The theoretical and practical use of the turnpike property for Model Predictive Control

Chair of Applied Mathematics
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Turnpike in a Nutshell, Chair of Applied Analysis, Erlangen

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Outline

1. Connection of MPC and Turnpike
2. Turnpike and exponential decay of discretization errors
3. Goal oriented error estimation for MPC

Contents

Infinite horizon optimal control problem

Consider a state $y \in Y$ und control $u \in U$ and a system

$$\dot{y} = f(y,u) \qquad y(0) = y_0 \tag{1}$$

we want to control optimally w.r.t. a running cost $f^0 : Y \times U \to \mathbb{R}$.

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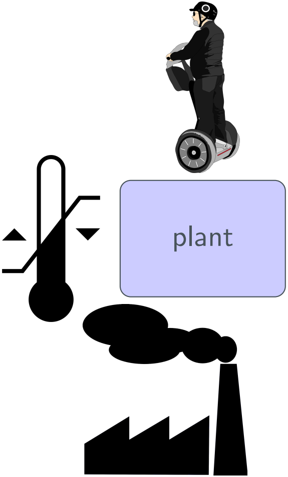
solution hard + not robust w.r.t. perturbations

Solution via model predictive control



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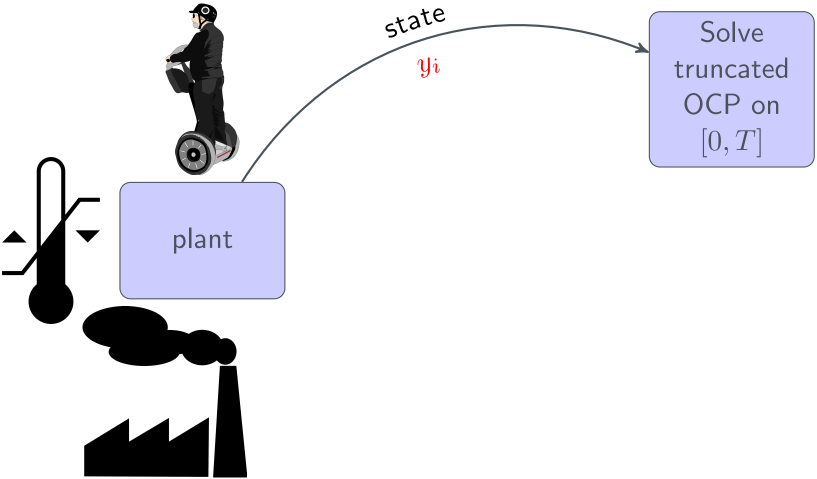

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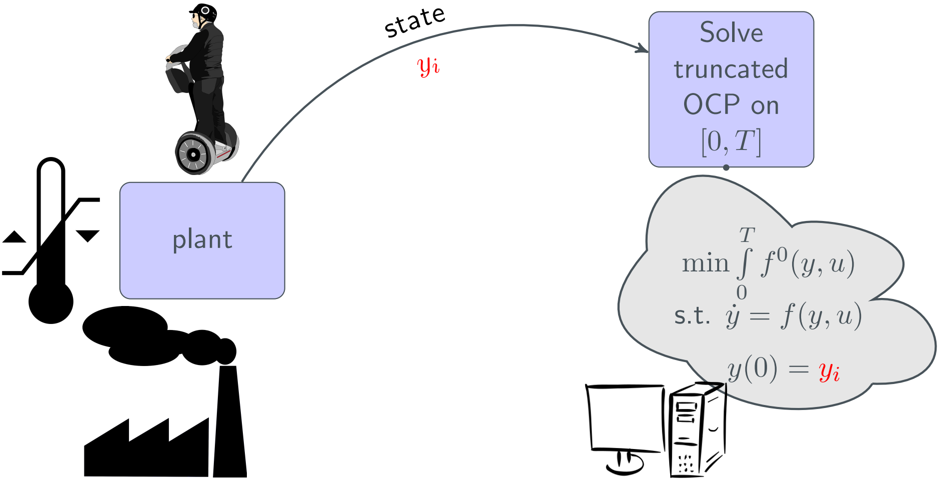

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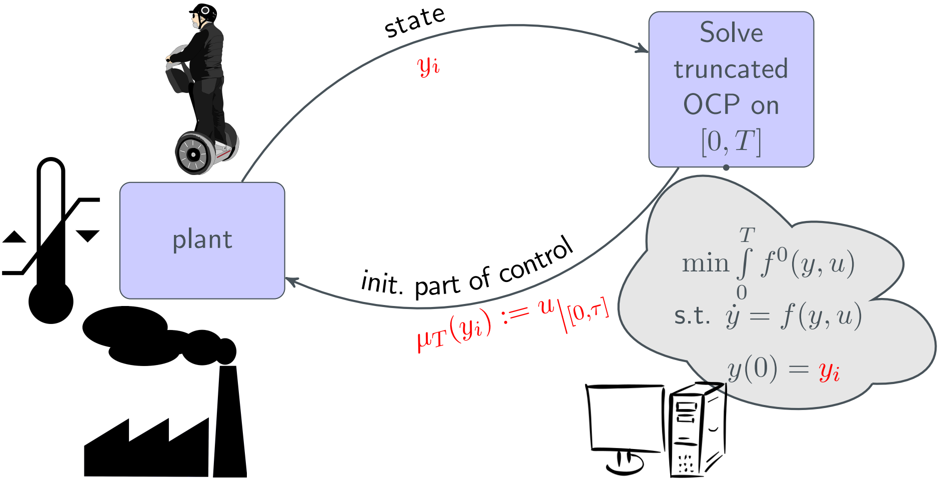

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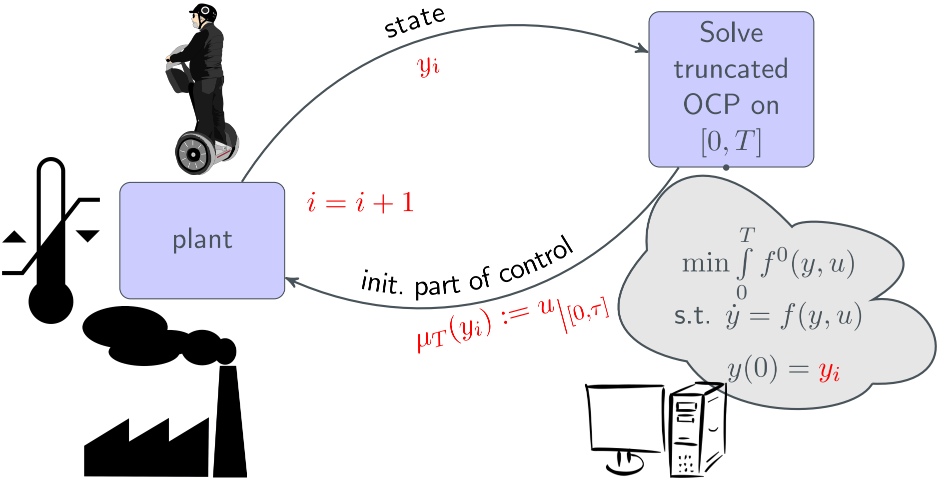
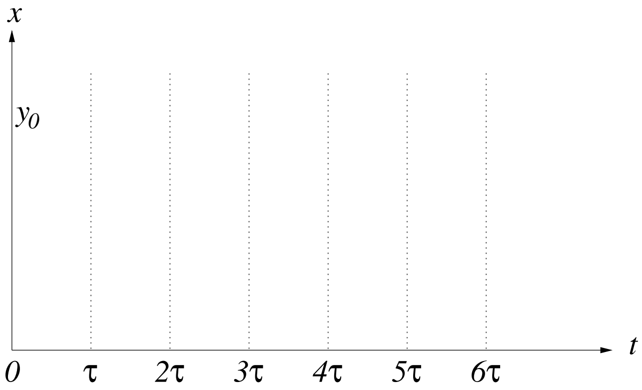


Usually $\tau \ll T < \infty$

Trajectories



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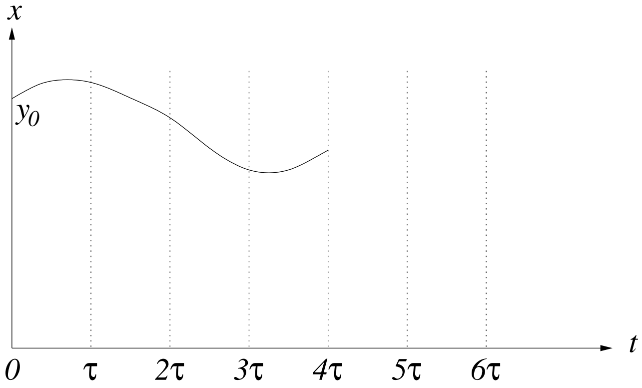
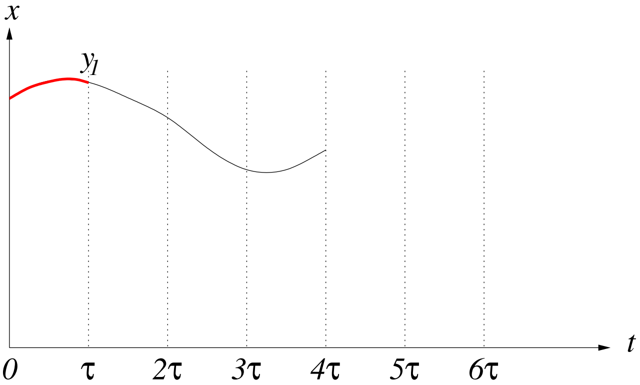


Black = Open loop

Trajectories



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Red = MPC closed loop

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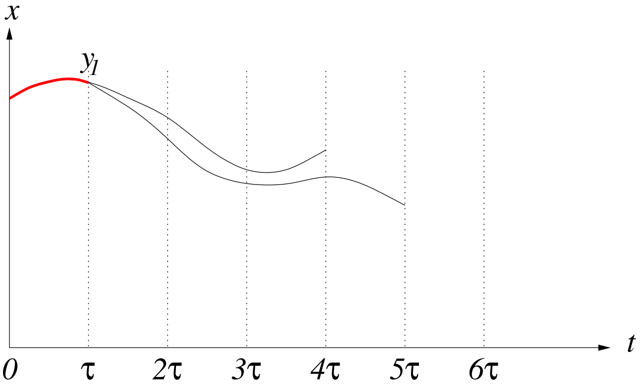

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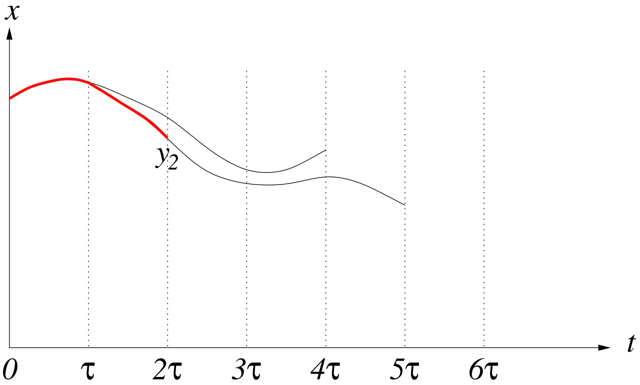

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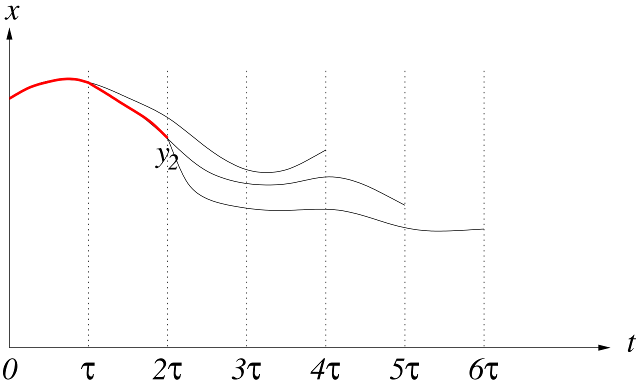

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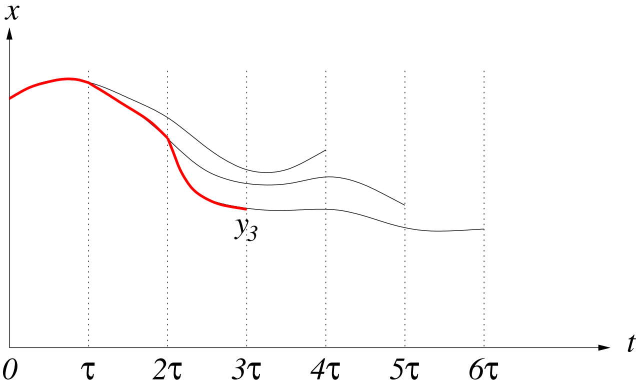

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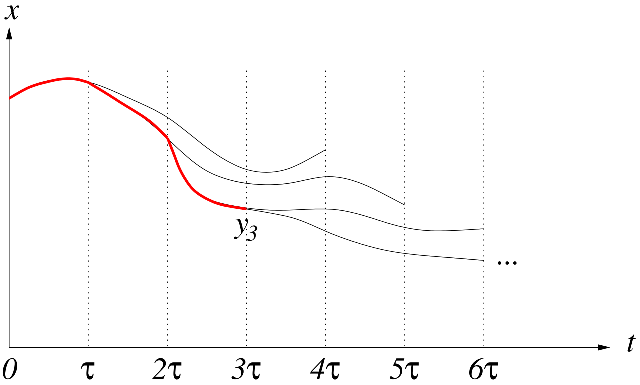

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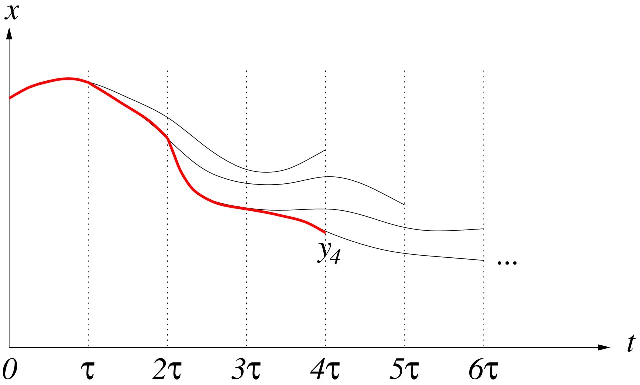

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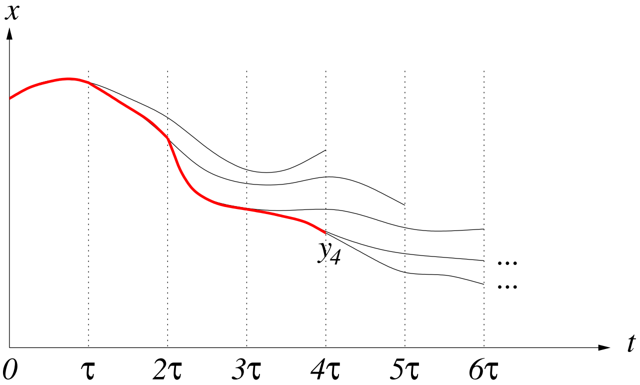

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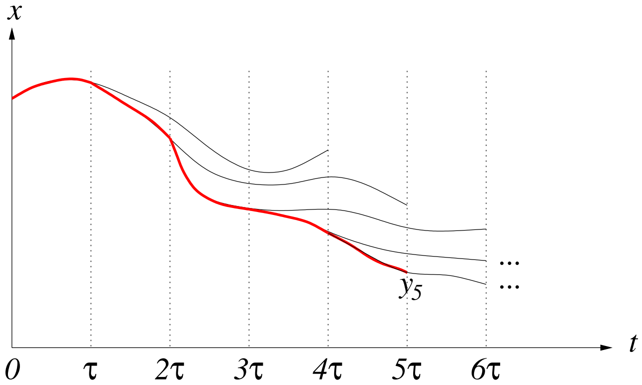

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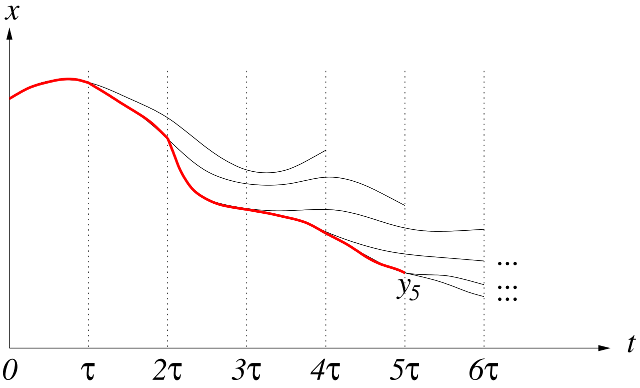

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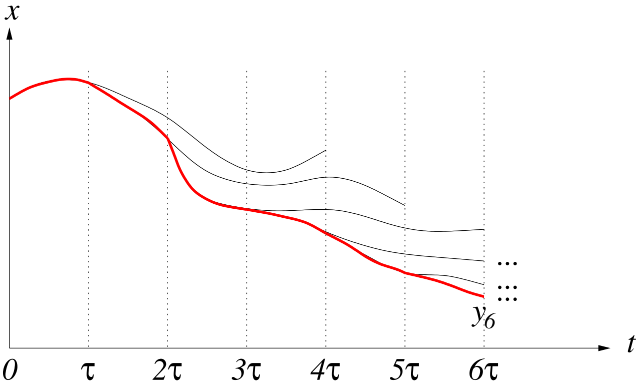

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MPC constructs feedback for infinite horizon problem by iteratively solving finite horizon problems.

Pros:

- Solving on $[0, T]$ not as hard as on $[0, \infty]$.
- Robust w.r.t. external influences, perturbation, no need for full future information
- control and state constraints can be handled easily.

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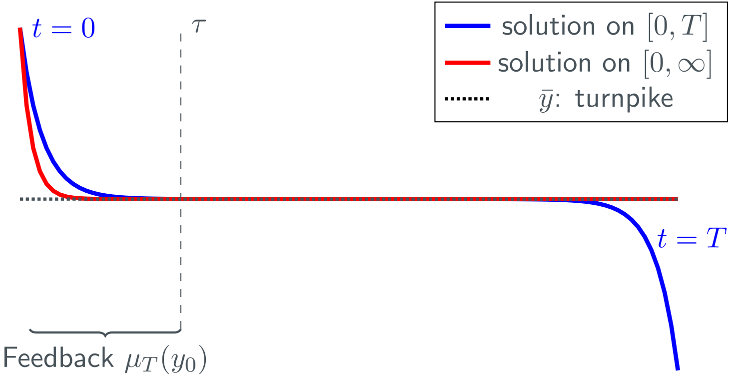
Cons:

- Why should the feedback $\mu_T(y_i)$ be optimal for the original problem on infinite horizon?

Turnpike Property



Performance estimates (Grüne & Pannek 2017)

$$\min J_\infty(y,u) := \int_0^\infty f^0(y(t),u(t))\,dt,$$
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- $J_K^{\mathsf{cl}}(y_0, u_{\mathsf{MPC},T}) := \int_0^K f^0(y_{u_{\mathsf{MPC},T}}, u_{\mathsf{MPC},T})$ (think of $K = n\tau$)
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- $\delta(T) \to 0$ as $T \to \infty$

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- $J_K^{\text{cl}}(y_0, u_{\text{MPC},T}) \le \inf_{u\in\tilde{U}} J_K(y_0,u) + \delta_1(T) + K\delta_2(T) + \delta_3(K)$

Necessary and sufficient conditions

OCP has turnpike property $\Rightarrow$ MPC-closed loop approximately optimal on infinite horizon (Grüne '13, Grüne/Stieler '14, Grüne/Pirkelmann '18)

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- strict dissipativity $\Longrightarrow$ Turnpike (Carlson et al. '91, Grüne '13, Grüne/Stieler/Pirkelmann '18)

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- System stabilizable: Turnpike $\Longleftrightarrow$ Detectability (Grüne/Guglielmi '18)
- Stabilizability and Detectability $\Rightarrow$ Turnpike for OC of PDEs (Trélat, Zhang, Zuazua, Poretta, Gugat, Zamorano, Breiten et al. ... '13–'19)

Recap

- MPC approximates infinite horizon problem by iteratively solving finite horizon subproblems if turnpike holds

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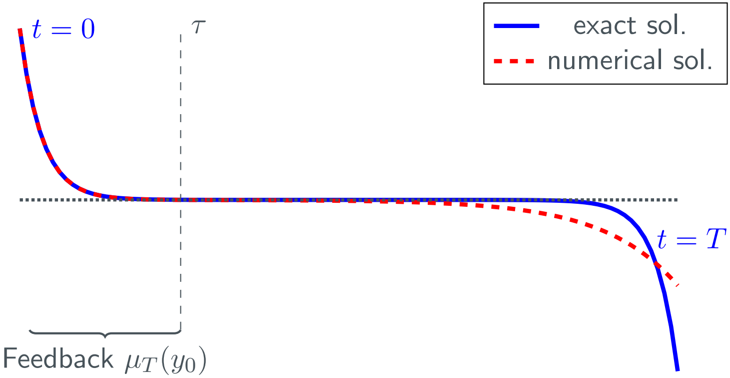
- MPC feedback given by part of optimal control $u\big|_{[0,\tau]}$ in every loop

Using the turnpike property numerically?



Goal: Discretization errors towards T have little influence on MPC-Feedback, if T is large.

Contents

Linear-quadratic OCP

Optimal Control problem.

$$\min_{y,u} \frac{1}{2}\int_0^T \|C(y(t)-y_d)\|_Y^2 + \|R(u(t)-u_d)\|_U^2 \, dt$$

$$\text{s.t.} \quad \dot{y} = Ay + Bu + f$$
$$y(0) = y_0 \in X$$

with

- X, Y, U Hilbert Spaces
- A generates $C0$-semigroup on X, B admissible control operator, $C \in L(X,Y)$
- $R \in L(U,U)$, $\|Ru\|_U^2 \geq \alpha\|u\|_U^2$ for $\alpha > 0$

Optimality conditions

(y, u) optimal, Lagrange multiplier $\lambda \in C(0, T; X)$ s.t.

$$\begin{aligned} C^*Cy - \lambda' - A^*\lambda &= C^*Cy_d, \qquad \lambda(T) = 0 \\ R^*Ru - B^*\lambda &= R^*Ru_d, \\ y' - Ay - Bu &= f, \qquad y(0) = y_0 \end{aligned}$$

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- $(\bar{y}, \bar{u}, \bar{\lambda})$ solves steady state problem

Theorem (Grüne, S., Schiela 2018)

Let (A, B) be exponentially stabilizable, (A, C) be exponentially detectable. Then there is $\mu, c > 0$ ind. of T, such that

$$\|y(t) - \bar{y}\| + \|u(t) - \bar{u}\| + \|\lambda(t) - \bar{\lambda}\| \sim c(e^{-\mu t} + e^{-\mu(T-t)})$$

Decay of discretization errors

- $(\tilde{y}, \tilde{u}, \tilde{\lambda})$ numerical solution of optimality system on space/time grid

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- If (A, B) exactly controllable, then one can also impose an end time condition on the state and the statements remain true.

Contents

Refinement for a quantity of interest (Meidner&Vexler '07)

Given: Quantity of interest $I(y,u) : Y \times U \to \mathbb{R}$.
Wanted: Time and space grid, such that numerical solution $(\tilde{y}, \tilde{u})$ on grid has small error w.r.t. I, i.e.

$$I(y,u) - I(\tilde{y}, \tilde{u}) < tol$$

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In the particular MPC-case, one is only interested in $u_{\big|[0,\tau]}$, so we choose

$$I^\tau(y,u) = \int_0^\tau f^0(y,u)\, dt$$

A numerical example

A priori analysis suggests, that to obtain small error w.r.t. $I^\tau(y, u)$, only refinement on $[0, \tau]$ is needed.

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Example (Distributed control of heat)

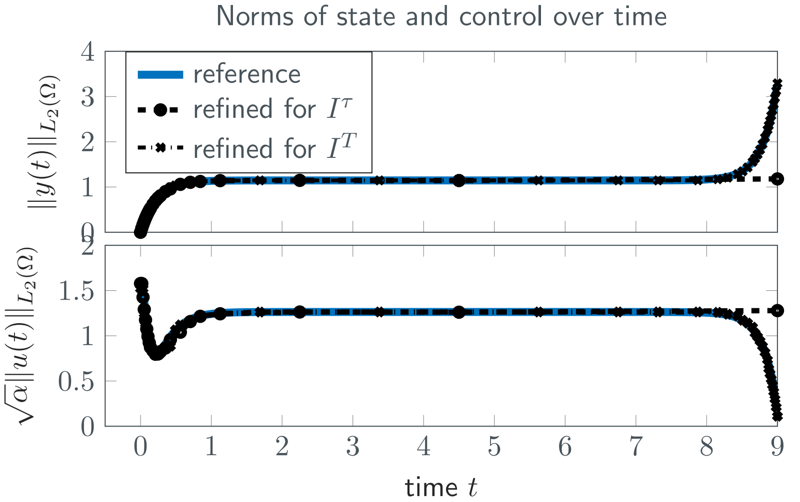
$$\min \frac{1}{2}\int_0^T \|y(t) - y_{\mathrm{ref}}\|^2_{L_2(\Omega)} + \alpha\|u(t)\|^2_{L_2(\Omega)}\,dt$$

$$\dot{y} = 0.1\Delta y + sy + u \qquad \text{on } \Omega$$

$$y = 0 \qquad \text{on } \partial\Omega$$

$$y(0) = 0 \qquad \text{on } \Omega$$

Time adaptivity - open loop ($\alpha = 10^{-1}, s = 5, \tau = 0.5$)



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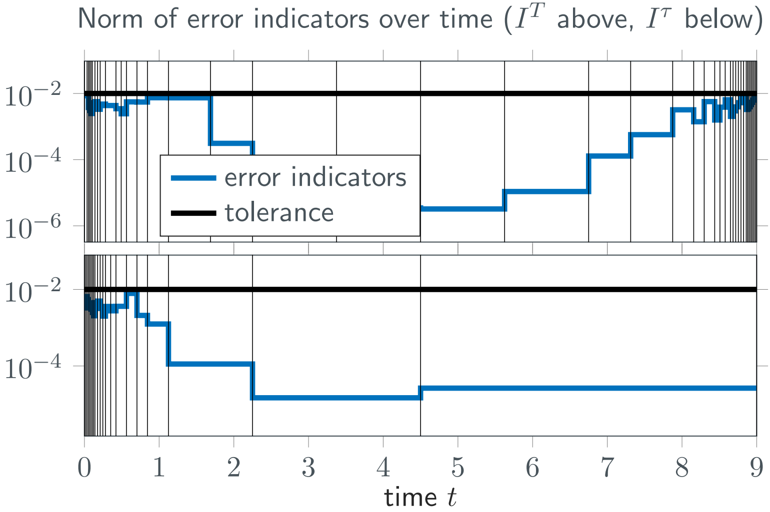
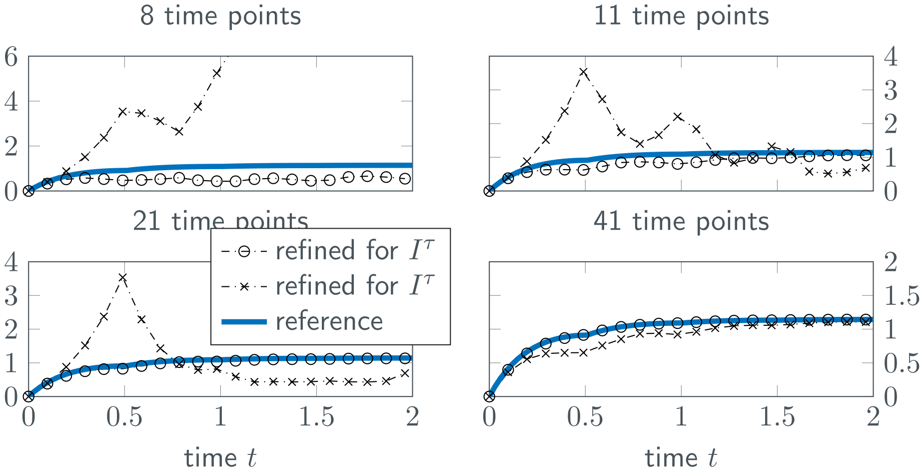


Time adaptivity - MPC-closed loop



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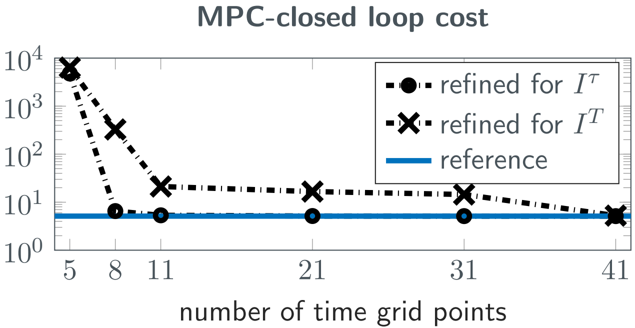
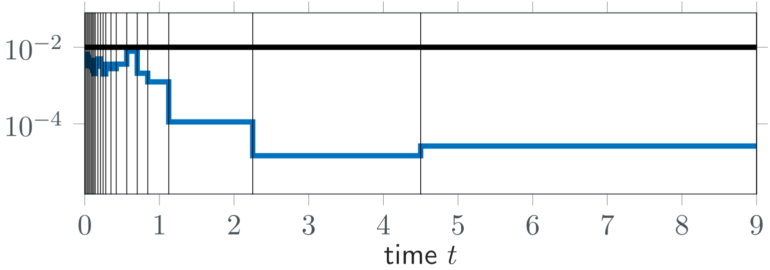


Intermezzo: Decay of error indicators

Error indicators for $I^\tau(y,u)$



- $I^\tau(y,u) = \int_0^{\tau} \|y(t) - y_{\text{ref}}\|^2 + \alpha\|u\|^2$, here $\tau = 0.5$.

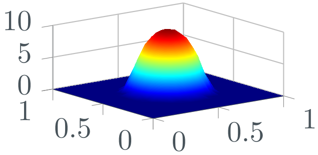
Theorem (S. 2019)

(A,B) stabilizable, (A,C) detectable. Then with η^τ error indicators for I^τ

$$\|\eta^\tau(t)\| \sim c(\tau)e^{-\mu t} \qquad c(\tau), \mu > 0 \text{ ind. of } T.$$

Space adaptivity - open loop ($\alpha = 10^{-3}, s = 0$)

$y_{\text{ref}}(x_1, x_2)$:



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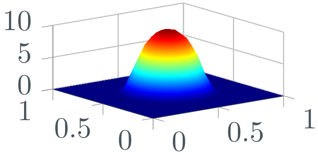
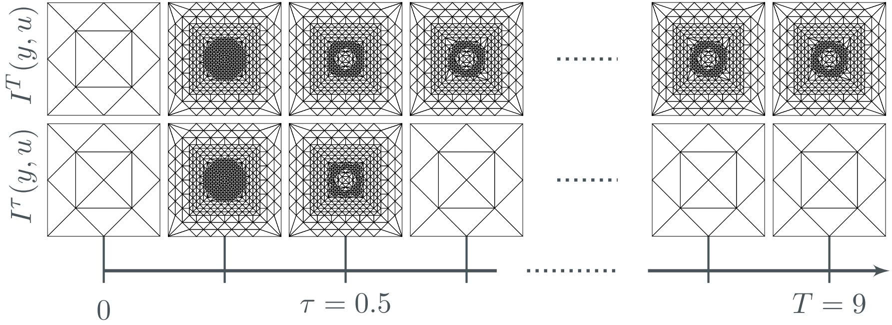
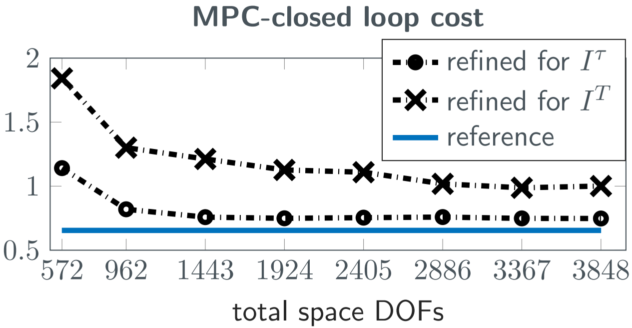


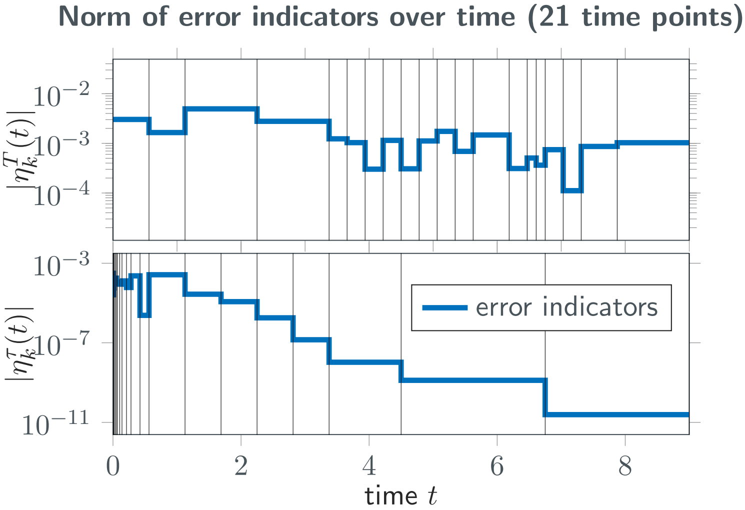
Space grids over time



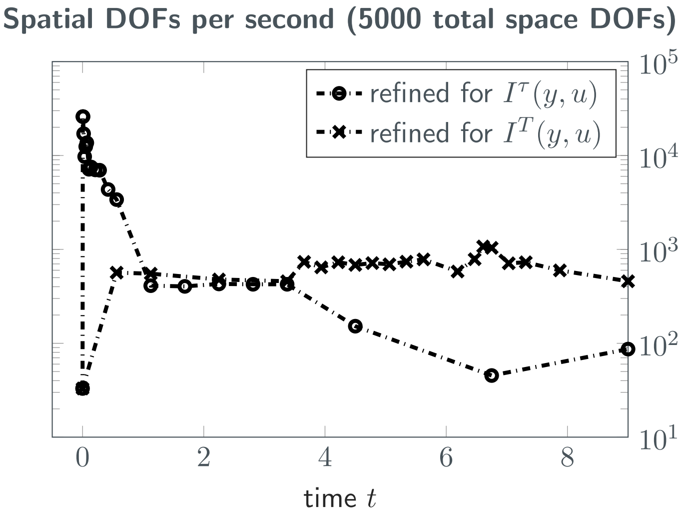
Space adaptivity - MPC-closed loop



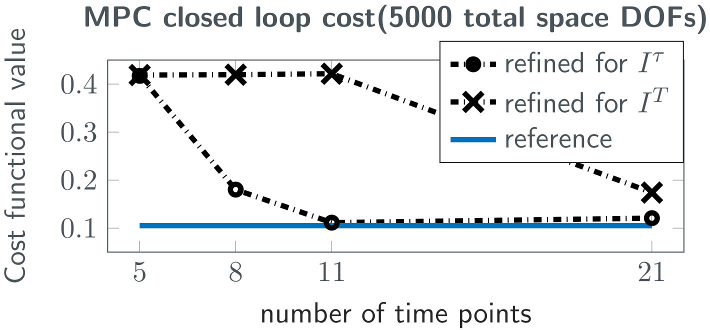
Space-time adaptivity - open loop $(y_{\text{ref}} = y_{\text{ref}}(t, x))$



Space-time adaptivity - open loop



Space-time adaptivity - MPC-closed loop



Recap

- If stabilizable+detectable, then
  - Turnpike property.
  - Influence of discretization errors decays exponentially in time
    $\hookrightarrow$ in MPC-context: coarsening of grids towards T.
- Goal oriented space-time error estimation techniques confirm these findings

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Ongoing work:

- Combine nonlinear solver with error estimation techniques and MPC-controller in an efficient manner

References

- Grüne, S., Schiela: Sensitivity analysis of optimal control for a class of parabolic PDEs motivated by model predictive control, SICON 2019
- Grüne, S., Schiela: Exponential sensitivity and turnpike analysis for linear quadratic optimal control of general evolution equations, Dec 2018, submitted
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Thank you for your attention!

Perturbations stay local in time

Notation:

- $\|M^{-1}\|_{L_2,C} = \|M^{-1}\|_{(L_2(0,T;X)\times X)^2 \to C(0,T;X)^2}$
- $\|M^{-1}\|_{L_2,L_2} = \|M^{-1}\|_{(L_2(0,T;X)\times X)^2 \to L_2(0,T;X)^2}$

Theorem (Grüne, S., Schiela, 2018)

$(\tilde{y}, \tilde{u}, \tilde{\lambda})$ *computed solution,* (y, u, λ) *exact solution and*

- $(\delta y, \delta u, \delta\lambda) := (\tilde{y}, \tilde{u}, \tilde{\lambda}) - (y, u, \lambda)$
- $0 \leq \mu < \frac{1}{\|M^{-1}\|_{(L_2,L_2)}}$
- $\|e^{-\mu \cdot}\varepsilon_1\|_{L_2(0,T;X)} + \|e^{-\mu \cdot}\varepsilon_2\|_{L_2(0,T;X)} \leq \rho, \quad \rho \geq 0$

Then, there is a constant $C \geq 0$ *indep. of* T *s.t.*

$$\|e^{-\mu \cdot}\delta y\|_{C(0,T;X)} + \|e^{-\mu \cdot}\delta u\|_{L_2(0,T;U)} + \|e^{-\mu \cdot}\delta\lambda\|_{C(0,T;X)} \leq C\rho\|M^{-1}\|_{L_2,C}$$