Turnpike theory for PDE: Introduction

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Turnpike in a Nutshell, FAU, October 2019

[1]Funded by an AvH Professorship & ERC Advanced Grant DYCON

Often the mathematical treatment of real life and industrial problems not only involves

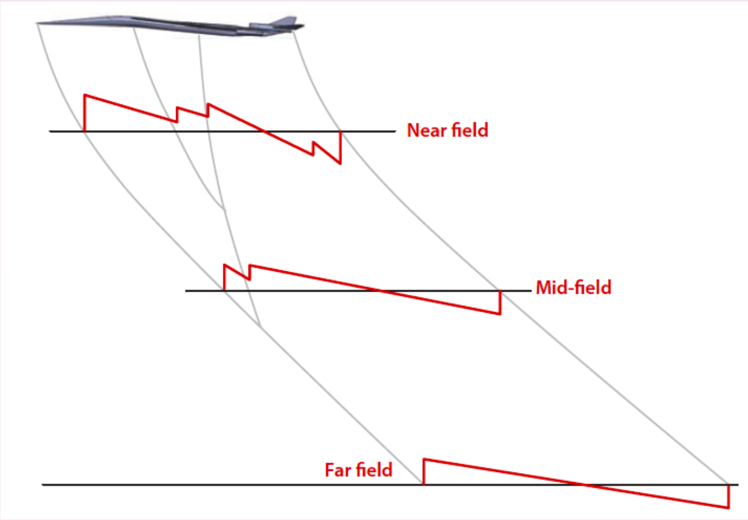
1. Modelling
2. Analysis
3. Simulation

but also

1. Design
2. Optimisation
3. Parameter identification
4. Uncertainty quantification
5. Control

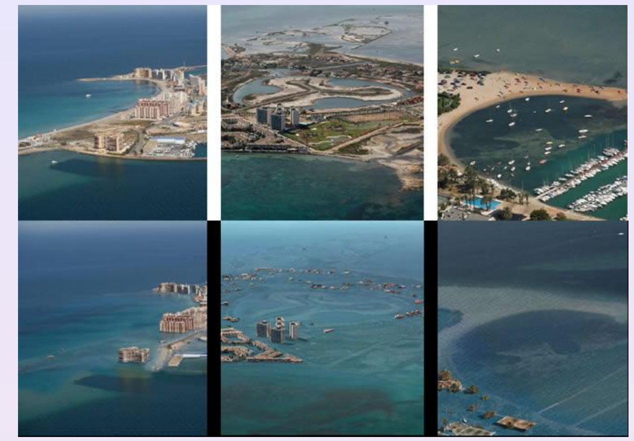
Motivation: Sonic boom

- Goal: the development of supersonic aircrafts, sufficiently quiet to be allowed to fly supersonically over land.
- The pressure signature created by the aircraft must be such that, when reaching ground, (a) it can barely be perceived by humans, and (b) it results in admissible disturbances to man-made structures.

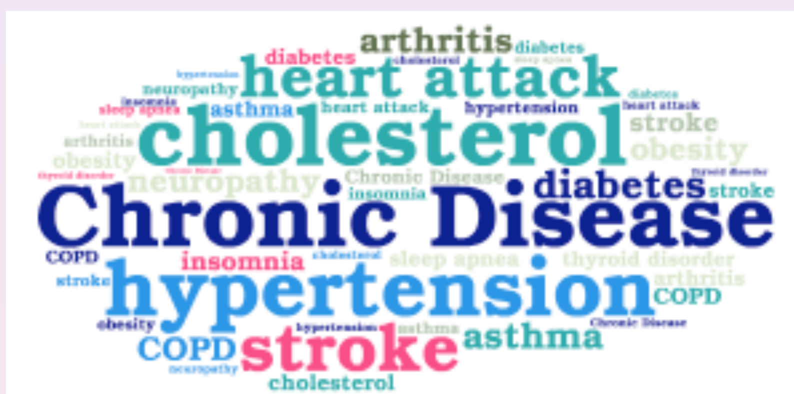


Juan J. Alonso and Michael R. Colonno, Multidisciplinary Optimization with Applications to Sonic-Boom Minimization, Annu. Rev. Fluid Mech. 2012, 44:505 – 526.

Many other examples in climate and earth sciences, biomedicine, social sciences, economics, lead to natural questions of control in long time.



Sustainable growth is a long-term challenge.

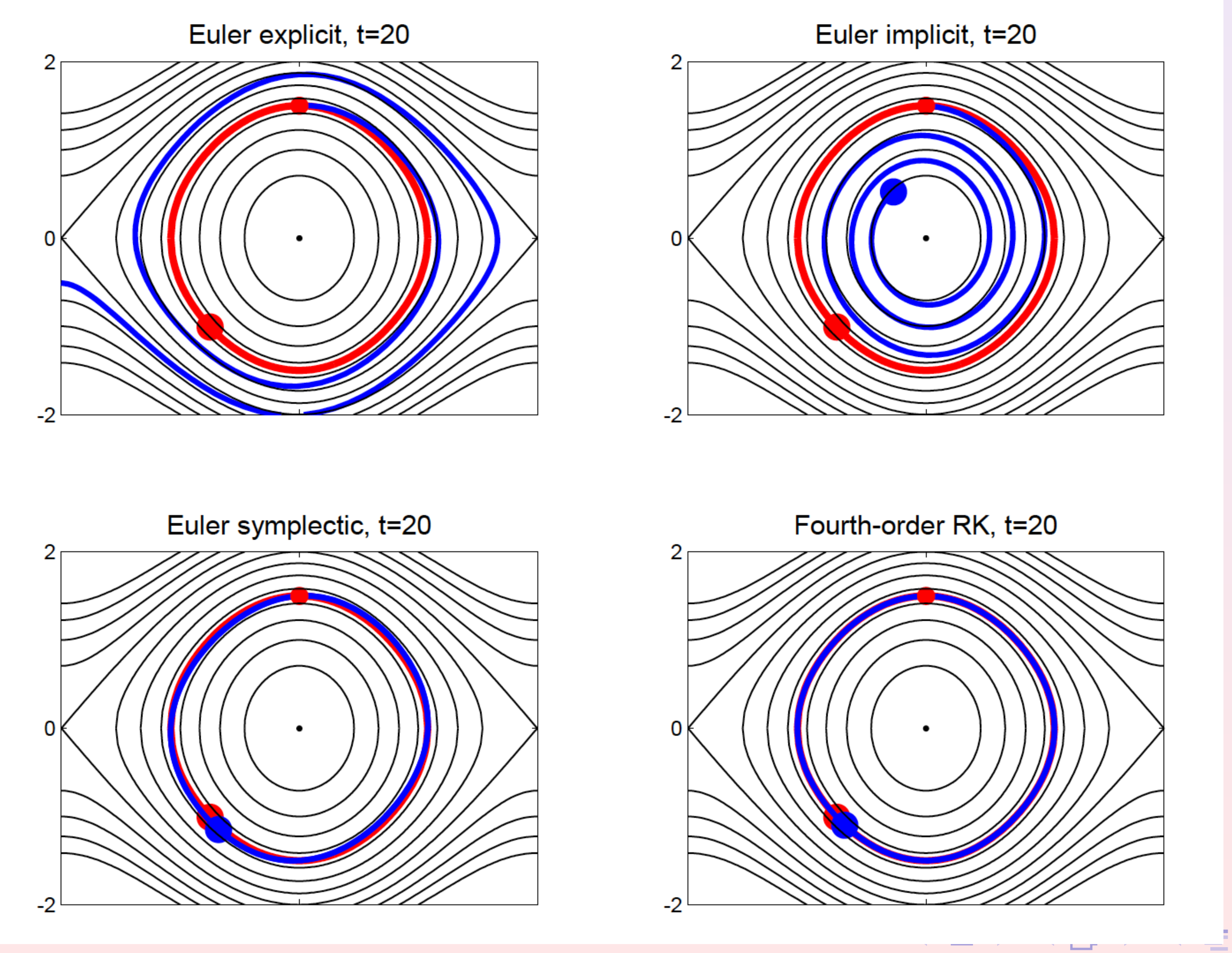


Two key issues :

- Specific control tools for long time control horizons.
- Numerical schemes capable of reproducing accurately the (control) dynamics in long time intervals.

Long time numerics: Geometric/Symplectic integration
Numerical integration of the pendulum (A. Marica)[a]

[a] HAIRER, E., LUBICH, Ch., WANNER, G.. Geometric Numerical Integration. Structure-Preserving Algorithms for Ordinary Differential Equations. 2nd ed. Berlin : Springer, 2006, 644 p.



Viscous/inviscid conservation laws

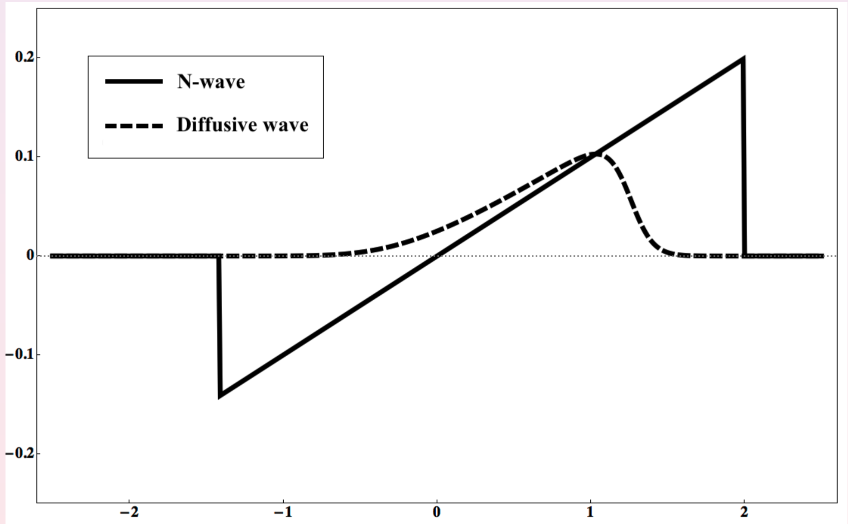
Consider the 1-D conservation law with or without viscosity:

$$u_t + \left[u^2\right]_x = \varepsilon u_{xx}, \quad x \in \mathbb{R}, t > 0.$$

Then [2] [3] :

- If $\varepsilon = 0$, $u(\cdot, t) \sim N(\cdot, t)$ as $t \to \infty$;
- If $\varepsilon > 0$, $u(\cdot, t) \sim u_M(\cdot, t)$ as $t \to \infty$,

u_M is the constant sign self-similar solution of the viscous Burgers equation (defined by the mass M of u_0), while N is the so-called hyperbolic N-wave.



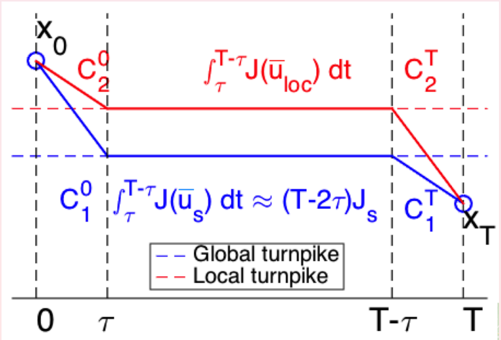
[2] L. Ignat, A. Pozo & E. Z, Math of Computation, 2014

[3] Y. J. Kim & A. E. Tzavaras, *Diffusive N-Waves and Metastability in the Burgers Equation*, SIAM J. Math. Anal. **33**(3) (2001), 607–633.

Turnpike: Origins

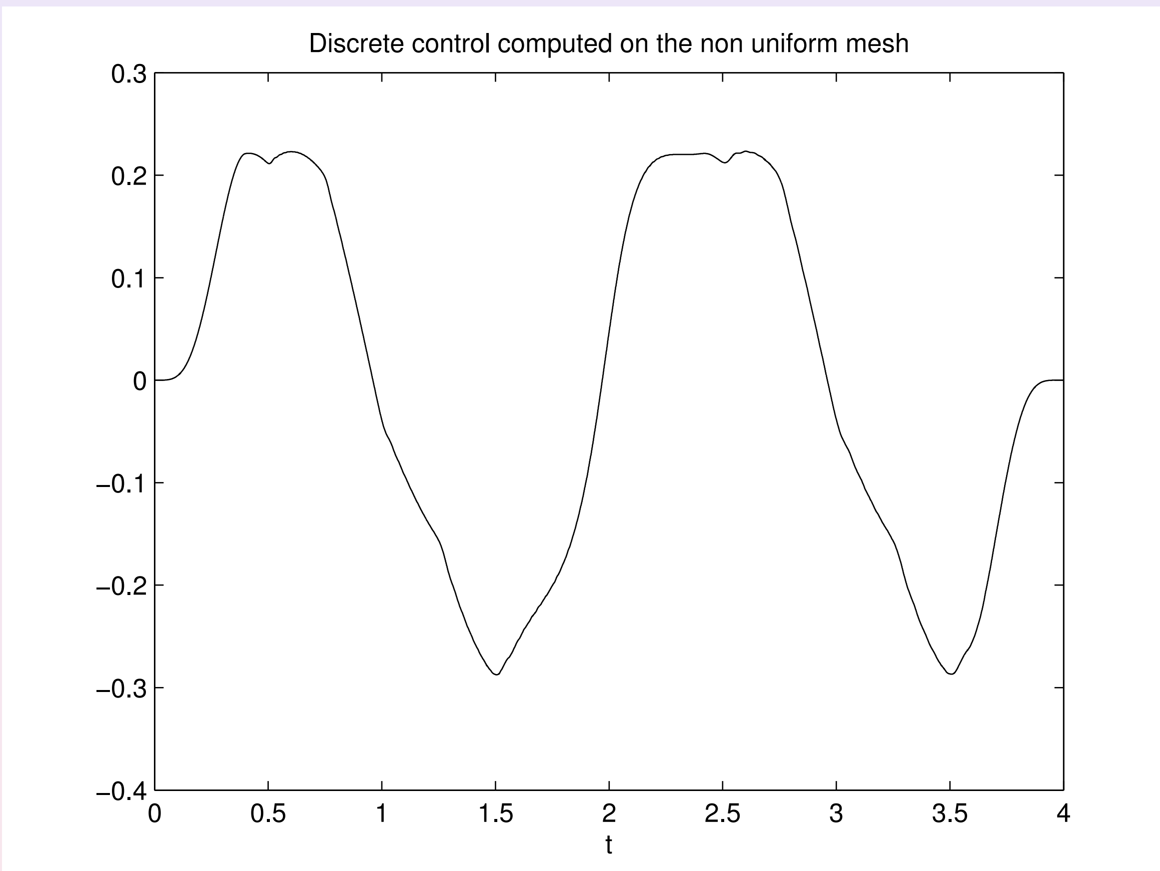
Although the idea goes back to John von Neumann in 1945, Lionel W. McKenzie traces the term to Robert Dorfman, Paul Samuelson, and Robert Solow's "Linear Programming and Economics Analysis" in 1958, referring to an American English word for a Highway:

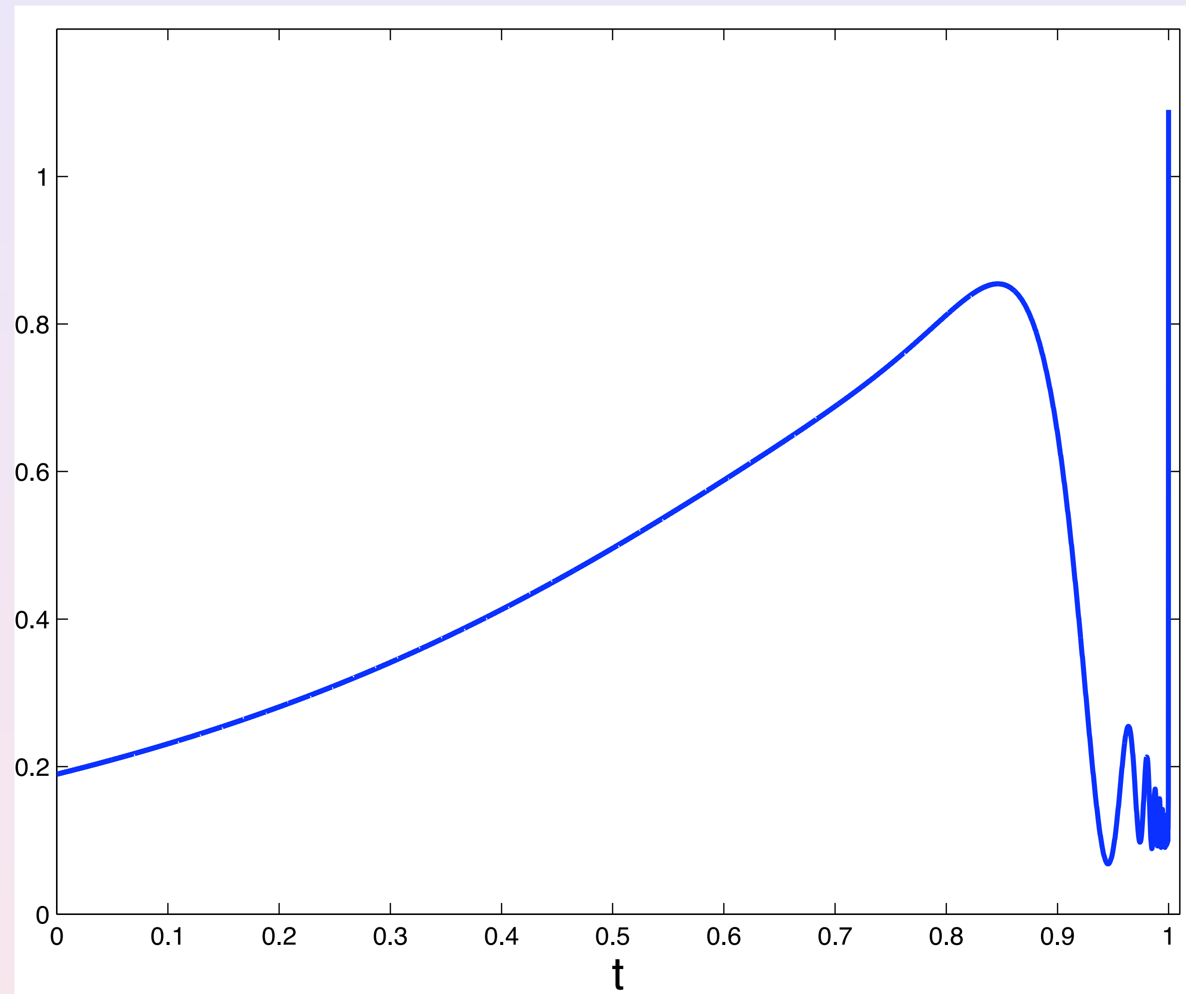
> *... There is a fastest route between any two points; and if the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But if the origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.*




Belgium
Frankfurt
Erl
Luxembourg
Mun
93 h
1.692 km
93 h
1.665 km
95 h
1.733 km
Milan
ay of Biscay
Monaco
Bilbao
Andorra

Examples where controls seem to fail the turnpike property





Typical dynamics of controls for **wave** and **heat** like equations, as solutions of the corresponding adjoint systems.

The control problem

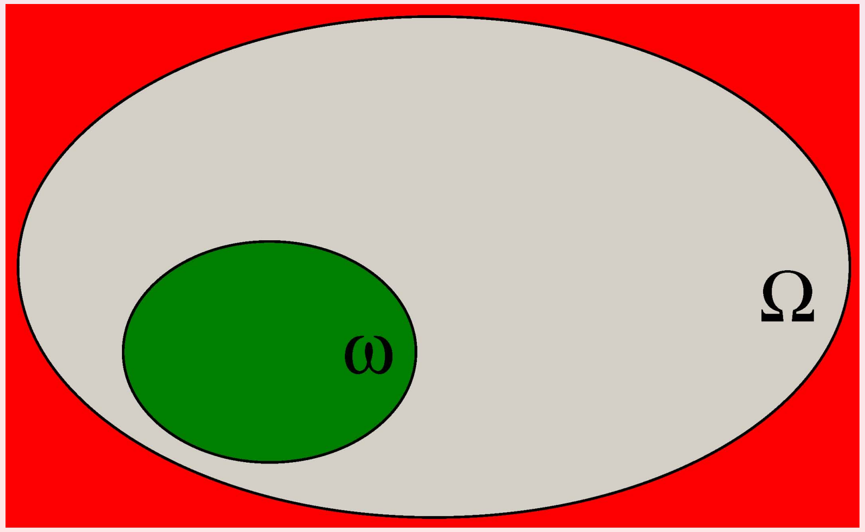
Let $n \geq 1$ and $T > 0$, Ω be a simply connected, bounded domain of $\mathbb{R}^n$ with smooth boundary Γ, $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Gamma$:

$$\begin{cases} y_t - \Delta y = f 1_\omega & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{1}$$

1_ω = the characteristic function of ω of Ω where the control is active.
We assume that $y^0 \in L^2(\Omega)$ and $f \in L^2(Q)$ so that (6) admits a unique solution

$$y \in C\left([0, T]\,; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right).$$

$$y = y(x,t) = solution = state, \; f = f(x,t) = control$$



We want to minimise the cost:

$$J(f) = \frac{1}{2}\int_0^T \int_\omega f^2 dxdt + \frac{1}{2}\int_\Omega |y(x,T) - y_d|^2 dx \qquad (2)$$

making a compromise between reaching the target u_d and energy consumption f.

The classical optimality system (Pontryaguin's principle) guarantees that the control is of the form

$$f = \varphi$$

where φ is the solution of the adjoint equation:

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in } Q \\ \varphi = 0 & \text{on } \Sigma \\ \varphi(T) = y(T) - y_d & \text{in } \Omega. \end{cases} \qquad (3)$$

Lack of turnpike!

Better balanced controls

Let us now consider the control f minimising a compromise between the norm of the state and the control among the class of admissible controls:

$$min\,\frac{1}{2}\Big[\int_0^T\int_\Omega |y|^2 dxdt + \int_0^T\int_\omega |f|^2 dxdt + \frac{1}{2}\int_\Omega |y(x,T)-y_d|^2 dx\Big].$$

Then the Optimality System reads

$$y_t - \Delta y = -\varphi 1_\omega \text{ in } Q$$

$$y = 0 \text{ on } \Sigma$$

$$y(x,0) = y^0(x) \text{ in } \Omega$$

$$-\varphi_t - \Delta\varphi = y \text{ in } Q$$

$$\varphi = 0 \text{ on } \Sigma.$$

$$\varphi(T) = y(T) - y_d \quad \text{ in } \Omega.$$

We now observe a coupling between φ and y on the adjoint state equation!

New Optimality System Dynamics

What is the dynamic behaviour of solutions of the new fully coupled OS?
For the sake of simplicity, assume $\omega = \Omega$.
The dynamical system now reads

$$y_t - \Delta y = -\varphi$$

$$\varphi_t + \Delta \varphi = -y$$

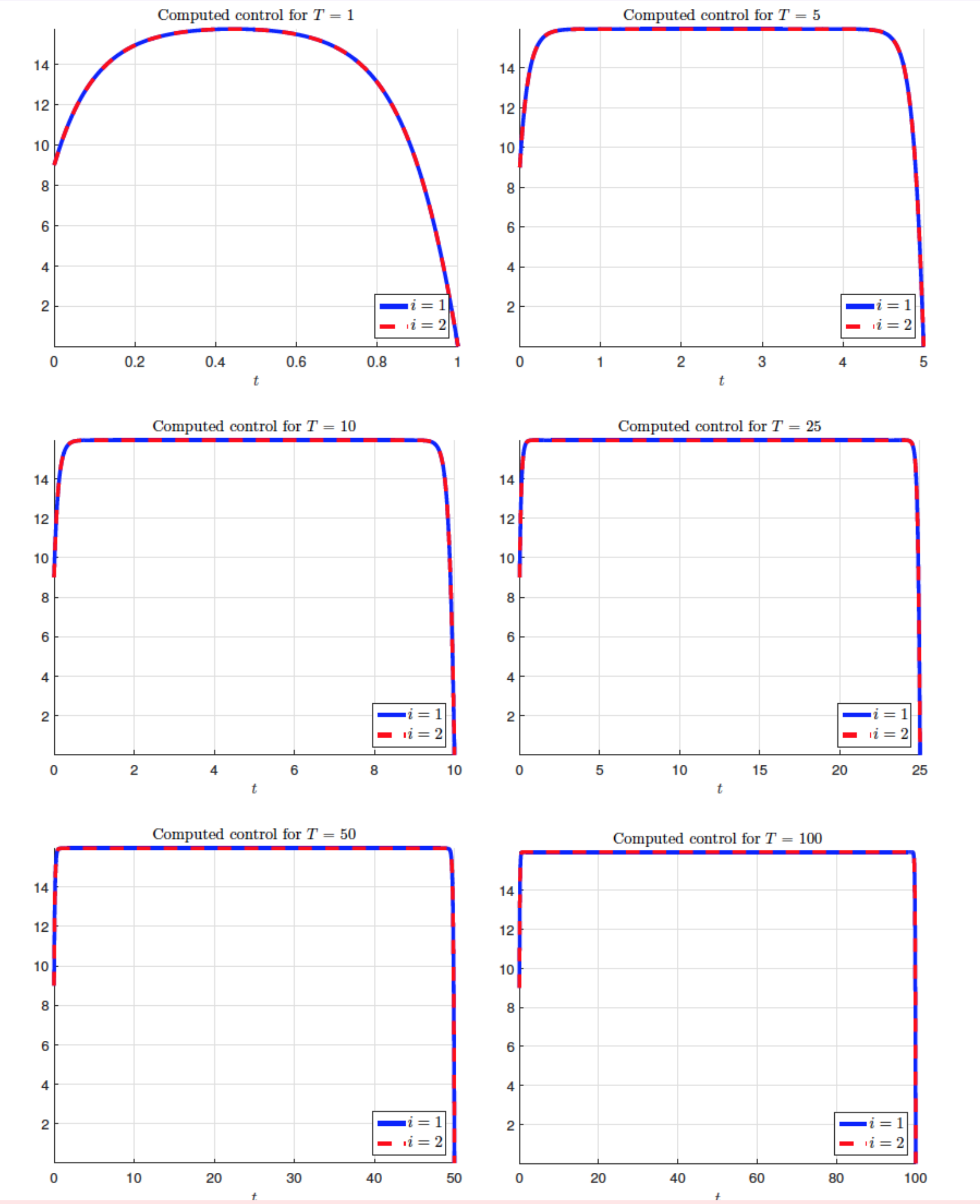
This is a forward-backward parabolic system.
A spectral decomposition exhibits the characteristic values

$$\mu_j^{\pm} = \pm\sqrt{1+\lambda_j^2}$$

where $(\lambda_j)_{j\geq 1}$ are the (positive) eigenvalues of $-\Delta$.
Thus, the system is the superposition of growing + diminishing real exponentials.

The turnpike property for the heat equation

This new dynamic behaviour, combining exponentially stable and unstable branches, is compatible with the turnpike behavior.
Controls and trajectories exhibit the expected dynamics:



Optimal pairs $(f(t), y(t))$ are exponentially close to the steady-state optimal pair characterised as the minima for the functional

$$J_s(g) = \frac{1}{2}\int_\omega g^2 dx + \frac{1}{2}\int_\Omega z^2 dx + \frac{1}{2}\int_\Omega |z(x) - y_d|^2 dx \qquad (4)$$

where $z = z(x)$ solves

$$\begin{cases} -\Delta z = g 1_\omega & \text{in} \quad \Omega \\ z = 0 & \text{on} \quad \partial\Omega, \end{cases} \qquad (5)$$

Namely

$$||y(t) - z|| + ||f(t) - g|| \leq C[\exp(-\mu t) + \exp(-\mu(T - t))]$$

if $T >> 1$.[4] [5]

[4]A. Porretta & E. Z. , Long time versus steady state optimal control, SIAM J. Control Optim., 51 (6) (2013), 4242-4273.

[5]E. Trélat & E. Z., The turnpike property in finite-dimensional nonlinear optimal control, JDE, 218 (2015) , 81-114.

What is the reason?

- The same is true for the wave equation. [6]
- The property is rather independent of the PDE under consideration.
- Optimal controls and trajectories exhibit the turnpike pattern in long times if the cost functional penalizes sufficiently both state and control (the accurate explanation requires of some in-depth understanding of controllability issues)

[6] M. Gugat, E. Trélat, E. Zuazua, Systems and Control Letters, 90 (2016), 61-70.

A major technical difficulty for nonlinear problems

Consider now the semilinear heat equation:

$$\begin{cases} y_t - \Delta y + y^3 = f 1_\omega & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x,0) = y^0(x) & \text{in} \quad \Omega \end{cases} \tag{6}$$

$$\min_f \Big[\frac{1}{2}\int_0^T \int_\Omega |y - y_d|^2 dxdt + \int_0^T \int_\omega f^2 dxdt\Big].$$

The optimality system reads:

$$y_t - \Delta y + y^3 = -\varphi 1_\omega \text{ in } Q$$

$$y = 0 \text{ on } \Sigma$$

$$y(x,0) = y^0(x) \text{ in } \Omega$$

$$-\varphi_t - \Delta\varphi + 3y^2\varphi = y - y_d \text{ in } Q$$

$$\varphi = 0 \text{ on } \Sigma$$

$$\varphi(x,T) = 0 \text{ in } \Omega.$$

And the linearised optimality system, around the optimal steady solution $(\bar{y}, \bar{\varphi})$ is as follows:

$$z_t - \Delta z + 3(\bar{y})^2 z = -\psi 1_\omega \text{ in } Q$$

$$z = 0 \text{ on } \Sigma$$

$$z(x, 0) = 0 \text{ in } \Omega$$

$$-\psi_t - \Delta \psi + 3(\bar{y})^2 \psi + 6\bar{y}\varphi z = z \text{ in } Q$$

$$\psi = 0 \text{ on } \Sigma$$

$$\psi(x, T) = 0 \text{ in } \Omega.$$

The equations describing the dynamics of the linearised optimality system red as follows:

$$z_t - \Delta z + 3(\bar{y})^2 z = -\psi 1_\omega$$

$$-\psi_t - \Delta\psi + 3(\bar{y})^2\psi = (1 - 6\bar{y}\varphi)z$$

This is the optimality system for a LQ control problem of the model

$$z_t - \Delta z + 3(\bar{y})^2 z = f 1_\omega$$

and the cost

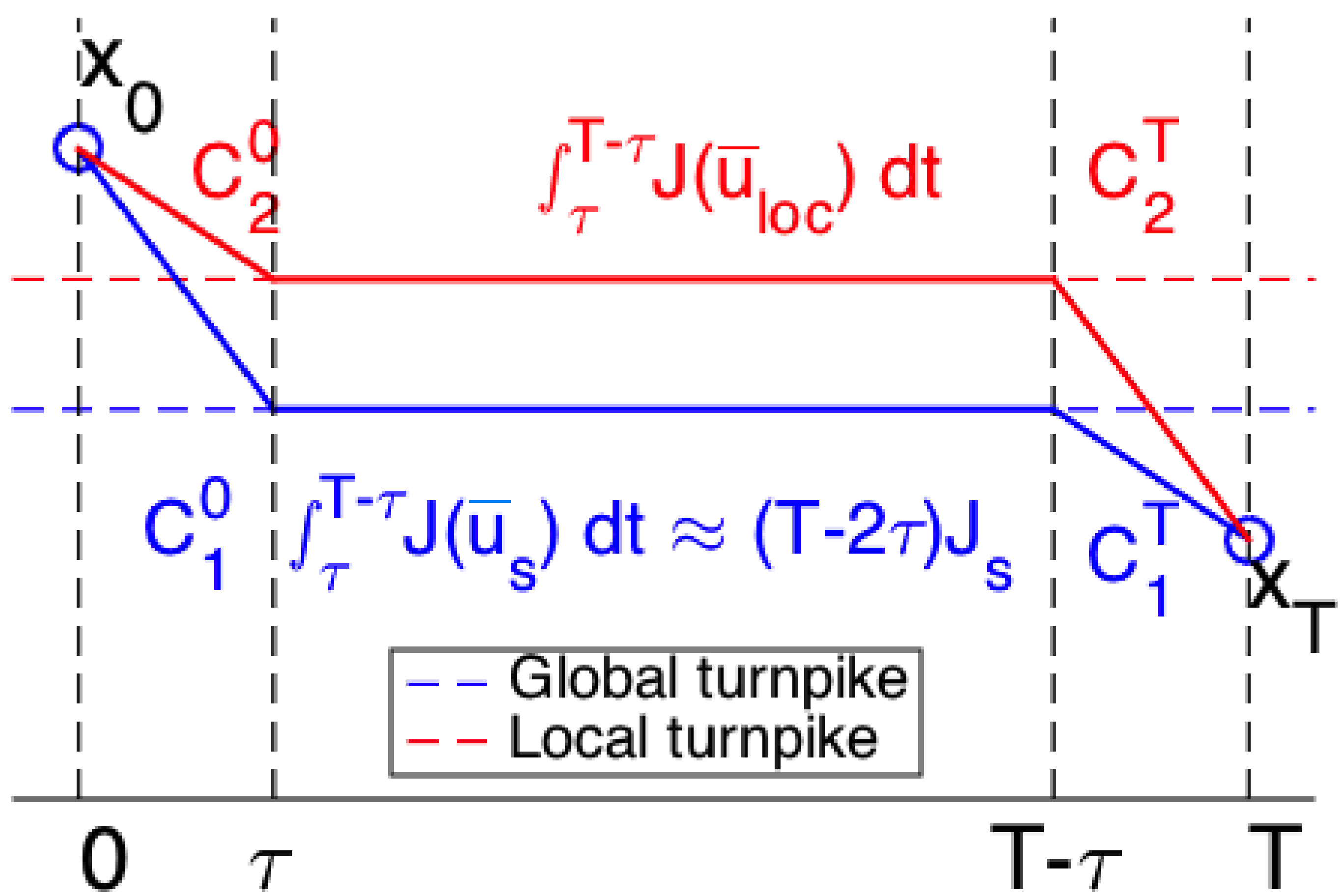
$$\min_f \left[\frac{1}{2}\int_0^T \int_\Omega |z|^2 dxdt + \int_0^T \int_\omega \rho(x) f^2 dxdt\right]$$

$$\rho(x) = 1 - 6\bar{y}(x)\varphi(x).$$

And the turnpike property holds as soon as $\rho(x) \geq \delta > 0$.
This holds if $\bar{y}$ and φ are small enough, and this requires the **smallness of the target**.

Heuristic explanation

Numerical simulations show that the turnpike property is quite robust and the smallness of the target does not seem to be needed.
In applications, and in daily life, we use a quasi-turnpike principle that is very robust and universal too.



Perspectives and open problems

Perspectives

- The turnpike theory or principle can be used in various different manners. It serves to handle more simply issues such as parameter dependent optimal control problems, model reduction, etc.
- Turnpike offers a natural way of obtaining easily a first approximation to the control: Just chose the steady state optimal strategy!

Open problems

- Computationally one observes the turnpike property to hold for nonlinear problems, much beyond the LQR frame where theory is well developed. A new approach to deal with nonlinearity?
- Turnpike for shape and optimal design problems. Systematically used in practice without a proof. Much harder since the control enters through the geometry of the domain, or coefficients of the PDE.