n}{1+\alpha}} \cdot t^{\frac{\alpha\cdot n\cdot d-\gamma\cdot n}{(1+\alpha)(4+n\cdot d)}}\Big|_{t=t_0}^{t=T} \\
\geq & \left(\int_{\mathbb{R}^d} u^{1+\alpha}(.,t_0)|x-x_0|^\gamma \, dx\right)^{-\frac{n}{1+\alpha}}
\end{aligned}$$

(note that the exponent at t is nonnegative since $\alpha \in (-1,0]$ and since $\gamma \leq -d$) which in turn in case $T \geq 2t_0$ is implied by

$$\begin{aligned}
&T^{\frac{\alpha\cdot n\cdot d-\gamma\cdot n}{(1+\alpha)(4+n\cdot d)}} \\
\geq & C(n,\alpha,d,\gamma,\epsilon)||u_0||_{L^1(\mathbb{R}^d)}^{\frac{n}{4+n\cdot d}\cdot\left(\gamma+4\frac{1+\alpha}{n}+d\right)\cdot\frac{n}{1+\alpha}} \cdot \left(\int_{\mathbb{R}^d} u^{1+\alpha}(.,t_0)|x-x_0|^\gamma \, dx\right)^{-\frac{n}{1+\alpha}} .
\end{aligned}$$

This proves our lemma since blowup of the entropy cannot occur before T^* as we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} u^{1+\alpha}(.,t)|x-x_0|^\gamma \, dx \\
\leq & \left(\int_{\mathbb{R}^d} u(.,t) \, dx\right)^{1+\alpha}\left(\int_{\{|x-x_0|\geq \operatorname{dist}(x_0,\operatorname{supp} u(.,t))\}} |x-x_0|^{-\frac{\gamma}{\alpha}} \, dx\right)^{-\alpha}
\end{aligned}$$

(note that the right-hand side is finite for $t < T^*$ since $\int_{\mathbb{R}^d} u(.,t) \, dx = ||u_0||_{L^1(\mathbb{R}^d)}$ and since the second integral is finite due to $\alpha \in (-1,0)$ and $\gamma \leq -d$). □

4.3 Estimate on entropy production

Our next aim is to bound the entropy $\int u^{1+\alpha}(.,t_0)\,dx$ from below; the estimate for this quantity will provide the starting point for an application of the results from the previous section.

Lemma 26. *Assume $d \leq 3$ and $n \in (1,3)$. Let $u_0 \in H^1(\mathbb{R}^d)$ be nonnegative and compactly supported. Let u either be a strong solution of the Cauchy problem for the thin-film equation constructed as in [12] and $n \in (1,2)$ or let u be a strong energy solution of the Cauchy problem for the thin-film equation and $n \in \left(2-\sqrt{\frac{8}{8+d}}, 3\right)$. Assume that u satisfies the α entropy estimate, where $\alpha \in (-1,0)$. Set $R_0 := \operatorname{diam}[\operatorname{supp} u_0]$. For any $\epsilon > 0$ the following assertion holds: If the condition*

$$t_0 \geq \epsilon R_0^{4+n\cdot d} ||u_0||_{L^1(\mathbb{R}^d)}^{-n}$$

is satisfied, then there exists a constant $c(d,n,\alpha,\epsilon) > 0$ such that

$$\int_{\mathbb{R}^d} u^{1+\alpha}(.,t_0)\,dx \geq c(d,n,\alpha,\epsilon)||u_0||_{L^1(\mathbb{R}^d)}^{1+\alpha-\frac{\alpha\cdot n\cdot d}{4+n\cdot d}} t_0^{\frac{-\alpha\cdot d}{4+n\cdot d}}\ .$$

Proof. By the α entropy inequality we have

$$\int_{\mathbb{R}^d} u^{1+\alpha}(.,t_0)\,dx \geq \int_{\mathbb{R}^d} u_0^{1+\alpha}\,dx + c(\alpha,n)\int_0^{t_0}\int_{\mathbb{R}^d} |\nabla u^{\frac{1+n+\alpha}{4}}|^4\,dx\,dt$$

We have $\operatorname{diam}[\operatorname{supp} u(.,t)] \leq C(d,n)(R_0 + ||u_0||_{L^1(\mathbb{R}^d)}^{\frac{n}{4+n\cdot d}} t^{\frac{1}{4+n\cdot d}})$; this is a consequence of Theorem 10. Moreover we have $\int_{\mathbb{R}^d} u(.,t)\,dx = ||u_0||_{L^1(\mathbb{R}^d)}$. This implies by the Poincare-Sobolev inequality on the ball with radius $C(d,n)(R_0 + ||u_0||_{L^1(\mathbb{R}^d)}^{\frac{n}{4+n\cdot d}} t^{\frac{1}{4+n\cdot d}})$ (note that $4 > d$ which implies that any L^p norm of $u^{\frac{1+n+\alpha}{4}}$ may be estimated in terms of the L^4 norm of $\nabla u^{\frac{1+n+\alpha}{4}}$) that for a.e. $t > 0$

$$\begin{aligned}
||u_0||_{L^1(\mathbb{R}^d)}^{1+n+\alpha} &= \left(\int_{\mathbb{R}^d} u(.,t)\,dx\right)^{1+n+\alpha}\\
&\leq C(d,n,\alpha)\left(R_0 + ||u_0||_{L^1(\mathbb{R}^d)}^{\frac{n}{4+n\cdot d}} t^{\frac{1}{4+n\cdot d}}\right)^{4+(n+\alpha)\cdot d}\int_{\mathbb{R}^d} |\nabla u^{\frac{1+n+\alpha}{4}}|^4\,dx\ .
\end{aligned}$$

Putting these inequalities together, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} u^{1+\alpha}(.,t_0)\,dx\\
\geq& c(d,n,\alpha)||u_0||_{L^1(\mathbb{R}^d)}^{1+n+\alpha}\int_0^{t_0}\left(R_0 + ||u_0||_{L^1(\mathbb{R}^d)}^{\frac{n}{4+n\cdot d}} t^{\frac{1}{4+n\cdot d}}\right)^{-4-(n+\alpha)\cdot d} dt\\
\geq& c(d,n,\alpha,\epsilon)||u_0||_{L(\mathbb{R}^d)}^{1+n+\alpha}\int_{\frac{t_0}{2}}^{t_0}\left(||u_0||_{L^1(\mathbb{R}^d)}^{\frac{n}{4+n\cdot d}} t^{\frac{1}{4+n\cdot d}}\right)^{-4-(n+\alpha)\cdot d} dt\\
\geq& c(d,n,\alpha,\epsilon||u_0||_{L^1(\mathbb{R}^d)}^{1+\alpha-\frac{\alpha\cdot n\cdot d}{4+n\cdot d}} t_0^{\frac{-\alpha\cdot d}{4+n\cdot d}}
\end{aligned}$$

where in the second step we have used the assumption $t_0 \geq \epsilon R_0^{4+n\cdot d}||u_0||_{L^1(\mathbb{R}^d)}^{-n}$ and in the third step we have used the fact that $\frac{-\alpha\cdot d}{4+n\cdot d} > 0$. □

4.4 Optimal lower bounds on asymptotic support propagation rates

We are now in position to prove our main result on asymptotic support propagation.

Proof of Theorem 11. Define $r := \operatorname{dist}(x_0, \operatorname{supp} u_0) + \operatorname{diam}(\operatorname{supp} u_0)$. Set $t_0 := r^{4+n\cdot d}||u_0||_{L^1(\mathbb{R}^d)}^{-n}$. For $\alpha \in (-1,0)$, we obtain by Lemma 26

$$\int_{\mathbb{R}^d} u^{1+\alpha}(.,t_0)\ dx \geq c(d,n,\alpha)||u_0||_{L^1(\mathbb{R}^d)}^{1+\alpha} r^{-\alpha\cdot d}\ .$$

Let $y \in \operatorname{supp} u_0$. We now know by Theorem 10 that $\operatorname{supp} u(.,t_0) \subset B_{R(t_0)}(y)$, where $R(t_0) = \operatorname{diam}(\operatorname{supp} u_0) + C(d,n)||u_0||_{L^1(\mathbb{R}^d)}^{\frac{n}{4+d\cdot n}} t_0^{\frac{1}{4+d\cdot n}}$. Putting these considerations together, we obtain in case $\gamma < 0$

$$\begin{aligned}\int_{\mathbb{R}^d} u^{1+\alpha}(.,t_0)|x-x_0|^\gamma\ dx \geq& c(d,n)r^\gamma \int_{\mathbb{R}^d} u^{1+\alpha}(.,t_0)\ dx \\ \geq& c(d,n,\alpha)||u_0||_{L^1(\mathbb{R}^d)}^{1+\alpha} r^{-\alpha\cdot d} r^\gamma\ .\end{aligned}$$

If we can find $\alpha \in (-1,0]$ and $\gamma < 0$ such that Lemma 25 is applicable (with $\epsilon = 1$), we get the estimate

$$T^* \leq \max\left(2t_0, C(d,n,\alpha,\gamma)||u_0||_{L^1(\mathbb{R}^d)}^{\frac{4+4\alpha+n(d+\gamma)}{-\gamma+\alpha\cdot d}} \left[||u_0||_{L^1(\mathbb{R}^d)}^{1+\alpha} r^{-\alpha\cdot d} r^\gamma\right]^{-\frac{4+n\cdot d}{-\gamma+\alpha\cdot d}}\right)$$

which gives

$$T^* \leq C(d,n,\alpha,\gamma)||u_0||_{L^1(\mathbb{R}^d)}^{-n} r^{4+d\cdot n}\ .$$

Since α and γ only depend on n and d, the result is then established.

Thus it remains to find admissible values for α and γ. We first treat the case $n \in (1.5, 2)$. Set $b := \frac{49}{40} + \frac{11}{20}(n - \frac{3}{2}) = \frac{11}{20}n + \frac{16}{40}$ and $\gamma := -d$. This implies $\alpha = \frac{16}{40} - \frac{9}{20}n$. Conditions (H1), (H2), (H3), (H6), (H7) are immediate. Condition (H4) is equivalent to

$$\frac{18n-16}{40} \cdot \frac{2n+16}{40} \cdot \frac{22n-24}{40} \cdot \frac{64-22n}{40} \geq \frac{1}{4}\left(\frac{12n-64}{40}\right)^2 \left(\frac{22n-24}{40}\right)^2$$

which is equivalent to (since $n \geq 1.5$)

$$4(18n-16)(2n+16)(64-22n) \geq (12n-64)^2(22n-24)\ .$$

Factorization (e.g. using a computer algebra system) leads to

$$64(2-n)(99n^2 - 176n + 256) \geq 0\ .$$

As the second polynomial factor is immediately seen to be strictly positive, condition (H4) is satisfied since $n \leq 2$. Finally, condition (H5a) is satisfied for some $\tau > 0$ if the inequality

$$\left(2b - \frac{n}{2}\right) \cdot \frac{(-d-4+d)(-d+\frac{d-4}{3})}{(-d-2)(-d-2+d)} > b+1$$

holds. Simplifying, this inequality becomes

$$\frac{24n+32}{40} \cdot \frac{4}{3} > \frac{22n+56}{40} ,$$

which is satisfied for $n = 2$ and for $n = 1.5$. Thus the inequality is satisfied in the whole interval $(1.5, 2]$ (as the difference of both sides of the inequality is an affine function).

We now deal with the case $n \in (1, 1.5]$. We set $b := \frac{49}{40} + \frac{9}{20}(n - \frac{3}{2}) = \frac{9}{20}n + \frac{22}{40}$ and $\gamma := -d$. This implies $\alpha = \frac{22}{40} - \frac{11}{20}n$. Conditions (H1), (H2), (H3), (H6) and (H7) are verified immediately. Condition (H4) is now equivalent to

$$\frac{22n-22}{40} \cdot \frac{22-2n}{40} \cdot \frac{18n-18}{40} \cdot \frac{58-18n}{40} \geq \frac{1}{4}\left(\frac{28n-88}{40}\right)^2 \left(\frac{18n-18}{40}\right)^2 ,$$

which is equivalent to

$$4 \cdot 22 \cdot (22-2n) \cdot (58-18n) \geq (28n-88)^2 \cdot 18 .$$

Rearranging the latter inequality, we obtain

$$11 \cdot (11-n) \cdot (29-9n) \geq 9 \cdot (7n-22)^2 .$$

The last condition is seen to be equivalent to

$$-342n^2 + 1364n - 847 \geq 0 .$$

This condition is true for all $n \in [1, 1.5]$ since for all such n we have $-342n^2 + 1364n - 847 \geq (1364 - 1.5 \cdot 342)n - 847 \geq 1364 - 1.5 \cdot 342 - 847 = 4$. It remains to verify (H5a). Condition (H5a) is seen to be satisfied for some $\tau > 0$ if

$$\frac{16n+44}{40} \cdot \frac{4}{3} > \frac{18n+62}{40} .$$

This inequality holds for $n = 1.5$; for $n = 1$, equality holds. Thus, since the difference of the functions on both sides of the inequality is an affine function, the inequality holds for every $n \in (1, 1.5]$.

Finally, we treat the case $n \in [2, \frac{32}{11})$. In this parameter range, we are fine with the choice $b := \frac{9}{20}n + \frac{12}{20}$, i.e. $\alpha = -\frac{11}{20}n + \frac{12}{20}$, and $\gamma := -d$. For these choices, conditions (H1) to (H4) have been verified in the proof of Theorem

3. Conditions (H6) and (H7) are immediate. It remains to check (H5a). We see that (H5a) is satisfied for some $\tau > 0$ if the inequality

$$\left(2b - \frac{n}{2}\right) \cdot \frac{(-d-4+d)(-d+\frac{d-4}{3})}{(-d-2)(-d-2+d)} > b+1$$

holds. Using our choice of b, this condition becomes

$$\frac{8n+24}{20} \cdot \frac{4}{3} > \frac{9n+32}{20} \ .$$

For $n = 3$, this condition is satisfied; for $n = 2$, it is also satisfied. Both sides of the inequality being affine functions, the inequality holds for all $n \in [2, 3]$. This finishes our proof. $\square$

Proof of Corollary 12. The assertion follows directly from Theorem 11: Taking into account that $\operatorname{supp} u(., t_1) \subset \operatorname{supp} u(., t_2)$ for all $0 \le t_1 \le t_2$, fixing $x_0 \in \mathbb{R}^d$ we see that $\operatorname{dist}(x_0, \operatorname{supp} u(., t))$ is a nonincreasing function of t. This yields $\operatorname{dist}(x_0, \operatorname{supp} u(., t)) = 0$ for $t > T^*$ (with T^* as defined in Theorem 11), which in turn implies $x_0 \in \operatorname{supp} u(., t)$ for $t > T^*$.

Rearranging the estimate on T^*, we see that it is equivalent to the inequality $\operatorname{dist}(x_0, \operatorname{supp} u_0) \ge c(d, n) ||u_0||_{L^1(\mathbb{R}^d)}^{\frac{n}{4+d\cdot n}} (T^*)^{\frac{1}{4+d\cdot n}} - \operatorname{diam}(\operatorname{supp} u_0)$. Given $T \ge 0$, we thus have $T^* < T$ for all points x_0 satisfying $\operatorname{dist}(x_0, \operatorname{supp} u_0) < R(T)$. Using $x_s \in \operatorname{supp} u_0$ which implies $\operatorname{dist}(x_0, x_s) \ge \operatorname{dist}(x_0, \operatorname{supp} u_0)$, the proof is finished. $\square$

5 Proof of the infinite speed of propagation of solutions to the Derrida-Lebowitz-Speer-Spohn equation

We now proceed to the proof of infinite speed of support propagation for solutions of the DLSS equation. We first prove a Hardy-like inequality which will be required in the multidimensional case.

Lemma 27 (Hardy-like inequality)**.** *Let* $\psi \in C^\infty\left(\mathbb{R}^d \setminus \{0\}\right)$ *be given with* $\Delta\psi > 0$ *on* $\mathbb{R}^d \setminus \{0\}$*. For any nonnegative* $v \in H^2(\mathbb{R}^d)$ *with* $\operatorname{supp} v \subset\subset \mathbb{R}^d \setminus \{0\}$*, the inequality*

$$\int_{\mathbb{R}^d} v\Delta\psi \; dx \leq \int_{\mathbb{R}^d} \frac{|\Delta v|^2}{v} \cdot \frac{|\psi|^2}{\Delta\psi} \; dx$$

holds.

Proof. For smooth compactly supported nonnegative v we calculate

$$\int_{\mathbb{R}^d} v\Delta\psi \; dx = \int_{\mathbb{R}^d} \Delta v \quad \psi \; dx$$
$$\leq \left(\int_{\operatorname{supp} v} (v+\epsilon)\Delta\psi \; dx\right)^{\frac{1}{2}} \left(\int_{\operatorname{supp} v} \frac{|\Delta v|^2}{v+\epsilon} \cdot \frac{|\psi|^2}{\Delta\psi} \; dx\right)^{\frac{1}{2}} .$$

By approximation (convolution of v with a smoothing kernel and passing to the limit) this formula remains valid for any v with the properties stated in the lemma.

Passing to the limit $\epsilon \to 0$ using the monotone convergence theorem, we finish the proof. □

We now prove our main result in the one-dimensional case with periodic boundary conditions.

Proof of Theorem 14. Let ψ be a smooth 1-periodic function on $\mathbb{R}$. We rearrange the weak formulation of the DLSS equation (6) with test function ψ using integration by parts; this yields

$$\int_0^T \langle \partial_t u, \psi \rangle \; dt = 2\int_0^T \int_{(0,1)} |\sqrt{u}_x|^2 \psi_{xx} \; dx \; dt - \frac{1}{2}\int_0^T \int_{(0,1)} u \; \psi_{xxxx} \; dx \; dt \; .$$

We now argue by contradiction. Extend u from $(0,1) \times I$ to $\mathbb{R} \times I$ by periodicity. Suppose that there exist $x_0 \in [0,1]$, $\delta > 0$, $0 < t_S < t_E$ such that $\operatorname{supp} u \cap (B_\delta(x_0) \times [t_S, t_E]) = \emptyset$. Without loss of generality, we may assume $x_0 = 0$. We then would like to choose $\psi(x) := x^\gamma$ as a test function in our previous formula, where $\gamma < -4$. Note that ψ is not admissible (since we would need it to be smooth and 1-periodic); however, by our assumption $u(.,t)$ vanishes in $(0,\delta)\cup(1-\delta,1)$ for a.e. $t\in[t_S,t_E]$, so instead of using ψ as a test function we may use a smooth function $\tilde\psi$ with $\operatorname{supp}\tilde\psi\subset(\frac{\delta}{3},1-\frac{\delta}{3})$ and with $\psi\equiv\tilde\psi$ on $(\frac{\delta}{2},1-\frac{\delta}{2})$; this function $\tilde\psi$ may be extended to a smooth 1-periodic function and inserting $\tilde\psi$ we get the same formula as we would if we would use ψ as a test function. Thus we obtain for a.e. $t_0,T\in[t_S,t_E]$ with $t_0<T$ and for all $\gamma<-4$

$$\begin{aligned}&\int_{(0,1)} u(.,t)x^\gamma\,dx\Big|_{t_0}^{T}\\ =&2\gamma(\gamma-1)\int_{t_0}^T\int_{(0,1)}|\sqrt{u}_x|^2x^{\gamma-2}\,dx\,dt\\ &-\frac{1}{2}\gamma(\gamma-1)(\gamma-2)(\gamma-3)\int_{t_0}^T\int_{(0,1)}u(.,t)x^{\gamma-4}\,dx\,dt\ .\end{aligned}\tag{45}$$

For a.e. $t\in[t_S,t_E]$, by assumption $u(.,t)$ vanishes in some neighbourhood of 0 and 1. We set $v(x):=\sqrt{u}(x,t)$ for $x\in(0,1)$ and $v(x):=0$ elsewhere; we then have $v\in H^1(\mathbb{R})$ for a.e. $t\in(t_S,t_E)$. Applying Lemma 19 to v with weight $|x|^{\gamma-2}$, we obtain

$$\int_{(0,1)}\sqrt{u}^2x^{\gamma-4}\,dx\le\frac{4}{(\gamma-3)^2}\int_{(0,1)}|\sqrt{u}_x|^2x^{\gamma-2}\,dx$$

for a.e. $t\in[t_S,t_E]$ and all $\gamma<-4$. With (45) it follows that

$$\begin{aligned}&\int_{(0,1)} u(.,t)x^\gamma\,dx\Big|_{t_0}^{T}\\ \ge&\frac{1}{2}\gamma(\gamma-1)\left((\gamma-3)^2-(\gamma-2)(\gamma-3)\right)\int_{t_0}^T\int_{(0,1)}u(.,t)x^{\gamma-4}\,dx\,dt\\ =&-\frac{1}{2}\gamma(\gamma-1)(\gamma-3)\int_{t_0}^T\int_{(0,1)}u(.,t)x^{\gamma-4}\,dx\,dt\\ \ge&-\frac{1}{2}\gamma(\gamma-1)(\gamma-3)\int_{t_0}^T\int_{(0,1)}u(.,t)x^{\gamma}\,dx\,dt\end{aligned}$$

where in the last step we have utilized $x^{-4}\ge1$ on $(0,1)$. Gronwall's Lemma now implies that for a.e. $t_0,T\in[t_S,t_E]$ with $t_0<T$ and for all $\gamma<-4$

$$\begin{aligned}\int_{(0,1)}u(.,T)x^\gamma\,dx\ge&e^{-\frac{1}{2}\gamma(\gamma-1)(\gamma-3)(T-t_0)}\int_{(0,1)}u(.,t_0)x^\gamma\,dx\\ \ge&e^{-\frac{1}{2}\gamma(\gamma-1)(\gamma-3)(T-t_0)}||u_0||_{L^1(\Omega)}\end{aligned}\tag{46}$$

where we have used $x^\gamma\ge1$ on $(0,1)$ (recall that $\gamma<0$) and conservation of mass. We now note that

$$\int_{(0,1)}u(.,T)x^\gamma\,dx\le\delta^\gamma||u(.,T)||_{L^1(\Omega)}=\delta^\gamma||u_0||_{L^1(\Omega)}$$

for a.e. $T \in [t_S, t_E]$ and all $\gamma < -4$, where we have used conservation of mass and the fact that $\operatorname{supp} u(.,T) \cap B_\delta(0) = \emptyset$ for a.e. $T \in [t_S, t_E]$. Putting this estimate and the inequality (46) together, we obtain for a.e. $t_0, T \in [t_S, t_E]$ with $t_0 < T$ and for all $\gamma < -4$

$$\delta^\gamma ||u_0||_{L^1(\Omega)} \geq e^{-\frac{1}{2}\gamma(\gamma-1)(\gamma-3)(T-t_0)} ||u_0||_{L^1(\Omega)}$$

or equivalently since $||u_0||_{L^1(\Omega)} > 0$

$$1 \geq e^{-\gamma[\log\delta + \frac{1}{2}(\gamma-1)(\gamma-3)(T-t_0)]}$$

which yields a contradiction if we let $\gamma \to -\infty$ for $T > t_0$ fixed (with $t_S < t_0, T < t_E$). This is possible as $\gamma < -4$ is arbitrary; recall also that $t_E > t_S$. Thus our assertion is proved. □

The following result has been established in [25].

Theorem 28. *Let $d \leq 3$ and let u be a weak solution to the DLSS equation on $\Omega = (0,1)^d$ with periodic boundary conditions. Provided that u has the additional regularity $u^{\frac{1}{4}} \in L^2_{loc}(I; H^2_{per}(\Omega))$, we have for any $T > 0$ and any $\psi \in L^\infty(I; W^{2,\infty}_{per}(\Omega)) \cap W^{1,1}(I; L^\infty(\Omega))$ with $\psi(.,T) \equiv 0$*

$$\begin{aligned} &-2\int_0^T \int_\Omega \sqrt{u}\psi_t \, dx \, dt - 2\int_\Omega \sqrt{u_0}\psi(.,0) \, dx \qquad (47) \\ &= \int_0^T \int_\Omega \frac{\psi}{\sqrt{u}} |\Delta\sqrt{u}|^2 \, dx \, dt - \int_0^T \int_\Omega \Delta\sqrt{u}\Delta\psi \, dx \, dt \, . \end{aligned}$$

Note that by the following lemma, the term $\frac{|\Delta\sqrt{u}|^2}{\sqrt{u}}$ is well-defined and belongs to $L^1_{loc}(I; L^1(\Omega))$. Therefore our formula (47) implies that we have $\sqrt{u} \in W^{1,1}_{loc}(I; [H^2_{per}(\Omega)]')$ which yields $\sqrt{u} \in C^0_{loc}(I; [H^2_{per}(\Omega)]')$. In connection with conservation of mass (which implies $\sqrt{u} \in L^\infty(I; L^2(\Omega))$) we deduce that $\sqrt{u} \in C^0_{loc}(I; L^2_w(\Omega))$, where L^2_w denotes the space L^2 equipped with its weak topology.

Lemma 29. *Let $d \leq 3$. Given $v \in H^2_{per}((0,1)^d)$ with $v \geq 0$ and some 1-periodic $\phi \in C^\infty(\mathbb{R}^d)$, we have $v^{\frac{1}{2}} \in W^{1,4}_{per}(\Omega)$ with the estimate*

$$\int_{(0,1)^d} |\nabla v^{\frac{1}{2}}|^4 \, dx \leq C(d) \int_{(0,1)^d} |D^2 v|^2 \, dx \, .$$

Moreover, for $v \in H^2_{per}((0,1)^d)$ with $v \geq 0$ and $\sqrt{v} \in H^2_{per}((0,1)^d)$ we have $\Delta v = 0$ a.e. on $\{v = 0\}$ as well as

$$\frac{|\Delta v|^2}{v} = 4\chi_{\{v \neq 0\}} \cdot \left(\Delta\sqrt{v} + 4|\nabla v^{\frac{1}{4}}|^2\right)^2 \, .$$

In particular we have the estimate

$$\left|\left| \frac{|\Delta v|^2}{v} \right|\right|_{L^1((0,1)^d)} \leq C(d) ||\sqrt{v}||^2_{H^2((0,1)^d)} \, .$$

Proof. We calculate for smooth strictly positive 1-periodic v

$$\int_{(0,1)^d} |\partial_i v^{\frac{1}{2}}|^4 \, dx = \frac{1}{16}\int_{(0,1)^d} v^{-2}|\partial_i v|^4 \, dx = \frac{3}{16}\int_{(0,1)^d} v^{-1}|\partial_i v|^2 \partial_{ii}^2 v \, dx \ .$$

By Young's inequality we obtain

$$\int_{(0,1)^d} |\partial_i v^{\frac{1}{2}}|^4 \, dx \leq C \int_{(0,1)^d} |\partial_{ii}^2 v|^2 \, dx \ .$$

Taking the sum with respect to i, we obtain the first assertion of the lemma for smooth strictly positive v. By approximation the inequality carries over to the case of stricly positive $v \in H^2_{per}((0,1)^d)$. Considering $v+\epsilon$ and passing to the limit $\epsilon \to 0$, by lower semicontinuity of the $W^{1,4}$ norm (with respect to convergence in L^4) we finally see that the first assertion of the lemma holds for any nonnegative $v \in H^2_{per}((0,1)^d)$.

Regarding the second assertion of the lemma, we first note that the first assertion of the lemma applied to $\sqrt{v}$ yields $v^{\frac{1}{4}} \in W^{1,4}_{per}((0,1)^d)$. For smooth strictly positive v we calculate

$$\frac{|\Delta v|^2}{v} = 4\frac{|\nabla \cdot (\sqrt{v}\nabla\sqrt{v})|^2}{v} = 4\frac{(\sqrt{v}\Delta\sqrt{v} + |\nabla\sqrt{v}|^2)^2}{v}$$

which implies

$$\frac{|\Delta v|^2}{v} = 4\left(\Delta\sqrt{v} + 4|\nabla v^{\frac{1}{4}}|^2\right)^2 \ .$$

By approximation, the latter identity remains valid a.e. for strictly positive $v \in H^2_{per}((0,1)^d)$ (note that by $d \leq 3$ we have $v \in C^0((0,1)^d)$). To prove the identity for all nonnegative $v \in H^2_{per}((0,1)^d)$ with $\sqrt{v} \in H^2_{per}((0,1)^d)$, we consider $v+\epsilon$ in place of v and deduce by the chain rule for Sobolev functions (note that $\Delta(v+\epsilon) = \Delta v$)

$$\frac{|\Delta v|^2}{v+\epsilon} = 4\left(\frac{\sqrt{v}}{\sqrt{v+\epsilon}}\Delta\sqrt{v} + \frac{\epsilon}{(v+\epsilon)^{\frac{3}{2}}}|\nabla\sqrt{v}|^2 + 4\frac{v^{\frac{3}{2}}}{(v+\epsilon)^{\frac{3}{2}}}|\nabla v^{\frac{1}{4}}|^2\right)^2 \ .$$

Passing to the limit $\epsilon \to 0$, the desired assertions are obtained as by dominated convergence the right-hand side converges strongly in $L^1((0,1)^d)$ to the desired limit (note that $\nabla\sqrt{v} = 2v^{\frac{1}{4}}\nabla v^{\frac{1}{4}}$).

The last assertion of the lemma follows by applying the first formula of the lemma to $\sqrt{v}$ and using the second formula. □

We are now in position to prove the infinite speed of propagation result in up to three spatial dimensions. We first show that the support of the solution immediately reaches any point $x_0 \in (0,1)^d$ with $\operatorname{dist}(x_0, \operatorname{supp} u_0) < \operatorname{dist}(x_0, \partial[0,1]^d)$; the general case will be seen to follow below.

Lemma 30. *Let $d \leq 3$ and $\Omega = (0,1)^d$. Suppose that u is a weak solution of the DLSS equation with periodic boundary conditions. Assume that u has the additional regularity $u^{\frac{1}{4}} \in L^2_{loc}(I; H^2_{per}(\Omega))$. Suppose that* $\operatorname{supp}\sqrt{u_0} \neq \emptyset$*.*

Given $x_0 \in \Omega$ with $\operatorname{dist}(x_0, \operatorname{supp}\sqrt{u_0}) < \operatorname{dist}(x_0, \partial\Omega)$*, for any $\delta > 0$ we have* $\operatorname{supp}\sqrt{u} \cap (B_\delta(x_0) \times [0,\delta)) \neq \emptyset$*.*

Proof. We argue by contradiction. Assume that there exists $\delta > 0$ such that $(B_\delta(x_0) \times [0,\delta)) \cap \operatorname{supp}\sqrt{u} = \emptyset$. Set $r := \operatorname{dist}(x_0, \operatorname{supp}\sqrt{u_0})$ and $R := \operatorname{dist}(x_0, \partial\Omega)$; by assumption we have $r < R$. Take a smooth nonnegative cutoff ϕ with $\phi \equiv 1$ on $B_{\frac{2r+R}{3}}(x_0)$ and $\phi \equiv 0$ on $\mathbb{R}^d \backslash B_{\frac{r+2R}{3}}(x_0)$. Let $\xi \in C^\infty(\mathbb{R})$ be a smooth monotonous function with $\xi(s) = 1$ for $s < 0$ and $\xi(s) = 0$ for $s > 1$. Let $t_0 \in (0,\delta)$ and $\gamma < -d-4$. We use $\psi(x,t) := \phi^4(x)|x-x_0|^\gamma \cdot \xi\left(\frac{t-t_0}{\epsilon}\right)$ as a test function in formula (47) (Theorem 28 is applicable by our assumptions). The function $\phi^4(x) \cdot |x-x_0|^\gamma$ is smooth on some neighbourhood of $\bigcup_{t\in[0,\delta)} \operatorname{supp}\sqrt{u}(.,t)$; moreover it vanishes on a neighbourhood of $\partial\Omega$, i.e. it belongs to $W^{2,\infty}_{per}((0,1)^d)$. Thus our test function is admissible.

Letting $\epsilon \to 0$ and recalling that $\sqrt{u} \in C^0_{loc}(I; L^2_w(\Omega))$ (here L^2_w denotes the L^2 space with its weak topology), we obtain for any $t_0 \in I$

$$
\begin{aligned}
&2\int_\Omega \sqrt{u(.,t_0)}\phi^4|x-x_0|^\gamma\,dx - 2\int_\Omega \sqrt{u_0}\phi^4|x-x_0|^\gamma\,dx \\
=&\int_0^{t_0}\int_\Omega \frac{\phi^4 \cdot |x-x_0|^\gamma}{\sqrt{u}}|\Delta\sqrt{u}|^2\,dx\,dt - \int_0^{t_0}\int_\Omega \Delta\sqrt{u}\Delta(\phi^4 \cdot |x-x_0|^\gamma)\,dx\,dt\;.
\end{aligned}
$$

Let $0 \leq t_1 < t_2 < \delta$. We obtain (since the previous formula holds for any $t_0 \in (0,\delta)$)

$$
\begin{aligned}
&2\int_\Omega \sqrt{u(.,t_2)}\phi^4|x-x_0|^\gamma\,dx \\
=&2\int_\Omega \sqrt{u(.,t_1)}\phi^4|x-x_0|^\gamma\,dx \\
&+\int_{t_1}^{t_2}\int_\Omega \frac{|x-x_0|^\gamma}{\phi^4\sqrt{u}}\left|\Delta[\phi^4\sqrt{u}]\right|^2\,dx\,dt - \int_{t_1}^{t_2}\int_\Omega \phi^4\sqrt{u}\Delta^2|x-x_0|^\gamma\,dx\,dt \\
&+\int_{t_1}^{t_2}\int_\Omega \frac{\phi^4|x-x_0|^\gamma}{\sqrt{u}}|\Delta\sqrt{u}|^2 - \frac{|x-x_0|^\gamma}{\phi^4\sqrt{u}}\left|\Delta[\phi^4\sqrt{u}]\right|^2\,dx\,dt \\
&-\int_{t_1}^{t_2}\int_\Omega \sqrt{u}\Delta^2[\phi^4|x-x_0|^\gamma] - \phi^4\sqrt{u}\;\Delta^2|x-x_0|^\gamma\,dx\,dt\;.
\end{aligned}
\tag{48}
$$

Note that $\phi^4\sqrt{u} \in L^2_{loc}(I; H^2_{per}(\Omega))$ and $\sqrt{\phi^4\sqrt{u}} = \phi^2 u^{\frac{1}{4}} \in L^2_{loc}(I; H^2_{per}(\Omega))$; this implies by Lemma 29 that the expression $\frac{1}{\sqrt{\phi^4\sqrt{u}}}\Delta[\phi^4\sqrt{u}]$ is well-defined and belongs to $L^2_{loc}(I; L^2(\Omega))$.

Lemma 27 applied to $v := \phi^4 \cdot \sqrt{u}$ (where v is extended by 0 to $\mathbb{R}^d$) and $\psi := \Delta|x-x_0|^\gamma$ for $\gamma < 1-d$ states that

$$\frac{(\gamma-2)(\gamma-4+d)}{\gamma(\gamma-2+d)} \int_\Omega \phi^4\sqrt{u}\Delta^2|x-x_0|^\gamma \, dx \leq \int_\Omega \frac{\left|\Delta[\phi^4\sqrt{u}]\right|^2}{\phi^4\sqrt{u}}|x-x_0|^\gamma \, dx \; .$$

Note that for $t \in [0,\delta)$, by definition of δ the function $\phi^4\sqrt{u}$ vanishes on some neighbourhood of x_0; moreover, we have $\operatorname{supp}\phi^4\sqrt{u} \subset\subset (0,1)^d$ and therefore (for a.e. $t \in [0,\delta)$) $v \in H^2(\mathbb{R}^d)$. Thus Lemma 27 is indeed applicable for a.e. $t \in [0,\delta)$.

Using this inequality in (48), we get

$$\begin{aligned}
&2\int_\Omega \sqrt{u(.,t_2)}\phi^4|x-x_0|^\gamma \, dx \\
\geq &2\int_\Omega \sqrt{u(.,t_1)}\phi^4|x-x_0|^\gamma \, dx \\
&+(\gamma-2)(\gamma-4+d)(-4\gamma-2d+8)\int_{t_1}^{t_2}\int_\Omega \phi^4\sqrt{u}\;|x-x_0|^{\gamma-4} \, dx \, dt \\
&+\int_{t_1}^{t_2}\int_\Omega \frac{\phi^4|x-x_0|^\gamma}{\sqrt{u}}|\Delta\sqrt{u}|^2 - \frac{|x-x_0|^\gamma}{\phi^4\sqrt{u}}\left|\Delta[\phi^4\sqrt{u}]\right|^2 \, dx \, dt \\
&-\int_{t_1}^{t_2}\int_\Omega \sqrt{u}\Delta^2[\phi^4|x-x_0|^\gamma] - \phi^4\sqrt{u}\Delta^2|x-x_0|^\gamma \, dx \, dt \; .
\end{aligned} \tag{49}$$

Note that the difference in the latter two integrals is nonzero only on $\operatorname{supp}\nabla\phi$, i.e. it is zero outside of $B_{\frac{r+2R}{3}}(x_0) \setminus B_{\frac{2r+R}{3}}(x_0)$. Moreover we have

$$\begin{aligned}
&\left|\left|\frac{1}{\phi^4\sqrt{u}}\left|\Delta[\phi^4\sqrt{u}]\right|^2\right|\right|_{L^1(\Omega)} + \left|\left|\frac{1}{\sqrt{u}}\left|\Delta\sqrt{u}\right|^2\right|\right|_{L^1(\Omega)} \\
\leq &C(d)||\phi^2 u^{\frac14}||^2_{H^2(\Omega)} + C(d)||u^{\frac14}||^2_{H^2(\Omega)} \\
\leq &C(d,\phi)||u^{\frac14}||^2_{H^2(\Omega)}
\end{aligned} \tag{50}$$

(the first inequality is a consequence of Lemma 29 applied to $v := \phi^4\sqrt{u}$ and $v := \sqrt{u}$; the second inequality follows by the product rule and smoothness of ϕ). Note also that

$$\left|\Delta(\phi^4\sqrt{u}) - \phi^4\Delta\sqrt{u}\right| \leq C\phi^3|\nabla\phi|u^{\frac14}|\nabla u^{\frac14}| + C\phi^2|\nabla\phi|^2\sqrt{u} + C\phi^3|\Delta\phi|\sqrt{u}$$

which implies

$$\left|\frac{1}{\phi^2 u^{\frac14}}\Delta(\phi^4\sqrt{u}) - \frac{\phi^2}{u^{\frac14}}\Delta\sqrt{u}\right| \leq C(r,R)\chi_{\{x:\frac{2r+R}{3}\leq|x-x_0|\leq\frac{r+2R}{3}\}} \cdot (u^{\frac14} + |\nabla u^{\frac14}|) \; . \tag{51}$$

Moreover, we have

$$
\begin{aligned}
&\left|\phi^4\Delta^2|x-x_0|^\gamma-\Delta^2(\phi^4|x-x_0|^\gamma)\right| \qquad (52)\\
\leq &C(r,R)\chi_{\{x:\frac{2r+R}{3}\leq|x-x_0|\leq\frac{r+2R}{3}\}}\cdot\Big(|x-x_0|^\gamma+|\gamma|\cdot|x-x_0|^{\gamma-1}\\
&\qquad+(|\gamma|^2+1)\cdot|x-x_0|^{\gamma-2}+(|\gamma|^3+1)\cdot|x-x_0|^{\gamma-3}\Big)\ .
\end{aligned}
$$

Putting (50), (51), and (52) together, we therefore obtain (note that the constant does not depend on γ)

$$
\begin{aligned}
&\left|\int_{t_1}^{t_2}\int_\Omega\frac{\phi^4|x-x_0|^\gamma}{\sqrt{u}}|\Delta\sqrt{u}|^2-\frac{|x-x_0|^\gamma}{\phi^4\sqrt{u}}\left|\Delta[\phi^4\sqrt{u}]\right|^2\ dx\ dt\right|\\
&+\left|\int_{t_1}^{t_2}\int_\Omega\sqrt{u}\Delta^2[\phi^4|x-x_0|^\gamma]-[\phi^4\sqrt{u}]\Delta^2|x-x_0|^\gamma\ dx\ dt\right|\\
\leq &C(d,r,R,\phi)(1+|\gamma|^3)\int_{t_1}^{t_2}\left[\frac{2r+R}{3}\right]^\gamma\left|\left|u^{\frac{1}{4}}\right|\right|_{H^2(\Omega)}^2\ dt\ .
\end{aligned}
$$

Using this inequality in (49), this proves that

$$
\begin{aligned}
&\int_\Omega\sqrt{u(.,t_2)}\phi^4|x-x_0|^\gamma\ dx\\
\geq&\int_\Omega\sqrt{u(.,t_1)}\phi^4|x-x_0|^\gamma\ dx\\
&+\frac{1}{2}(\gamma-2)(\gamma-4+d)(-4\gamma-2d+8)\int_{t_1}^{t_2}\int_\Omega\phi^4\sqrt{u}\ |x-x_0|^{\gamma-4}\ dx\ dt\\
&-C(d,r,R,\phi)(1+|\gamma|^3)\left[\frac{2r+R}{3}\right]^\gamma\int_{t_1}^{t_2}\left|\left|u^{\frac{1}{4}}\right|\right|_{H^2(\Omega)}^2\ dt\\
\geq&\int_\Omega\sqrt{u(.,t_1)}\phi^4|x-x_0|^\gamma\ dx \qquad (53)\\
&+\frac{1}{2}(\gamma-2)(\gamma-4+d)(-4\gamma-2d+8)R^{-4}\int_{t_1}^{t_2}\int_\Omega\phi^4\sqrt{u}\ |x-x_0|^{\gamma}\ dx\ dt\\
&-C(d,r,R,\phi)(1+|\gamma|^3)\left[\frac{2r+R}{3}\right]^\gamma\int_{t_1}^{t_2}\left|\left|u^{\frac{1}{4}}\right|\right|_{H^2(\Omega)}^2\ dt\ ,
\end{aligned}
$$

where in the last step we have used $\operatorname{supp}\phi\subset B_R(x_0)$. Recalling that $r=\operatorname{dist}(x_0,\operatorname{supp}\sqrt{u_0})$ we see that $M_0:=\int_{B_{\frac{3r+R}{4}}(x_0)}\sqrt{u_0}\ dx>0$ and therefore

$$\int_\Omega\sqrt{u(.,0)}\phi^4|x-x_0|^\gamma\ dx\geq\left[\frac{3r+R}{4}\right]^\gamma M_0>0\ .$$

Here we have used that $\phi\equiv 1$ on $B_{\frac{2r+R}{3}}(x_0)$. We now apply Gronwall's inequality to (53); using our assumption $(B_\delta(x_0)\times[0,\delta))\cap\operatorname{supp}\sqrt{u}=\emptyset$ we get for $t \in (0, \delta)$

$$\begin{aligned}
&\delta^\gamma \int_\Omega \sqrt{u(.,t)}\phi^4 \, dx \\
\geq &\int_\Omega \sqrt{u(.,t)}\phi^4 |x-x_0|^\gamma \, dx \\
\geq &\Bigg(\int_\Omega \sqrt{u(.,0)}\phi^4 |x-x_0|^\gamma \, dx \\
&\qquad - C(d,r,R,\phi)(1+|\gamma|^3) \left[\frac{2r+R}{3}\right]^\gamma \int_0^t \left|\left|u^{\frac{1}{4}}\right|\right|_{H^2(\Omega)}^2 \, dt \Bigg) \\
&\quad \cdot \exp\left(\frac{1}{2}R^{-4}(\gamma-2)(\gamma-4+d)(-4\gamma-2d+8)t\right) \\
\geq &\Bigg(\left[\frac{3r+R}{4}\right]^\gamma M_0 \\
&\qquad - C(d,r,R,\phi)(1+|\gamma|^3) \left[\frac{2r+R}{3}\right]^\gamma \int_0^\delta \left|\left|u^{\frac{1}{4}}\right|\right|_{H^2(\Omega)}^2 \, dt \Bigg) \\
&\quad \cdot \exp\left(\frac{1}{2}R^{-4}(\gamma-2)(\gamma-4+d)(-4\gamma-2d+8)t\right) .
\end{aligned}$$

For $-\gamma$ large enough (depending on d, r, R, ϕ, M_0, and u), the first factor on the right-hand side can be estimated from below by $\frac{1}{2}\left[\frac{3r+R}{4}\right]^\gamma M_0$ since $\frac{3r+R}{4} < \frac{2r+R}{3}$. Dividing both sides of the inequality by δ^γ we obtain for $t \in (0, \delta)$

$$\begin{aligned}
&\int_\Omega \sqrt{u(.,t)}\phi^4 \, dx \\
\geq &\frac{1}{2} M_0 \exp\Bigg(\frac{1}{2}R^{-4}(\gamma-2)(\gamma-4+d)(-4\gamma-2d+8)t \\
&\qquad\qquad + \gamma \log \frac{3r+R}{4} - \gamma \log \delta \Bigg) .
\end{aligned}$$

Letting $\gamma \to -\infty$ we get $\int_\Omega \sqrt{u(.,t)}\phi^4 \, dx = \infty$ for any $t \in (0, \delta)$. With ϕ being smooth, this contradicts the fact that $u \in L^\infty(I; L^1(\Omega))$ and therefore finishes the proof. □

Proof of Theorem 15. It is sufficient to show that for any $\delta > 0$, any $x_0 \in (0,1)^d$ and any $t \geq 0$ we have $\operatorname{supp} \sqrt{u} \cap (B_\delta(x_0) \times [t, t+\delta)) \neq \emptyset$. W.l.o.g. we may assume $t = 0$.

Thus it suffices to prove that for any $\delta > 0$ and any $x_0 \in (0,1)^d$ we have $\operatorname{supp} \sqrt{u} \cap (B_\delta(x_0) \times [0, \delta)) \neq \emptyset$.

In Lemma 30 this assertion has been shown if the additional condition $\operatorname{dist}(x_0, \operatorname{supp} \sqrt{u_0}) < \operatorname{dist}(x_0, \partial\Omega)$ is satisfied. The general case follows easily by iterating the result of the lemma: Given an arbitrary point $x_0 \in \Omega$ and a point $y \in \Omega$ with $\mathrm{dist}(y, \operatorname{supp} \sqrt{u_0}) < \mathrm{dist}(y, \partial\Omega)$ (such a point exists as otherwise $\operatorname{supp} \sqrt{u_0} = \emptyset$), there exists a path $\alpha : [0,1] \to \Omega$ with $\alpha(0) = y$ and $\alpha(1) = x_0$. By compactness of the interval $[0,1]$ and continuity of α, for any n sufficiently large we have

$$\mathrm{dist}\left(\alpha\left(\frac{k}{n}\right), \alpha\left(\frac{k+1}{n}\right)\right) + \frac{1}{n} < \mathrm{dist}\left(\alpha\left([0,1]\right), \partial\Omega\right)$$

for all $k \in \{0, \ldots, n-1\}$. We additionally require $n \geq \frac{1}{\delta}$.

Since $\mathrm{dist}(y, \operatorname{supp} \sqrt{u_0}) < \mathrm{dist}(y, \partial\Omega)$, by the previous lemma there exists $t_0 \in \left[0, \frac{\delta}{n+1}\right)$ such that the inequality $\mathrm{dist}(\alpha(0), \operatorname{supp} \sqrt{u}(., t_0)) < \frac{1}{n}$ holds. This implies

$$\begin{aligned}
&\mathrm{dist}\left(\alpha\left(\frac{1}{n}\right), \operatorname{supp} \sqrt{u}(., t_0)\right) \\
\leq &\mathrm{dist}\left(\alpha\left(\frac{0}{n}\right), \alpha\left(\frac{1}{n}\right)\right) + \mathrm{dist}\left(\alpha\left(\frac{0}{n}\right), \operatorname{supp} \sqrt{u}(., t_0)\right) \\
\leq &\mathrm{dist}\left(\alpha\left(\frac{0}{n}\right), \alpha\left(\frac{1}{n}\right)\right) + \frac{1}{n} \\
< &\mathrm{dist}(\alpha([0,1]), \partial\Omega) \\
\leq &\mathrm{dist}\left(\alpha\left(\frac{1}{n}\right), \partial\Omega\right) \quad .
\end{aligned}$$

We can therefore apply the previous lemma again (starting at time t_0 instead of 0) to obtain $t_1 \in \left[t_0, \frac{2\delta}{n+1}\right)$ satisfying

$$\mathrm{dist}\left(\alpha\left(\frac{1}{n}\right), \operatorname{supp} \sqrt{u}(., t_1)\right) < \frac{1}{n} \ .$$

More generally, let $k \in \{1, \ldots, n-1\}$ and assume that there exists $t_k \in \left[0, \frac{\delta(k+1)}{n+1}\right)$ such that

$$\mathrm{dist}\left(\alpha\left(\frac{k}{n}\right), \operatorname{supp} \sqrt{u}(., t_k)\right) < \frac{1}{n} \ .$$

Then we deduce that

$$\begin{aligned}
&\mathrm{dist}\left(\alpha\left(\frac{k+1}{n}\right), \operatorname{supp} \sqrt{u}(., t_k)\right) \\
\leq &\mathrm{dist}\left(\alpha\left(\frac{k+1}{n}\right), \alpha\left(\frac{k}{n}\right)\right) + \mathrm{dist}\left(\alpha\left(\frac{k}{n}\right), \operatorname{supp} \sqrt{u}(., t_k)\right) \\
\leq &\mathrm{dist}\left(\alpha\left(\frac{k+1}{n}\right), \alpha\left(\frac{k}{n}\right)\right) + \frac{1}{n} \\
< &\mathrm{dist}(\alpha([0,1]), \partial\Omega) \\
\leq &\mathrm{dist}\left(\alpha\left(\frac{k+1}{n}\right), \partial\Omega\right) \quad .
\end{aligned}$$

Thus, applying the previous lemma again with initial time t_k instead of 0, we see that there exists $t_{k+1} \in \left[t_k, t_k + \frac{\delta}{n+1}\right)$ (in particular $t_{k+1} \in [0, \frac{\delta(k+2)}{n+1})$) with

$$\operatorname{dist}\left(\alpha\left(\frac{k+1}{n}\right), \operatorname{supp}\sqrt{u}(., t_{k+1})\right) < \frac{1}{n} .$$

Arguing by induction, we finally obtain $t_n \in [0, \delta)$ for which the inequality $\operatorname{dist}(\alpha(1), \operatorname{supp}\sqrt{u}(., t_n)) < \frac{1}{n} \leq \delta$ is satisfied (recall that $n \geq \frac{1}{\delta}$). This finishes the proof since $\alpha(1) = x_0$. $\square$

6 Discussion

In this thesis, we have obtained sharp bounds on waiting times for solutions of the thin-film equation for $n \in (2, \frac{32}{11})$; in the regime of weak slippage $n \in (2, \frac{32}{11})$ the thin-film equation is seen to induce support spreading of solutions exactly as predicted by the order of degeneracy of the operator. The critical exponent for the occurrence of a waiting time is $\frac{4}{n}$.

However, for $n \leq 2$ the situation changes drastically: for $n = 2$ we can only prove nonexistence of waiting times for initial data with growth steeper than $x_+^2 |\log x|^{\frac{5}{2}}$, whereas the existence of a waiting time has only be shown for growth like x_+^2 or slower. This gap becomes significantly larger when $n < 2$; both the minimal growth exponent $\frac{4}{n}$ known to be sufficient for the existence of a waiting time and the maximal exponent known to be sufficient for the nonexistence of a waiting time move away from 2 in opposite directions. Although the entropy estimates are a powerful tool providing compactness for the construction of solutions and being the base of studies of qualitative properties of solutions to the thin-film equation, it may be that they provide only partial information on qualitative properties of the thin-film equation in the regime $n \leq 2$.

As a second result, in this thesis we have shown that the upper bounds on asymptotic support propagation rates for the thin-film equation by Bernis [4], by Hulshof and Shishkov [39], by Bertsch, Dal Passo, Garcke and Grün [12] and by Grün [35] are optimal for any initial data: we have derived lower bounds on asymptotic support propagation rates which coincide with these upper bounds up to a constant factor for any solution of the thin-film equation. To the best of our knowledge, this is the first lower bound on large-time support propagation for solutions to the Cauchy problem for the thin-film equation for $n \neq 1$.

While we have shown that for large times solutions to the thin-film equation display support spreading at the rate suggested by the behaviour of the corresponding self-similar solution, one may hope to prove polynomial decay of any solution to the self-similar solution as done by Carrillo and Toscani [16] in case $n = 1$ and $d = 1$. However, proving the latter assertion for $n \neq 1$ currently seems out of reach: for $n \neq 1$, an entropy useful for proving decay to the self-similar solution must differ significantly from the entropy used in [16], since for $n \neq 1$ there is no explicit formula available for the self-similar solution.

Regarding the Derrida-Lebowitz-Speer-Spohn equation, we have shown that solutions to the DLSS equation displays infinite speed of support propagation; more precisely, viewing a solution u of the DLSS equation as a function of both space and time, the support of u has been seen to be either empty or equal to $\Omega \times [0, \infty)$. Unfortunately, our approach does not yield any result on the support of u at any fixed time. Numerical results seem to indicate that the support of any solution to the DLSS equation is nondecreasing with respect to time [42]. However, as comparison methods are unavailable for higher-order equations and our methods do not seem to yield any stronger assertion, we do not know how to prove estimates on $\operatorname{supp} u(.,t)$ at a fixed time $t > 0$. If $d = 1$ and if at a certain time t_0 the quantity $\log u$ becomes globally integrable, formal calculations suggest that it will stay integrable for all $t > t_0$; see e.g. [44]. However, a localization of this result is presently out of reach.

Future work based on the methods developed in this thesis may involve the extension of our upper bounds on waiting times to the full range $n \in (2, 3)$ (as opposed to $n \in (2, \frac{32}{11})$ in the present thesis; note that $\frac{32}{11} \approx 2.909$) or an improved analysis of the waiting time behaviour of the thin-film equation for $n \in (1, 2)$; this is currently work in progress.

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